The Existence of Optimal Regulators

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Abstract

We show the existence of a local solution to a parametrized family of infinite horizon optimal control problems such as those that arise in nonlinear regulation, disturbance rejection, gain schelduling and linear parameter varying control

Key words: Parametrized optimal control, nonlinear regulation, nonlinear disturbance rejection, gain scheduling, linear parameter varying control.

1 Introduction

Consider a smooth nonlinear plant

$$\begin{array}{rcl} \dot{x} & = & f(x,u,\bar{x}) \\ & = & Ax+Bu+F\bar{x} \\ & & +f^{[2]}(x,u,\bar{x})+O(x,u,\bar{x})^3 \\ e & = & h(x,u,\bar{x}) \\ & = & Cx+Du+H\bar{x} \\ & & +h^{[2]}(x,u,\bar{x})+O(x,u,\bar{x})^3 \end{array} \tag{1.1}$$

which is perturbed by a smooth nonlinear exosystem

$$\begin{array}{rcl} \dot{\bar{x}} & = & \bar{f}(\bar{x}) \\ & = & \bar{A}\bar{x}+\bar{f}^{[2]}(\bar{x})+O(\bar{x})^3 \end{array} \tag{1.2}$$

where superscript $[d]$ denotes terms composed of homogeneous polynomials of degree d. The dimensions of $x, u, \bar{x}, e$ are $n, m, \bar{n}, p$ respectively.

The goal of regulation is to use a combination of feedforward and feedback control $u = \alpha(x, \bar{x})$ so that the output of the plant asymptotically goes to 0,

$$e(t) \longrightarrow 0$$

for every $x(0)$, $\bar{x}(0)$. The plant should also be internally stable. The exosystem could be a system whose output we wish the plant to track, a noise source whose disturbance we wish the plant to reject or static and/or dynamic parameters to be used for schelduling the controller of the plant. A linear parameter varying (LPV) system falls into the last category. We make the reasonable assumptions that linear part of the plant is stabilizable and detectable when $\bar{x} = 0$ and the linear part of the the exosystem is not unstable.

The solution of the regulator problem is in two steps. The first is to use feedforward from the exosystem state to insure exact tracking when the initial conditions of the plant and the exosystem permit this. The linear version of the problem was solved by Francis [5] and its nonlinear generalization is due to Isidori and Byrnes [7]. One seeks $\theta(\bar{x})$, $\beta(\bar{x})$ satisfying the Francis-Byrnes-Isidori (FBI) PDE

$$\begin{array}{rcl} f(\theta(\bar{x}),\beta(\bar{x}),\bar{x}) & = & \frac{\partial\theta}{\partial\bar{x}}(\bar{x})\bar{f}(\bar{x}) \\ h(\theta(\bar{x}),\beta(\bar{x}),\bar{x}) & = & 0 \end{array} \tag{1.3}$$

If the FBI PDE is solvable then the control $u = \beta(\bar{x})$ makes $x = \theta(\bar{x})$ an invariant manifold of the combined system consisting of plant and exosystem. And on this manifold, exact tracking occurs, $e = 0$.

One can attempt to solve the FBI equations term by term. Suppose

$$\begin{array}{rcl} \theta(\bar{x}) & = & T\bar{x}+\theta^{[2]}(\bar{x})+O(\bar{x})^3 \\ \beta(x,\bar{x}) & = & L\bar{x}+\beta^{[2]}(\bar{x})+O(\bar{x})^3. \end{array}$$

The linear part of the FBI equations are the Francis equations

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} T \\ L \end{bmatrix} - \begin{bmatrix} T \\ O \end{bmatrix}\bar{A} = -\begin{bmatrix} F \\ H \end{bmatrix}$$

These equations are solvable for any F, H iff no output zero of the plant is a pole of the exosystem [6, 8, 9]. In other words, the exosystem should not excite those frequencies that the plant cannot produce.

The output zeros of the plant are those complex numbers s for which there exist complex n and p vectors ξ and ζ such that

$$\begin{bmatrix} \xi & \zeta \end{bmatrix}\begin{bmatrix} A-sI & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

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There may be a finite or infinite number of output zeros. For example if $m = p$ then there are either $n + m$ zeros or every s is a zero. The poles $\lambda_1, \ldots, \lambda_{\bar{n}}$ of the exosystem are the eigenvalues of $\bar{A}$.

If there is a resonance between a pole and zero the equations will still be solvable for some F, H. The solvability depends on the direction of the zero ξ, ζ and the eigenvector of the pole.

The higher degree equations are linear and depend on the solutions of the lower degree equations. They are solvable for arbitrary higher degree terms iff the harmonics of the exosystem don't resonate with the zeros of the plant [6, 8, 9].

For example the degree two equations are

$$\begin{aligned} A\theta^{[2]}(\bar{x}) &+ B\beta^{[2]}(\bar{x}) - \frac{\partial\theta}{\partial\bar{x}}(\bar{x})\left(\bar{A}\bar{x}\right) \\ &= -f^{[2]}(T\bar{x}, L\bar{x}, \bar{x}) + T\bar{f}^{[2]}(\bar{x}) \\ C\theta^{[2]}(\bar{x}) &+ D\beta^{[2]}(\bar{x}) \\ &= -h^{[2]}(T\bar{x}, L\bar{x}, \bar{x}) \end{aligned}$$

These are solavble for arbitrary $f^{[2]}, h^{[2]}$ iff no ouput zero of the plant equals the sum of two poles of the exosystem, $\lambda_k \neq s_i + s_j$. If there is a resonance they are solvable for some $f^{[2]}, h^{[2]}$.

Now suppose that the FBI equations have been solved. The second step is to use additional feedforward and feedback to insure that the closed loop system converges to the tracking manifold $x = \theta(\bar{x})$ where $e = 0$. This can be achieved locally by linear pole placement techniques but an alternative approach is to use optimal control methods to achieve a nonlinear solution [8, 9]. Define transverse coordinates z, v by

$$\begin{aligned} z &= x - \theta(\bar{x}) = x - T\bar{x} - \theta^{[2]}(\bar{x}) + O(\bar{x})^3 \\ v &= u - \beta(\bar{x}) = u - L\bar{x} - \beta^{[2]}(\bar{x}) + O(\bar{x})^3 \end{aligned} \quad (1.4)$$

In these coordinates the plant and exosystem are of the form

$$\begin{aligned} \dot{z} &= \tilde{f}(z, v, \bar{x}) = Ax + Bu + \tilde{f}^{[2]}(z, v, \bar{x}) + O(z, v, \bar{x})^3 \\ \dot{\bar{x}} &= \bar{f}(\bar{x}) = \bar{A}\bar{x} + \bar{f}^{[2]}(\bar{x}) + O(\bar{x})^3 \\ e &= \tilde{h}(z, v, \bar{x}) = Cz + Dv + \tilde{h}^{[2]}(z, v, \bar{x}) + O(z, v, \bar{x})^3 \end{aligned} \quad (1.5)$$

where

$$\begin{aligned} \tilde{f}(z, v, \bar{x}) &= f(z + \theta(\bar{x}), v + \beta(\bar{x}), \bar{x}) - f(\theta(\bar{x}), \beta(\bar{x}), \bar{x}) \\ \tilde{h}(z, v, \bar{x}) &= h(z + \theta(\bar{x}), v + \beta(\bar{x}), \bar{x}) \end{aligned} \quad (1.6)$$

Notice that the linear part of the z dynamics and the linear part of the output are unaffected by $\bar{x}$. The linear part is stabilizable and detectable by assumption. Furthermore

$$\begin{aligned} \tilde{f}(0, 0, \bar{x}) &= f(\theta(\bar{x}), \beta(\bar{x}), \bar{x}) - f(\theta(\bar{x}), \beta(\bar{x}), \bar{x})) = 0 \\ \tilde{h}(0, 0, \bar{x}) &= h(\theta(\bar{x}), \beta(\bar{x}), \bar{x}) = 0 \end{aligned} \quad (1.7)$$

A stabilizing feedback can be found by minimizing

$$\tfrac{1}{2}\int_0^\infty \|e\|^2 + \|v\|^2 dt \quad (1.8)$$

subject to the dynamics (1.5). Other costs criterions can be used. Let $\pi(z, \bar{x})$ denote the optimal cost and $\kappa(z, \bar{x})$ the optimal feedback then π, γ satisfy the Hamiliton-Jacobi-Bellman (HJB) PDE

$$\begin{aligned} 0 &= \tfrac{\partial\pi}{\partial z}(z, \bar{x})\tilde{f}(z, \gamma(z, \bar{x}), \bar{x}) + \tfrac{\partial\pi}{\partial\bar{x}}(z, \bar{x})\bar{f}(\bar{x}) \\ &\quad + l(z, \gamma(z, \bar{x}), \bar{x}) \\ 0 &= \tfrac{\partial\pi}{\partial z}(z, \bar{x})\tfrac{\partial\tilde{f}}{\partial v}(z, \gamma(z, \bar{x}), \bar{x}) + \tfrac{\partial l}{\partial v}(z, \gamma(z, \bar{x}), \bar{x}) \end{aligned} \quad (1.9)$$

where

$$\begin{aligned} l(z, v, \bar{x}) &= \tfrac{1}{2}(\|e\|^2 + \|v\|^2) \\ &= \tfrac{1}{2}(z'Qz + 2z'Sv + v'Rv) \\ &\quad + l^{[3]}(z, v, \bar{x}) + O(z, v, \bar{x})^4 \end{aligned} \quad (1.10)$$

for some matrices Q, R, S and some cubic polynomial $l^{[3]}$.

By generalizing Al'brecht's method [1], we can solve the HJB PDE term by term [8]. Since

$$\begin{aligned} \tilde{f}(z, v, \bar{x}) &= O(z, v) \\ \tilde{h}(z, v, \bar{x}) &= O(z, v) \\ l(z, v, \bar{x}) &= O(z, v)^2 \end{aligned} \quad (1.11)$$

we expect that

$$\begin{aligned} \pi(z, \bar{x}) &= O(z)^2 \\ \gamma(z, \bar{x}) &= O(z). \end{aligned} \quad (1.12)$$

In particular, we expect that

$$\begin{aligned} \pi(z, \bar{x}) &= \frac{1}{2}z'Pz + \pi^{[3]}(z, \bar{x}) + O(z, \bar{x})^4 \\ \gamma(z, \bar{x}) &= Kz + \gamma^{[2]}(z, \bar{x}) + O(z, \bar{x})^3 \end{aligned}$$

The lowest degree terms in the HJB equations are the familiar Riccati equation and the formula for the optimal linear feedback

$$\begin{aligned} 0 &= A'P + PA + Q - (PB + S)R^{-1}(PB + S)' \\ K &= -R^{-1}(PB + S)' \end{aligned} \quad (1.13)$$

At each higher degree $d > 1$, the equations are linear in the unknowns $\pi^{[d+1]}$, $\gamma^{[d]}$ and depend on the lower order terms of the solution. They are solvable if the linear part of the plant is stabilizable and the linear part of the exosystem is not unstable. For example, to find the next terms $\pi^{[3]}(z, \bar{x})$, $\gamma^{[2]}(z, \bar{x})$ one plugs the first two terms of π, γ into HJB equations and collects the next terms (degree 3 from the first HJB equation and degree 2 from the second HJB equation)

$$\begin{aligned} 0 &= \tfrac{\partial\pi^{[3]}}{\partial z}(z, \bar{x})(A + BK)z + \tfrac{\partial\pi^{[3]}}{\partial\bar{x}}(z, \bar{x})(\bar{A}\bar{x}) \\ &\quad + z'P\tilde{f}^{[2]}(z, Kz, \bar{x}) + l^{[3]}(z, Kz, \bar{x}) \\ 0 &= \tfrac{\partial\pi^{[3]}}{\partial z}(z, \bar{x})B + z'P\tfrac{\partial\tilde{f}^{[2]}}{\partial v}(z, Kz, \bar{x}) \\ &\quad + \gamma^{[2]}(z, \bar{x})'R + \tfrac{\partial l^{[3]}}{\partial v}(z, Kz, \bar{x}) \end{aligned} \quad (1.14)$$

Notice that the first equation involves only $\pi^{[3]}$, the other unknown $\gamma^{[2]}$ does not appear. This equation is solvable if $A + BK$ is asymptotically stable and $\bar{A}$ is not unstable. Given the solution $\pi^{[3]}$ then we can solve the second equation for $\gamma^{[2]}$

$$\begin{aligned}\gamma^{[2]}(z,\bar{x}) &= -R^{-1}\left(\frac{\partial \pi^{[3]}}{\partial z}(z,\bar{x})B\right.\\ &\qquad \left. +z'P\frac{\partial \tilde{f}^{[2]}}{\partial v}(z,Kz,\bar{x}) + \frac{\partial l^{[3]}}{\partial v}(z,Kz,\bar{x})\right)'\end{aligned}$$

The higher degree terms are found in a similar fashion.

Given the solutions of the FBI and HJB equations, the desired feedforward/feedback for exosystem matching is

$$\begin{aligned} u &= \alpha(x,\bar{x}) \\ &= \beta(\bar{x}) + \gamma(x - \theta(\bar{x}), \bar{x})\end{aligned}$$

In this paper we shall show using results from [3], [4] that the HJB PDE (1.9) is locally solvable. Furthermore its Taylor series expansion can be computed term by term as described above. To do so we shall use a invariant manifold theorem that we shall discuss in the next section. In Section 3 we use this theorem to show the local existence of the solution to the HJB equation (1.9).

2 Stable and Partial Center Manifold Theorem

The following theorem was proven by Aulbach, Flockerzi and Knobloch [3], [4].

Theorem 2.1 *Given*

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix} A_1 & 0 & 0\\ 0 & A_2 & 0\\ 0 & 0 & A_3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}f_1(x)\\ f_2(x)\\ f_3(x)\end{bmatrix} \tag{2.15}$$

where $x_i \in \mathbf{R}^{n_i}$, $n = n_1 + n_2 + n_3$ and $f_i(x)$ is C^k for $k \geq 2$.
Suppose that the eigenvalues of A_1 have negative real part, the eigenvalues of A_2 have nonnegative real part, the eigenvalues of A_3 have nonpositive real part and

$$f(0) = 0 \tag{2.16}$$

$$\frac{\partial f}{\partial x}(0) = 0 \tag{2.17}$$

$$f_i(0, x_2, 0) = 0,\ i = 1, 3. \tag{2.18}$$

Then there exists around $x = 0$ a local C^{k-2} invariant manifold

$$x_3 = \phi(x_1, x_2) \tag{2.19}$$

where

$$\phi(0, x_2) = 0 \tag{2.20}$$

$$\frac{\partial \phi}{\partial x}(0,0) = 0 \ \textit{if}\ k > 2 \tag{2.21}$$

Remarks: The condition (2.18) implies that $\{x_1 = 0,\ x_3 = 0\}$ is an invariant manifold. When the spectrum of A_2 lies on the imaginary axis, we call this a partial center manifold as it corresponds to only some of the eigenvalues on the imaginary axis. We call (2.19) a local stable and partial center manifold because it contains the local stable manifold and part of the local center manifold. The partial center manifold provides the needed gap between the eigenvalues associated to the invariant manifold and those that are not. The necessity of the existence of the partial center manifold to the existence of the stable and partial center manifold could be argued as follows. If the stable and partial center manifold exists then its intersection with the center manifold should yield the partial center manifold. The flaw in this argument is that center manifolds are not necessarily unique and the intersection of manifolds is not necessarily a manifold. Still it is plausible.

Since (2.19) defines a C^{k-2} invariant manifold we can takes its time derivative to obtain the PDE

$$\begin{aligned} &A_3\phi(x_1,x_2) + f_3(x_1,x_2,\phi(x_1,x_2)) \\ &= \sum_{j=1}^{2} \frac{\partial \phi_i}{\partial x_j}(x_1,x_2)\left(A_j x_j + f_j(x_1,x_2,\phi(x_1,x_2))\right)\end{aligned} \tag{2.22}$$

We can develop a term by term approximation of the stable and partial center manifold.

Theorem 2.2 *Suppose the hypothesis of Theorem 2.1 hold for $k > 3$ and let $\phi(x_1,x_2)$ defines the C^{k-2} stable and partial center manifold. Suppose $\psi(x_1,x_2)$ is a C^{k-2} function satisfying (2.20) and the PDE (2.22) through terms of degree $k-3$,*

$$\begin{aligned} &A_3\psi(x_1,x_2) + f_3(x_1,x_2,\psi(x_1,x_2)) \\ &= \textstyle\sum_{j=1}^{2} \frac{\partial \psi}{\partial x_j}(x_1,x_2)\left(A_j x_j + f_j(x_1,x_2,\psi(x_1,x_2))\right) \\ &+ O(x_1,x_2)^{k-2}\end{aligned} \tag{2.23}$$

Then ϕ and ψ agree to degree $k-3$,

$$\psi(x_1,x_2) = \phi(x_1,x_2) + O(x_1,x_2)^{k-2} \tag{2.24}$$

Because of space limitations we omit the proof.

3 Local Solvability of the HJB PDE

The principle theorem of this paper is the following

Theorem 3.1 *Suppose the plant and exosystem are C^k, the linear part of the plant is stabilizable and detectable when $\bar{x} = 0$, the FBI PDE (1.3) has a C^k solution in some neighborhood of 0 in $\bar{x}$ space. Then in some neighborhood of $0, 0$ in $x, \bar{x}$ space there exists a C^{k-2} solution to HJB PDE (1.9) satisfying (1.12).*

Sketch of Proof: The proof generalizes the standard approach [10], [11] to showing the existence of local solutions to HJB PDE's. The graph of gradient of the solution π of the HJB PDE (1.9) is an invariant manifold of the associated Hamiltonian system of ODE's. In the standard case the Hamiltonian ODE's have a hyperbolic fixed point at the origin and the invariant manifold is the stable manifold of this fixed point. But in this case the Hamiltonian ODE's do not have a hyperbolic fixed point at the origin and the desired invariant manifold is a stable and partial center manifold.

Consider the Hamiltonian associated to the optimal control problem (1.8),

$$\begin{aligned} H(\lambda,\mu,z,\bar{x},v) &= \lambda \tilde{f}(z,v,\bar{x}) + \mu \bar{f}(\bar{x}) + l(z,v,\bar{x}) \\ &= \lambda \left(Az + Bv + \tilde{f}^{[2]}(z,v,\bar{x}) \right) \\ &\quad + \mu \left(\bar{A}\bar{x} + \bar{f}^{[2]}(\bar{x}) \right) \\ &\quad + \tfrac{1}{2} \left(z'Qz + 2z'Sv + v'Rv \right) \\ &\quad + \tilde{l}^{[3]}(z,v,\bar{x}) + O(\lambda,\mu,z,\bar{x},v)^4 \end{aligned} \tag{3.25}$$

The Pontryagin Maximum Principle asserts that the optimal control is

$$v = \alpha(\lambda,\mu,z,\bar{x}) = \operatorname{argmin}_v H(\lambda,\mu,z,\bar{x},v) \tag{3.26}$$

For small $\lambda, \mu, z, \bar{x}$ this is given by solving

$$\frac{\partial H}{\partial v}(\lambda,\mu,z,\bar{x},v) = 0$$

which yields

$$\begin{aligned} \alpha = & -R^{-1} \left(B'\lambda' + S'z + \left(\frac{\partial \tilde{f}^{[2]}}{\partial v} \right)' \lambda' + \left(\frac{\partial \tilde{l}^{[3]}}{\partial v} \right)' \right) \\ & + O(\lambda,\mu,z,\bar{x})^3 \end{aligned}$$

The HJB PDE (1.9) can be expressed in terms of the Hamiltonian as

$$H\left(\tfrac{\partial \pi}{\partial z}, \tfrac{\partial \pi}{\partial \bar{x}}, z, \bar{x}, \alpha\left(\tfrac{\partial \pi}{\partial z}, \tfrac{\partial \pi}{\partial \bar{x}}, z, \bar{x}\right)\right) = 0 \tag{3.27}$$

The Hamiltonian ODE's are

$$\begin{aligned} \dot{z}' &= \tfrac{\partial H}{\partial \lambda}(\lambda,\mu,z,\bar{x},\alpha(\lambda,\mu,z,\bar{x})) \\ \dot{\lambda} &= -\tfrac{\partial H}{\partial z}(\lambda,\mu,z,\bar{x},\alpha(\lambda,\mu,z,\bar{x})) \\ \dot{\bar{x}}' &= \tfrac{\partial H}{\partial \mu}(\lambda,\mu,z,\bar{x},\alpha(\lambda,\mu,z,\bar{x})) \\ \dot{\mu} &= -\tfrac{\partial H}{\partial \bar{x}}(\lambda,\mu,z,\bar{x},\alpha(\lambda,\mu,z,\bar{x})) \end{aligned} \tag{3.28}$$

and these are C^{k-1} since the Hamiltonian is C^k.

The linearization of this system around $0, 0, 0, 0$ is

$$\begin{bmatrix} \dot{z} \\ \dot{\lambda}' \\ \hline \dot{\bar{x}} \\ \dot{\mu}' \end{bmatrix} = \begin{bmatrix} \mathbf{H_{11}} & \mathbf{H_{12}} \\ \mathbf{H_{21}} & \mathbf{H_{22}} \end{bmatrix} \begin{bmatrix} z \\ \lambda' \\ \hline \bar{x} \\ \mu' \end{bmatrix} \tag{3.29}$$

where

$$\mathbf{H} = \left[\begin{array}{cc|cc} A - BR^{-1}S' & -BR^{-1}B' & 0 & 0 \\ -Q + SR^{-1}S' & -A' + SR^{-1}B' & 0 & 0 \\ \hline 0 & 0 & \bar{A} & 0 \\ 0 & 0 & 0 & -\bar{A}' \end{array} \right] \tag{3.30}$$

The column span of

$$\begin{bmatrix} I_{n \times n} & 0 \\ P & 0 \\ 0 & I_{\bar{n} \times \bar{n}} \\ 0 & 0 \end{bmatrix} \tag{3.31}$$

is an $n + \bar{n}$ dimensional stable and partial center manifold of the linear Hamiltonian system (3.29) where P is the unique positive definite solution of the algebraic Riccati equation (1.13). We know that such a solution exists because the linear part of the plant was assumed to be stabilizable and detectable. Half of the eigevalues of the upper left $2n \times 2n$ block $\mathbf{H_{11}}$ lie in the open left half plane and half lie in open right half plane. The asymptotically stable subspace is spanned by the first n columns of (3.31). As for the lower right $2\bar{n} \times 2\bar{n}$ block $\mathbf{H_{22}}$, by assumption the eigenvalues of $\bar{A}$ are in the closed left half plane and hence those of $-\bar{A}'$ are in the closed right half plane. A stable subspace is spanned by the last $\bar{n}$ columns of (3.31). Furthermore the submanifold $z = 0$, $\lambda = 0$, $\mu = 0$ is an invariant submanifold of the nonlinear Hamiltonian system (3.28) so the conditions of the Stable and Partial Center Manifold Theorem are satisfied. There exists an $n + \bar{n}$ dimensional stable and partial center manifold in the $2(n + \bar{n})$ dimensional $z, \lambda, \bar{x}, \mu$ space which is tangent to the column span of (3.31) at $0, 0, 0, 0$. Hence this manifold is given by

$$\begin{aligned} \lambda &= \phi(z, \bar{x}) \\ \mu &= \psi(z, \bar{x}) \end{aligned} \tag{3.32}$$

where ϕ, ψ are C^{k-3} and

$$\begin{aligned} \phi(0, \bar{x}) &= 0 \\ \psi(0, \bar{x}) &= 0 \end{aligned} \tag{3.33}$$

This submanifold is Lagrangian, i.e., a maximal dimension submanifold on which the canonical two form

$$\omega = d\lambda \, dz + d\mu \, d\bar{x}$$

vanishes. To see that it vanishes we note that ω is invariant under the Hamiltonian flow (3.28) and this flow is converging to the $\bar{n}$ dimensional submanifold $z = 0,\ \lambda = 0,\ \mu = 0$ where ω clearly vanishes. The submanifold (3.32) is of maximal dimension, $n + \bar{n}$, in $2(n + \bar{n})$ variables.

Hence the one form

$$\phi(z,\bar{x})\ dz + \psi(z,\bar{x})\ d\bar{x}$$

is closed so locally around $0, 0$ in $z, \bar{x}$ there exists a C^{k-2} function $\pi(z,\bar{x})$ such that

$$\begin{aligned} \frac{\partial \pi}{\partial z}(z,\bar{x}) &= \phi(z,\bar{x}) \\ \frac{\partial \pi}{\partial \bar{x}}(z,\bar{x}) &= \psi(z,\bar{x}) \\ \pi(0,\bar{x}) &= 0 \\ \frac{\partial \pi}{\partial z}(0,\bar{x}) &= 0 \\ \frac{\partial \pi}{\partial \bar{x}}(0,\bar{x}) &= 0 \end{aligned}$$

Note that π satisfies (1.12).

Differentiating (3.32) with repect to t along the Hamiltonian flow (3.28) yields

$$\begin{aligned} \frac{\partial H}{\partial \lambda}\frac{\partial^2 \pi}{\partial z^2} + \frac{\partial H}{\partial \mu}\frac{\partial^2 \pi}{\partial z \partial \bar{x}} + \frac{\partial H}{\partial z} &= 0 \\ \frac{\partial H}{\partial \lambda}\frac{\partial^2 \pi}{\partial z \partial \bar{x}} + \frac{\partial H}{\partial \mu}\frac{\partial^2 \pi}{\partial \bar{x}^2} + \frac{\partial H}{\partial z} &= 0 \end{aligned}$$

or equivalently

$$\begin{aligned} \frac{\partial}{\partial z}H(\frac{\partial \pi}{\partial z},\frac{\partial \pi}{\partial \bar{x}},z,\bar{x},\alpha(\frac{\partial \pi}{\partial z},\frac{\partial \pi}{\partial \bar{x}},z,\bar{x})) &= 0 \\ \frac{\partial}{\partial \bar{x}}H(\frac{\partial \pi}{\partial z},\frac{\partial \pi}{\partial \bar{x}},z,\bar{x},\alpha(\frac{\partial \pi}{\partial z},\frac{\partial \pi}{\partial \bar{x}},z,\bar{x})) &= 0 \end{aligned}$$

Clearly π satisfies the HJB PDE (3.27) at $z = 0,\ \bar{x} = 0$ so it satisfies it in a neighborhood. Moreover π is of the form

$$\pi(z,\bar{x}) = \frac{1}{2}z'Pz + O(z,\bar{x})^3 \qquad (3.34)$$

QED.

The next theorem shows that the solution to the HJB PDE (1.9) can be computed term by term.

Theorem 3.2 *Suppose the hypothesis of Theorem 3.1 hold for $k > 3$ and let $\pi(z,\bar{x})$ be the C^{k-2} solution of the HJB PDE (1.9) satisfying (1.12). Suppose $\phi(x_1,x_2)$ is a C^{k-2} function satisfying the HJB PDE through terms of degree $k-3$ and satisfying (1.12). Then π and ψ agree to degree $k-3$,*

$$\pi(z,\bar{x})) = \psi(z,\bar{x}) + O(z,\bar{x})^{k-2} \qquad (3.35)$$

Sketch of Proof: Clearly $\pi(z,\bar{x})$ satisfies the term by term equations so the result follows if we can show that these equations have unique solutions satisfying (1.12). We showed above that the quadratic terms agree, as for the cubic terms consider (1.14). The first equation is a linear equation for $\pi^{[3]}$. For simplicity assume that $A + BK$ and $\bar{A}$ have bases of left eigenvectors

$$\begin{aligned} \xi_i(A+BK) &= \lambda_i \xi_i & i &= 1,\ldots,n \\ \zeta_j \bar{A} &= \mu_j \xi_i & j &= 1,\ldots,\bar{n} \end{aligned} \qquad (3.36)$$

otherwise we use bases of generalized eigenvectors. Since the linear part of the plant is stabilizable and detectable, the linear part of the closed loop system is asymptotically stable, Re $\lambda_i < 0$, and by assumption the linear part of the exosystem is not unstable, Re $\mu_j \leq 0$.

Now any cubic polynomial $\pi^{[2]}(z,\bar{x})$ satisfying (1.12) can be expressed as

$$\begin{aligned} \pi^{[2]}(z,\bar{x}) &= \sum c_{i_1,i_2,i_3}\ \xi_{i_1}z\ \xi_{i_2}z\ \xi_{i_3}z \\ &+ \sum d_{i_1,i_2,j_3}\ \xi_{i_1}z\ \xi_{i_2}z\ \zeta_{j_3}\bar{x} \end{aligned}$$

and

$$\begin{aligned} &\frac{\partial \pi^{[3]}}{\partial z}(z,\bar{x})(A+BK)z + \frac{\partial \pi^{[3]}}{\partial \bar{x}}(z,\bar{x})(\bar{A}\bar{x}) \\ &= \sum c_{i_1,i_2,i_3}\ (\lambda_{i_1}+\lambda_{i_2}+\lambda_{i_3})\,\xi_{i_1}z\ \xi_{i_2}z\ \xi_{i_3}z \\ &+ \sum d_{i_1,i_2,j_3}\ (\lambda_{i_1}+\lambda_{i_2}+\mu_{j_3})\,\xi_{i_1}z\ \xi_{i_2}z\ \zeta_{j_3}\bar{x} \end{aligned}$$

It follows from (1.11) that

$$z'P\bar{f}^{[2]}(z,Kz,\bar{x}) + l^{[3]}(z,Kz,\bar{x}) = O(z,\bar{x})^2$$

so

$$\begin{aligned} &z'P\bar{f}^{[2]}(z,Kz,\bar{x}) + l^{[3]}(z,Kz,\bar{x}) \\ &= \sum k_{i_1,i_2,i_3}\ \xi_{i_1}z\ \xi_{i_2}z\ \xi_{i_3}z \\ &+ \sum l_{i_1,i_2,j_3}\ \xi_{i_1}z\ \xi_{i_2}z\ \zeta_{j_3}\bar{x} \end{aligned}$$

for some k, l's. Hence there is an unique $\pi^{[3]}$ satisfying (1.14) and (1.12) given by

$$\begin{aligned} c_{i_1,i_2,i_3} &= -\frac{k_{i_1,i_2,i_3}}{\lambda_{i_1}+\lambda_{i_2}+\lambda_{i_3}} \\ d_{i_1,i_2,j_3} &= -\frac{l_{i_1,i_2,j_3}}{\lambda_{i_1}+\lambda_{i_2}+\mu_{j_3}} \end{aligned} \qquad (3.37)$$

because the denominators are not zero, Re $\lambda_i < 0$ and Re $\mu_j \leq 0$.

The higher degree terms are handled in a similar fashion. **QED.**

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