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Nonlinear Asymptotic Observer Design, A Backstepping Approach

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Abstract

Nonlinear observer design based on backstepping method is introduced in this paper. The method is applicable to any nonlinear smooth systems. The observer asymptotically approaches to any given bounded trajectory in the observable set of the original system, provided that the initial estimation is not too far from the actual state value.

1 Introduction

This paper addresses a simple and efficient method of observer design for general nonlinear dynamical systems. It is known that, if a nonlinear system admits an observer normal form under change of coordinates and output injection, observers can be designed using linear observer theory (see [3], [4], [2] and references therein). Another existing approach uses high gain design for the observation of general nonlinear dynamical systems ([1]).

In this paper, a different approach based on backstepping is introduced. It does not require any extra assumption for the system, except smoothness and observability. So, the method is applicable to general systems whose error dynamics are not necessarily linearizable. Furthermore, it is not high gain design. The size of the observer gain at a given point depends on the partial derivatives of the vector fields at the same point. A formula of the observer gain is derived. The gain is a function of the state of the observer. The function can be derived off-line through an iterative algorithm.

The paper is organized as follows. In § 2, a two dimensional example is given to illustrate the fundamental idea of backstepping in observer design. The method used in this section is generalized to arbitrary system in observable form in § 3. In § 4, the derivation in section 3 is summarized as an iterative algorithm for the calculation of the observer gain.

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2 An incentive example

In the following example, a nonlinear asymptotic observer is designed for a two dimensional system in observable form. Consider

$$\begin{aligned} y &= x_1 \\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \phi(x_1, x_2) \end{aligned} \tag{1}$$

Given a bounded trajectory of (1), the problem is to find an asymptotic observer to track the states of (1) based on the measurement of $y(t)$. The observer has the following form

$$\begin{aligned} \dot{\bar{x}}_1 &= \bar{x}_2 + g_1(\bar{x})(\bar{x}_1 - x_1) \\ \dot{\bar{x}}_2 &= \phi(\bar{x}_1, \bar{x}_2) - g_2(\bar{x})(\bar{x}_1 - x_1) \end{aligned} \tag{2}$$

The variable $\bar{x}$ is an estimation of x. Define the error by $e = \bar{x} - x$. The goal of this paper is to develop a method of finding g_1 and g_2 so that $\lim_{t\to\infty} e(t) = 0$. The error dynamics is

$$\begin{aligned} \dot{e}_1 &= e_2 - g_1(\bar{x})e_1 \\ \dot{e}_2 &= \phi(\bar{x}) - \phi(x) + g_2(\bar{x})e_1 \end{aligned} \tag{3}$$

Notice that the error equation is dependent on both e and $\bar{x}$. Since

$$\bar{x}(t) = x(t) + e(t). \tag{4}$$

(3) can be treated as a time variant system of e for a given trajectory $x(t)$. In the following, we derive a Lyapunov function for the system. Then, the observer gain g_1 and g_2 are derived from the Lyapunov function.

Let $z_1 = e_1$, and let $V_1 = \frac{1}{2}z_1^2$. Then $\dot{V}_1 = -c_1 z_1^2 + z_1 z_2$, where $z_2 = c_1 e_1 + e_2 - g_1(\bar{x})e_1$. Define $V_2 = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2$. We have

$$\begin{aligned} \dot{V}_2 = & -c_1 z_1^2 - c_2 z_2^2 + z_2(c_2 z_2 + z_1 + \phi(\bar{x}) - \phi(x) \text{ [illegible]} \\ & + g_2(\bar{x})e_1 \text{ [illegible]} - c_1 g_1(\bar{x})e_1 \\ & + \dot{g}_1(\bar{x})e_1 - g_1(\bar{x})e_2 + (g_1(\bar{x}))^2 e_1) \end{aligned} \tag{5}$$

Define $\phi_1(x) = \frac{\partial \phi}{\partial x_1}(x)$, $\phi_2(x) = \frac{\partial \phi}{\partial x_2}(x)$, then

$$\phi(\bar{x}) - \phi(x) = \phi_1(x)e_1 + \phi_2(x)e_2 + O_3(e)^2 \tag{6}$$

where $O_x(e)^2$ is a function of (x,e) such that $\frac{O_x(e)^2}{\|e\|^2}$ is bounded for (x,e) in a neighborhood of $(0,0)$. Substituting (6) and the definition of z_1, z_2 into (5), we have

$$\begin{aligned}\dot{V}_2 &= -c_1z_1^2 - c_2z_2^2 + z_2((g_2(\bar{x}) + 1 + c_1c_2 + c_2g_1(\bar{x}) \\ &\quad +\phi_1(x) + c_1g_1(\bar{x}) + \dot{g}_1(\bar{x}) + (g_1(\bar{x}))^2)e_1 \\ &\quad +(c_1 + c_2 + \phi_2(x) + g_1(\bar{x}))e_2).\end{aligned} \tag{7}$$

To make (7) negative definite, we assume

$$\begin{aligned}g_1(\bar{x}) &= -(c_1 + c_2 + \phi_2(\bar{x})) \\ g_2(\bar{x}) &= -(1 + c_1c_2 + c_2g_1(\bar{x}) + \phi_1(\bar{x}) + c_1g_1(\bar{x}) \\ &\quad +\dot{g}_1(\bar{x}) + (g_1(\bar{x}))^2)\end{aligned} \tag{8}$$

Notice that $g_1(\bar{x})$ and $g_2(\bar{x})$ can not cancel the term $\phi_2(x)$ and $\phi_1(x)$. However, the function defined in (8) can cancel undesired quadratic terms. From (6), it is easy to show that

$(c_1 + c_2 + \phi_2(x) + g_1(\bar{x})) = O_x(e)$
$g_2(\bar{x}) + 1 + c_1c_2 + c_2g_1(\bar{x}) + \phi_1(x) + c_1g_1(\bar{x}) + \dot{g}_1(\bar{x}) +$
$(g_1(\bar{x}))^2 = O_x(e)$

$$\dot{V}_2 = -c_1z_1^2 - c_2z_2^2 + O_x(e)^3. \tag{9}$$

Suppose that $x(t)$, the trajectory of (1), is bounded. Then, $\dot{V}_2$ is locally negative definite.

This implies that the error dynamics (3) is locally asymptotically stable. Or equivalently, there exists a $\delta > 0$ so that $\lim_{t\to\infty} e(t) = 0$ if $\|e(0)\| < \delta$.

The observer (2)-(8) is global in the sense that it can track any bounded trajectory of (1). However, the initial estimation $\bar{x}(0)$ can not be arbitrary. It is required that $\|e(0)\| < \delta$ for some $\delta > 0$. This is because that the error dynamics (3) is not necessarily globally asymptotically stable.

3 Observer design for systems in observable form

In this section, observers are designed for the following system in observable form.

$$\begin{aligned}\dot{x}_i &= x_{i+1}, \quad for \ 1 \le i \le n-1, \\ \dot{x}_n &= \varphi(x)\end{aligned} \tag{1}$$

System (1) is a special case of (1) for $n = 2$. In this section, general backstepping approach is introduced. A family of asymptotic nonlinear observers are derived for systems in the form of (1).

The observer is in the following form

$$\begin{aligned}\dot{\bar{x}}_i &= \bar{x}_{i+1} + g_i(\bar{x})(\bar{x}_1 - x_1), \quad for \ 1 \le i \le n-1, \\ \dot{\bar{x}}_n &= \phi(\bar{x}) + g_n(\bar{x})(\bar{x}_1 - x_1)\end{aligned} \tag{2}$$

The problem of observer design is to find the gains $g_i(\bar{x})$, $1 \le i \le n$, so that $\lim_{t\to\infty} \bar{x}(t) - x(t) = 0$ if $\|\bar{x}(0) - x(0)\|$ is not large. In other words, the gains render the following error dynamics asymptotically stable. Define $e = \bar{x} - x$, the error dynamics are

$$\begin{aligned}\dot{e}_i &= e_{i-1} + g_i(\bar{x})e_1, \qquad for \ 1 \le i \le n-1, \\ \dot{e}_n &= \phi(\bar{x}) - \varphi(x) + g_n(\bar{x})e_1\end{aligned} \tag{3}$$

For a fixed trajectory $x(t)$ of (1), (3) can be considered as a time-variant nonlinear dynamical system of e because $\bar{x}(t) = e(t) + x(t)$. The fact that $x(t)$ is unknown makes the problem of observer design for nonlinear system difficult because the coefficients in the nonlinear functions of the error dynamics is time variant and they are not measurable. This is a fundamental difference between linear and nonlinear design. On the other

hand, the trianglular structure in all the systems (1), (2) and (3) suggests that backstepping method of finding a Lyapunov function might work for error dynamics. Let's introduce some notations used in the main theorem of this section.

Given any function $\alpha(x,\bar{x},e)$, the operator D transforms α into a

function by taking directional derivatives, i.e.,

$$\begin{aligned}&D(\alpha(x,\bar{x},e)) \\ &= \sum_{i=1}^{n-1}\frac{\partial\alpha}{\partial x_i}x_{i+1} + \frac{\partial\alpha}{\partial x_n}\phi(x) \\ &\quad + \sum_{i=1}^{n-1}\frac{\partial\alpha}{\partial\bar{x}_i}(\bar{x}_{i+1} + g_i(\bar{x})(\bar{x}_1 - x_1)) \\ &\quad + \frac{\partial\alpha}{\partial\bar{x}_n}(\phi(\bar{x}) + g_n(\bar{x})(\bar{x}_1 - x_1)) \\ &\quad + \sum_{i=1}^{n-1}\frac{\partial\alpha}{\partial e_i}(e_{i+1} + g_i(\bar{x})(e_1)) \\ &\quad + \frac{\partial\alpha}{\partial e_n}(\phi(\bar{x}) - \phi(x) + g_n(\bar{x})(e_1))\end{aligned} \tag{4}$$

In the definition of Lyapunov function, the variable z will be used, which is defined by the following iterative formulae.

$$\begin{aligned}z_1 &= e_1 \\ z_2 &= c_1z_1 + Dz_1 \\ z_k &= c_{k-1}z_{k-1} + Dz_{k-1} + z_{k-2}, \quad for \ 3 \le k \le n\end{aligned} \tag{5}$$

This function will be used to determine the observer gain in Theorem 1. The Lyapunove function is defined by

$$V = \frac{1}{2}(z_1^2 + z_2^2 + \cdots + z_n^2) \tag{6}$$

In (5), z represents a mapping from $(x,\bar{x},e) \in I\!\!R^n \times I\!\!R^n \times I\!\!R^n$ to $I\!\!R^n$. If $\bar{x}(t)$ is substituted by $x(t) + e(t)$, the Lyapunov function (6) is a time-variant function. However in the derivation of the formulae of z_i, we treat x, $\bar{x}$ and e as independent variables. The purely algebraic derivation avoids higher degree terms of e in the expression of z. This is also the reason why in the definition of D, [illegible] instead of [illegible] is used in the second summation. For the reason of space, all results in this section are introduced without proof.

Lemma 1 *The mapping (5) has the triangular structure*

$$z_i = a_{i\,1}(x,\bar{x})e_1 + a_{i\,2}(x,\bar{x})e_2 + \cdots + a_{i\,i-1}(x,\bar{x})e_{i-1} + e_i \qquad (7)$$

for $1 \le i \le n$. *Notice that the coefficient of* e_i *is* 1.

It is easy to check that

$$z_{n+1} = \phi(\bar{x}) - \phi(x) + \sum_{i=1}^{n} a_{n+1\,i}\, e_i \qquad (8)$$

for some coefficients $a_{n+1,j}$, $1 \le j \le n$. In the following, the coefficients $a_{i\,j}$ in (7) and (8) are grouped in the following way.

$$G_j = \{a_{j+1\,1}, a_{j+2\,2}, \cdots, a_{n+1\,n-j+1}\}, \quad 0 \le j \le n \qquad (9)$$

i.e. G_j consists of all $a_{s\,t}$, $1 \le t \le s \le n+1$, so that $s - t = j$. They represents the off diagonals in the matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ a_{2\,1} & 1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n\,1} & a_{n\,2} & a_{n\,3} & \cdots & 1 & 0 \\ a_{n+1\,1} & a_{n+1\,2} & a_{n+1\,3} & \cdots & a_{n+1\,n} & 0 \end{bmatrix}$$

In this matrix, G_0 consists of the diagonal elements (the last entry is zero since no e_{n+1} term in z_{n+1}). The off diagonal line is G_1, the second off diagonal line is G_2, etc.

Lemma 2 *For any* $a_{k+j,j} \in G_k$, $k \ge 1$, *it satisfies*

$$a_{k+j\,j} = g_k + h_{k+j\,j}(x,\bar{x}) \qquad (10)$$

where $h_{k+j\,j}$ *is determined by* $g_1, g_2, \cdots, g_{k-1}$.

Theorem 1 *Let* $g_1(x), g_2(x), \cdots, g_n(\bar{x})$ *be functions satisfying*

$$g_i(x) + h_{n+1\,n-i+1}(\bar{x},\bar{x}) + \phi_{n-i+1}(\bar{x}) = 0 \quad \textit{for} \;\; 1 \le i \le n \qquad (11)$$

where $\phi_i(x) = \frac{\partial \phi}{\partial x_i}(x)$. *For any given bounded trajectory* $x(t)$ *of (1),*

there exists a neighborhood U *of* $e = 0$ *so that* $e(0) \in U$ *implies that* $\lim_{t\to\infty}(x(t) - x(0)) = 0$.

4 Iterative algorithm for the calculation of $g_i(x)$

Lemma 3 *Suppose that* $G_0, G_1, \cdots, G_{k-1}$ *are known for some* $k > 2$. *Suppose that* $g_1, g_2, \cdots, g_k$ *are known. Then the elements of* G_k *satisfies*

$$a_{k+1\,1} = c_k a_{k\,1} + Da_{k\,1} + a_{k-1\,1} + \sum_{i=1}^{k} a_{k\,i}\, g_i(\bar{x}) \qquad (1)$$

$$a_{k+j\,j} = c_{k+j-1} a_{k+j-1\,j} + Da_{k+j-1\,j} + a_{k+j-2\,j} + a_{k+j-1\,j-1} \ [illegible] \qquad (2)$$

for $2 \le j \le n - k + 1$.

Notice that in the right sides of (1) and (2), the coefficients $a_{k\,1}$, $a_{k-1\,1}$, $a_{k+j-1\,j}$ and $a_{k+j-2\,j}$ are all in G_{k-1} or G_{k-2}. The function $a_{k+j-1\,j-1}$ is in G_k. The first coefficient $a_{k+1\,1}$ can be computed using (1), the equation (2) iteratively determines all the functions in G_k as j increases from 2 to $n-k+1$.

¿From Lemma 1, it is known that $a_{i\,i} = 1$ for $i = 1, \cdots, n$. It is obvious that $a_{n+1\,n+1} = 0$. So

$$G_0 = \{a_{1\,1} = 1, \cdots, a_{n\,n} = 1, a_{n+1\,n+1} = 0\} \qquad (3)$$

The functions in G_1 are given in the following result.

Lemma 4 *The functions in* G_1 *satisfies*

$$a_{j\,j-1} = c_1 + c_2 + \cdots + c_{i-1} + g_1 \qquad (4)$$

for $2 \le j \le n+1$.

¿From Theorem 1, the function $g_i(\bar{x})$ is determined by $c_1, c_2, \cdots, c_n$, $\phi(x)$ and $h_{n+1\,i}(x,\bar{x})$. The coefficients $c_1, c_2, \cdots, c_n$, are the weights of z_i^2 in V. They can be any set of positive numbers. The function $\phi(x)$ is a known function in the dynamical system. The results in Lemma 3 and

4 provides iterative formula for the calculation of $h_{n-1\,i}(x,\bar{x})$. In the following, we summarize all these in an iterative algorithm for the calculation of the observer gain. The notations $a_{i\,j}$ and $h_{i\,j}$ are defined in (7), (8) and (10).

Algorithm

Step 1 (The computation of $g_1(\bar{x})$) Set $k = 1$.

$$g_1(\bar{x}) = -(\sum_{i=1}^{n} c_i + \varphi_n(\bar{x})).$$

Step 2. (The computation of G_1) For $1 \le j \le n$.

$$a_{j+1\,j} = \sum_{i=1}^{j} c_i + g_1(\bar{x}).$$

Step 3. (The computation of $h_{i\,j}$ satisfying $i - j = k+1$) For $j = 1$,

$$h_{k+2\,1} = c_{k+1} a_{k+1\,1} + Da_{k+1\,1} + a_{k\,1} + \sum_{i=1}^{k} a_{k+1\,i}\, g_i(\bar{x})$$

For $2 \le j \le n - k$,

$$h_{k+j+1\,j} = c_{k+j} a_{k+j\,j} + Da_{k+j\,j} + h_{k+j\,j-1} + a_{k+j-1\,j}$$

Step 4. (The computation of $g_{k+1}(\bar{x})$)

$$g_{k+1} = -(h_{n+1\,n-k}(\bar{x},\bar{x}) + \varphi_{n-k}(\bar{x})).$$

Step 5. (The computation of G_{k+1}) If $k+1 = n$, stop. If $k+1 < n$, for $1 \le j \le n-k$,

$$a_{k+j+1\,j} = h_{k+j+1\,j} + g_{k+1}(\bar{x}).$$

Step 6. Define $k = k+1$ then go to step 3.

5 Conclusion

Nonlinear observers for general dynamical systems are derived based on backstepping method. The formulae for the calculation of the observer gain are given in an algorithm (§ 4). It can be programmed using softwares of symbolic computation such as Maple or Mathematica. The gain function can be derived off-line. The approach in § 4 is generalized to the case in which the system is not in observable form. The results will be given in a forthcoming paper. In this paper, the systems have no control input. Observer design for nonlinear control systems is one of the problems in our future research.

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