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A Lyapunov Theory of Nonlinear Observers

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ABSTRACT: We will develop a Lyapunov and a partial converse Lyapunov theory of nonlinear observers.

1 Convergent Observers

The problem of estimating the state of a dynamical system from partial and possibly noisy measurements has a long history. In its nonlinear state space form, one assumes that the dynamics satisfies a known nonlinear differential equation with unkown initial condition and the measurement is a known nonlinear function of the state

$$\begin{aligned} \dot{x} &= f(x) \\ x(0) &= x^0 \\ y &= h(x) \end{aligned} \tag{1}$$

The linear form of the problem is

$$\begin{aligned} \dot{x} &= Ax \\ x(0) &= x^0 \\ y &= Cx \end{aligned} \tag{2}$$

One is given an estimate $\hat{x}^0$ of x^0 and the observations $y(s),\ 0 \leq s \leq t$ up to time t. The problem is generate an estimate $\hat{x}(t)$ of $x(t)$ in real time, as the process evolves. The estimate should converge to the true state as $t \to \infty$. Ideally the estimation process should be robust to noise both in the dynamics and in the observations, to the initial state error and also to modeling errors in the functions $f,\ h$.

One way of approaching this problem is to assume that the dynamics, initial condition and observations are corrupted by noise with a known distribution and then to find the conditional density of the state given the past observations. When the dynamics and observations are linear functions of the state and the noises and initial condition are independent and jointly Gaussian then the conditional density is Gaussian and explicitly computable.

Wiener [We] solved this problem for stationary Gaussian processes using the method of spectral factorization. Kalman [Ka],[KB] extended this to nonstationary Gaussian processes and reduced the problem to solving off-line a Riccati equation and on-line a linear differential equation driven by the observations.

When the dynamics and/or observations are nonlinear then the unnormalized conditional density satisfies the Zakai equation, a parabolic PDE driven by the observations [DM]. Accurately computing its solution in real time for all but the smallest state dimensions is a very difficult task.

The Extended Kalman Filter [Ge] is a widely used alternative method for estimating the state of a nonlinear system. It is obtained by linearizing the nonlinear dynamics and the observation along the trajectory of the estimate. It requires the on-line solution of a Riccati differential equation and a linear differential equation driven by the observations. The Extended Kalman Filter is globally defined but it is only a local method, one expects the estimate to converge to the true state if the initial estimation error is not too large but there is no general proof of this.

There are several nonstochastic approaches to state estimation. For linear systems with linear observations, Luenberger [Lu] developed the concept of an observer. This is another linear system that is driven by the observations in such a way that the error dynamics is asymptotically stable.

Several nonstochastic methods have been proposed for nonlinear estimation. Some of these are surveyed by Misawa and Hendrick [MH]. Other methods include linearization [KI],[BZ],[KR], high gain observers [GHO] and H_∞ methods [Kr2].

Our approach to the Lyapunov theory of nonlinear observers follows that sketched out by Krener and Duarte [KD]. It parallels the excellent paper of Lin, Sontag and Wang [LSW], in which a Lyapunov and converse Lyapunov theory for the stability of nonlinear dynamical systems with respect to sets is presented. Many of our proofs are straight forward adaptations of theirs.

The systems that we are considering are of the form

$$\dot{x} = f(x,u) \tag{3}$$

$$y = h(x) \tag{4}$$

$$x(0) = x^0 \tag{5}$$

with state x, control u and observation y. The basic problem is to estimate the current state $x(t)$ from knowledge of the system and the past values of the control $u(\tau)$ and observation $y(\tau)$, $0 \leq \tau \leq t$ and some information about the initial state. For simplicity we assume that the information available about the initial state is an initial state estimate $\hat{x}^0$.

Definition 1 *An observer is a causal functional*

$$\begin{bmatrix} \hat{x}^0 \\ u(\tau) \\ y(\tau) \end{bmatrix} \longmapsto \hat{x}(t), \qquad 0 \leq \tau \leq t \tag{6}$$

from the initial state estimate, $\hat{x}^0$ and the past control and observation, $u(\tau)$, $y(\tau)$, $0 \le \tau \le t$, to the current state estimate, $\hat{x}(t)$ such that the estimate $\hat{x}(t)$ as a function of t is continuous from the left and limits exist from the right.

Recall a functional $\xi(\cdot) \longmapsto \zeta(\cdot)$ is causal if $\zeta(t)$ only depends on $\xi(\tau)$ for $\tau \le t$. We will need some terminology. A function $\alpha(r)$ is of class K if it is continuous, strictly increasing and $\alpha(0) = 0$. A function $\alpha(r)$ is of class K_∞ if it is of class K and in addition $\alpha(r)$ goes to ∞ as r goes to ∞. A function $\gamma(r,s)$ is of class K^2 if it is continuous in both variables, strictly increasing in each variable when the other is fixed and $\gamma(0,0) = 0$. A function $\gamma(r,s)$ is of class K^2_∞ if it is of class K^2 and goes to ∞ as either r or s goes to ∞. We introduce noises into the model to reflect our uncertainty about the dynamics $w(\cdot)$, the measurments $v(\cdot)$ and the error of the initial state estimate $\tilde{x}^0 = x^0 - \hat{x}^0$.

$$\dot{x} = f(x,u) + g(x)w \tag{7}$$

$$y = h(x) + v \tag{8}$$

$$x(0) = x^0 = \hat{x}^0 + \tilde{x}^0 \tag{9}$$

We will measure the size of the three noises using L_2 norms but other norms or ways of measuring size are possible. For example, the size of $\tilde{x}^0$ could be measured by a nonegative function $Q^0(x)$ which has a global minimum at $x = \hat{x}^0$. We denote the L_2 norm of the driving and observation noises on $[\tau, t]$ by

$$||w,v||_{[\tau,t]} = \left(\int_\tau^t |w(\sigma)|^2 + |v(\sigma)|^2 \, d\sigma \right)^{\frac{1}{2}}$$

Definition 2 *An observer is said to be convergent if there exist class K_∞ functions $\delta_i(\epsilon)$, $i = 1,2$, and a function $T(r,s,\epsilon)$ with the following properties. For any times $0 \le \tau_1 \le \tau_2 \le t$, any initial initial state, any initial state estimate $\hat{x}^0$, any control $u(\cdot)$ and any noises $w(\cdot)$, $v(\cdot)$ with support in $[\tau_1, \tau_2]$, the state trajectory and state estimate satisfy*
(1) for any $\epsilon > 0$ if

$$\begin{aligned} |x(\tau_1) - \hat{x}(\tau_1)| &\le \delta_1(\epsilon) \\ ||w,v||_{[\tau_1,\tau_2]} &\le \delta_2(\epsilon) \end{aligned}$$

then

$$|x(t) - \hat{x}(t)| \le \epsilon,$$

(2) for any r, s, $\epsilon > 0$, if

$$\begin{aligned} |x(\tau_1) - \hat{x}(\tau_1)| &< r \\ ||w,v||_{[\tau_1,\tau_2]} &< s \\ T(r,s,\epsilon) &\le t - \tau_2 \end{aligned}$$

then

$$|x(t) - \hat{x}(t)| < \epsilon$$

Condition (1) is the stability of the estimation error. Loosely speaking it states that the estimation error at a later time is a uniformly continuous function of the estimation error at a prior time and the driving and observation noises between the times. Condition (2) is the asymptotic stability of the noise free estimation error. It states that if the noises stop then the estimation error asymptotically converges to zero uniformly with respect to time, prior state, prior state estimate, control, prior driving noise and prior observation noise. Of course there are obvious local versions of the definition, e. g., (1) and (2) hold only if $|x(\tau_1) - \hat{x}(\tau_1)|$ and $||w, v||_{[\tau_1,\tau_2]}$ are sufficiently small.

Condition (1) implies that if an observer is convergent, if the estimate is exact at some time, $x(\tau) = \hat{x}(\tau)$ and if the noise is zero at subsequent times $w(\sigma) = 0,\ v(\sigma) = 0,\ \tau \leq \sigma \leq t$ then the estimate is exact at subsequent times $x(t) = \hat{x}(t)$ for all $t \geq \tau$.

A function $\beta(r, t)$ is of class KL if for fixed t, it is of class K and for fixed r, it is strictly decreasing to 0 as t goes to ∞. A function $\gamma(r, s, t)$ is of class K^2L if for fixed t, it is of class K^2 and for fixed r, s, it is strictly decreasing to 0 as t goes to ∞.

Theorem 1 *An observer is convergent iff there exists a function $\gamma(r, s, t)$ of class K^2L such that for any $0 \leq \tau_1 \leq \tau_2 \leq t$, any initial state x^0, control history $u(\cdot)$, and noise triple $\tilde{x}^0$, $w(\cdot)$, $v(\cdot)$, with support in $[\tau_1, \tau_2]$, the state trajectory and estimate satisfy*

$$|x(t) - \hat{x}(t)| \leq \gamma(|x(\tau_1) - \hat{x}(\tau_1)|, ||w, v||_{[\tau_1,\tau_2]}, t - \tau_2) \tag{10}$$

Proof: Suppose $\gamma(r, s, t)$ exists. First we show (1) of Definition 2 holds. Define a class K_∞ function $\bar{\gamma}(r) = \gamma(r, r, 0) + r$ and let $\delta(\epsilon) = \bar{\gamma}^{-1}(\epsilon)$. If

$$\begin{aligned} |x(\tau_1) - \hat{x}(\tau_1)| &\leq \delta(\epsilon) \\ ||w, v||_{[\tau_1,\tau_2]} &\leq \delta(\epsilon) \end{aligned}$$

then

$$\begin{aligned} |x(t) - \hat{x}(t)| &\leq \gamma(\delta(\epsilon), \delta(\epsilon), t - \tau_2) \\ &\leq \gamma(\delta(\epsilon), \delta(\epsilon), 0) \\ &\leq \bar{\gamma}(\delta(\epsilon)) = \epsilon \end{aligned}$$

for all $0 \leq \tau_1 \leq \tau_2$.

Given any $r,\ s,\ \epsilon$ choose $T = T(r, s, \epsilon)$ so that

$$\gamma(r, s, T) < \epsilon.$$

This is always possible because γ is K^2L. Then (2) holds.

To show the converse, we first show that the $T(r,s,\epsilon)$ can be chosen with the following properties;
(i) for fixed r, $s>0$, the mapping $\epsilon \mapsto T(r,s,\epsilon)$ is a continuous, strictly decreasing map from $(0,\infty)$ to $(0,\infty)$,
(ii) for fixed r, $\epsilon>0$, the mapping $s \mapsto T(r,s,\epsilon)$ is a strictly increasing map which goes to ∞ as $s \to \infty$,
(iii) for fixed s, $\epsilon>0$, the mapping $r \mapsto T(r,s,\epsilon)$ is a strictly increasing map which goes to ∞ as $r \to \infty$.
Define

$$A(r,s,\epsilon) = \{T \geq 0 : \text{Condition (2) is satified for } r,s,\epsilon\}.$$

This set is not empty for any r, s, $\epsilon>0$ and is decreasing in r, s and increasing in ϵ. Define $\bar{T}(r,s,\epsilon) = \inf A(r,s,\epsilon)$ then $\bar{T}$ is increasing in r, s and decreasing in ϵ. Because they are class K_∞, the functions $\delta_i(\epsilon) \to \infty$ as $\epsilon \to \infty$ so $\bar{T}(r,s,\epsilon) = 0$ for ϵ sufficiently large.
Define

$$T(r,s,\epsilon) = \frac{2}{\epsilon}\int_{\frac{\epsilon}{2}}^{\epsilon} T(r,s,\nu)\,d\nu + \frac{r+s}{\epsilon}$$

then T is strictly increasing in r, s, continuous and strictly decreasing in ϵ and

$$\begin{aligned} T(r,s,\epsilon) &> \bar{T}(r,s,\epsilon) \\ \lim_{\epsilon\to\infty} T(r,s,\epsilon) &= 0 \\ \lim_{\epsilon\to 0+} T(r,s,\epsilon) &= \infty \end{aligned}$$

Define a function $\phi(r,s) = \max\{\delta_1^{-1}(r),\ \delta_2^{-1}(s)\}$ then by Condition (1) for any noises with support in $[\tau_1,\tau_2]$ for any $t \geq \tau_2$

$$|x(t)-\hat{x}(t)| \leq \phi(|x(\tau_1)-\hat{x}(\tau_1)|, \|w,v\|_{[\tau_1,\tau_2]}) \tag{11}$$

Define a function $\psi(r,s,\cdot) = T^{-1}(r,s,\cdot)$ where $T(r,s,\epsilon)$ is as above. Then by Condition (2) for any noises with support in $[\tau_1,\tau_2]$ and any $t \geq \tau_2$

$$|x(t)-\hat{x}(t)| < \psi(r,s,t-\tau_2)$$

for any $r > |x(\tau_1)-\hat{x}(\tau_1)|$ and $s > \|w,v\|_{[\tau_1,\tau_2]}$.
Define

$$\bar{\psi}(r,s,t) = \min\left\{\phi(r,s),\ \inf_{\rho>r,\sigma>s} \psi(\rho,\sigma,t)\right\}$$

then by the above for any noises with support in $[\tau_1,\tau_2]$ and any $t \geq \tau_2$

$$|x(t)-\hat{x}(t)| \leq \bar{\psi}(|x(\tau_1)-\hat{x}(\tau_1)|, \|w,v\|_{[\tau_1,\tau_2]}, t-\tau_2) \tag{12}$$

If $\bar{\psi}$ was K^2L we would be done but it may not be. However we can construct a K^2L function which majorizes it. By its definition, $\bar{\psi}(r,s,t)$ is increasing in r, s and decreases to 0 as $t \to \infty$.

Define

$$\hat{\psi}(r,s,t) = \int_r^{r+1} \int_s^{s+1} \bar{\psi}(\rho,\sigma,t)\, d\sigma\, d\rho$$

then $\hat{\psi}(r,s,t)$ is continuously increasing in r, s and $\hat{\psi}(r,s,t) \geq \bar{\psi}(r,s,t)$. Note that since $\psi(r,s,t) \to \infty$ as $t \to 0$

$$\bar{\psi}(r,s,t) \leq \bar{\psi}(r,s,0) = \phi(r,s)$$

which is a continuous function. By the Lebesgue dominated convergence theorem, for any fixed $r, s \geq 0$

$$\lim_{t\to\infty} \hat{\psi}(r,s,t) = \int_r^{r+1} \int_s^{s+1} \lim_{t\to\infty} \bar{\psi}(\rho,\sigma,t)\, d\sigma\, d\rho$$

Since $\bar{\psi}(r,s,t)$ decreases to 0 as $t \to \infty$, so does $\hat{\psi}(r,s,t)$.
Define

$$\tilde{\psi}(r,s,t) = \hat{\psi}(r,s,t) + \frac{r+s}{(r+s+1)(t+1)}$$

then this function is continuous and strictly increasing in r, s for fixed t and for fixed r, s it strictly decreases to 0 as $t \to \infty$. Moreover from (12) for any noises with support in $[\tau_1, \tau_2]$ and any $t \geq \tau_2$

$$\begin{aligned} |x(t) - \hat{x}(t)| &\leq \bar{\psi}(|x(\tau_1) - \hat{x}(\tau_1)|, \|w,v\|_{[\tau_1,\tau_2]}, t - \tau_2) \\ &\leq \hat{\psi}(|x(\tau_1) - \hat{x}(\tau_1)|, \|w,v\|_{[\tau_1,\tau_2]}, t - \tau_2) \\ &\leq \tilde{\psi}(|x(\tau_1) - \hat{x}(\tau_1)|, \|w,v\|_{[\tau_1,\tau_2]}, t - \tau_2) \qquad (13) \end{aligned}$$

The only thing that is lacking for it to be K^2L is that $\tilde{\psi}(0,0,t)$ might not be 0. This can remedied that by defining

$$\gamma(r,s,t) = \sqrt{\phi(r,s)}\sqrt{\tilde{\psi}(r,s,t)}.$$

then (11,13)

$$|x(t) - \hat{x}(t)| \leq \gamma(|x(\tau_1) - \hat{x}(\tau_1)|, \|w,v\|_{[\tau_1,\tau_2]}, t - \tau_2)$$

□

Lemma 1 *Suppose α is a class K function then there exists a K^2L function $\gamma_\alpha(r,s,t)$ with the following property.*
If $z(\cdot)$, $\zeta(\cdot)$ are any absolutely continuous functions satisfying the differential inequality

$$\begin{aligned} \dot{z}(\tau) &\leq -\alpha(z(\tau)) + \dot{\zeta}(\tau) \\ z(0) &\geq 0 \\ \zeta(0) &= 0 \\ \dot{\zeta}(t) &\geq 0 \end{aligned}$$

where the support of $\dot{\zeta}(t)$ *is* $[0, \tau_2] \subseteq [0, t]$ *then*

$$z(t) \leq \gamma_\alpha(z(0), \zeta(\tau_2), t - \tau_2) \tag{14}$$

Proof: Define $\bar{\alpha}$ by $\bar{\alpha}(0) = 0$ and for $r > 0$

$$\bar{\alpha}(r) = \frac{1}{r} \int_0^r \min\{s^2, \alpha(s)\}\, ds$$

Then $\bar{\alpha}$ is a smooth class K function and for $r > 0$

$$\begin{aligned} \bar{\alpha}(r) &< \alpha(r) \\ \bar{\alpha}(r) &< r^2 \\ \frac{d}{dr}\bar{\alpha}(r) &= \frac{1}{r}\left(\min\{r^2, \alpha(r)\} - \bar{\alpha}(r)\right) > 0 \end{aligned}$$

Define for any $r > 0$

$$\eta(r) = -\int_1^r \frac{ds}{\bar{\alpha}(s)}$$

then

$$\lim_{r \to 0^+} \eta(r) = \infty$$

It is not hard to see that η is a strictly decreasing differentiable function mapping $(0, \infty)$ to some interval $(a, +\infty)$. Then η^{-1} is a strictly decreasing function from (a, ∞) to $(0, \infty)$.
Define

$$\beta_\alpha(r, t) = \begin{cases} 0, & r = 0 \\ \eta^{-1}(\eta(r) + t), & r > 0 \end{cases}$$

Note that $\beta_\alpha(r, t)$ is a class KL function which satisfies

$$\begin{aligned} \frac{d}{dt}\beta_\alpha(r, t) &= -\bar{\alpha}(\beta_\alpha(r, t)) \\ \beta_\alpha(r, 0) &= r \end{aligned}$$

Trivially

$$\dot{z}(\tau) \leq -\bar{\alpha}(z(\tau)) + \dot{\zeta}(\tau)$$

so $z(t) \leq \bar{z}(t)$ where

$$\begin{aligned} \frac{d}{d\tau}\bar{z}(\tau) &= -\bar{\alpha}(\bar{z}(\tau)) + \dot{\zeta}(\tau) \\ \bar{z}(0) &= z(0) \end{aligned}$$

Now consider the mapping

$$\theta(\tau) = \beta_\alpha(\bar{z}(\tau), \tau_2 - \tau)$$

then

$$\begin{aligned}\theta(0) &= \beta_\alpha(z(0),\tau_2)\\ \theta(\tau_2) &= \bar z(\tau_2)\\ \frac{d}{d\tau}\theta(\tau) &= \frac{d}{dz}\beta_\alpha(\bar z(\tau),\tau_2-\tau)\dot\zeta(\tau)\end{aligned}$$

so

$$\bar z(\tau_2) = \beta_\alpha(z(0),\tau_2) + \int_0^{\tau_2}\frac{d}{dz}\beta_\alpha(\bar z(\tau),\tau_2-\tau)\dot\zeta(\tau)d\tau$$

But

$$\frac{d}{dz}\beta_\alpha(\bar z(\tau),\tau_2-\tau)\le 1$$

because it satisfies the linear differential equation

$$\begin{aligned}\frac{d}{dt}\frac{d}{dz}\beta_\alpha(\bar z(\tau),\tau_2-\tau) &= -\frac{d}{dz}\bar\alpha(\bar z(\tau))\frac{d}{dz}\beta_\alpha(\bar z(\tau),\tau_2-\tau)\\ \frac{d}{dz}\beta_\alpha(r,0) &= 1\end{aligned}$$

with

$$\frac{d}{dz}\bar\alpha(\bar z(\tau))\ge 0$$

Hence

$$\bar z(\tau_2)\le\beta_\alpha(z(0),\tau_2)+\zeta(\tau_2)$$

and

$$\begin{aligned}\bar z(t) &\le \beta_\alpha(\beta_\alpha(z(0),\tau_2)+\zeta(\tau_2),t-\tau_2)\\ &\le \beta_\alpha(z(0)+\zeta(\tau_2),t-\tau_2)\\ &= \gamma_\alpha(z(0),\zeta(\tau_2),t-\tau_2)\end{aligned}$$

where γ_α is defined by

$$\gamma_\alpha(r,s,t)=\beta_\alpha(r+s,t)$$

□

Definition 3 *A Lyapunov function for the observer* $\hat x(t)$ *is a function* $Q(x,t)$ *with the following properties*
(1) $Q(x,t)$ *is a causal functional of the initial state estimate* $\hat x^0$ *and the past control and observation,* $u(\tau)$, $y(\tau)$, $0\le\tau\le t$ *such that* $Q(x,t)$ *has a minimum at* $x=\hat x(t)$ *and is smooth for all* $x\ne\hat x(t)$,
(2) there exist class K_∞ *functions* $\alpha_i(r)$, $i=1,2$ *which satisfy*

$$\alpha_1(|x-\hat x(t)|)\le Q(x,t)\le\alpha_2(|x-\hat x(t)|), \tag{15}$$

(3) there exist a class K funtion $\alpha_3(r)$ *which satisfies*

$$\frac{d}{dt}Q(x(t),t) \leq -\alpha_3(|x(t)-\hat{x}(t)|) + |w(t)|^2 + |v(t)|^2 \tag{16}$$

for any state trajectory $x(t)$ *and noises* $w(t)$, $v(t)$ *consistent with the observation* $y(t) = h(x(t)) + v(t)$.

Theorem 2 *The observer* $\hat{x}(t)$ *is convergent if there exist a Lyapunov function* $Q(x,t)$.

Proof: Suppose a Lyapunov function $Q(x,t)$ exists then

$$\begin{aligned}\frac{d}{dt}Q(x(t),t) &\leq -\alpha_3(|x(t)-\hat{x}(t)|) + |w(t)|^2 + |v(t)|^2 \\ &= -\alpha_3(\alpha_2^{-1}(\alpha_2(|x(t)-\hat{x}(t)|))) + |w(t)|^2 + |v(t)|^2 \\ &\leq -\alpha(Q(x(t),t)) + |w(t)|^2 + |v(t)|^2 \end{aligned} \tag{17}$$

where α is the class K function defined by

$$\alpha(r) \;=\; \alpha_3(\alpha_2^{-1}(r)) \tag{18}$$

By the above lemma, there exists a class K^2L function $\gamma_\alpha(r,s,t)$ such that

$$Q(x(t),t) \leq \gamma_\alpha(Q(x(\tau_1),\tau_1), \|w,v\|^2_{[\tau_1,\tau_2]}, t-\tau_2)$$

Hence the observer is convergent because

$$\begin{aligned}\alpha_1(|x(t)-\hat{x}(t)|) &\leq Q(x(t),t) \\ &\leq \gamma_\alpha(Q(x(\tau_1),\tau_1), \|w,v\|^2_{[\tau_1,\tau_2]}, t-\tau_2) \\ &\leq \gamma_\alpha(\alpha_2(|x(\tau_1)-\hat{x}(\tau_1)|), \|w,v\|^2_{[\tau_1,\tau_2]}, t-\tau_2)\end{aligned}$$

or as desired

$$|x(t)-\hat{x}(t)| \leq \gamma(|x(\tau)-\hat{x}(\tau)|, \|w,v\|_{[\tau_1,\tau_2]}, t-\tau_2)$$

where $\gamma(r,s,t)$ is the K^2L function defined by

$$\gamma(r,s,t) = \alpha_1^{-1}\left(\gamma_\alpha(\alpha_2(r), s^2, t)\right).$$

□

A partial converse to this is the following.

Theorem 3 *Suppose an observer* $\hat{x}(t)$ *is convergent then there exist a function* $Q(x,t)$ *such that*
(1) $Q(x,t)$ *is a causal functional of the initial state estimate* $\hat{x}^0$ *and the past control and observation,* $u(\tau)$, $y(\tau)$, $0 \leq \tau \leq t$ *such that* $Q(x,t)$ *has*

a minimum at $x = \hat{x}(t)$
(2) there exist class K_∞ *functions* $\alpha_i(r),\ i = 1, 2$ *which satisfy*

$$\alpha_1(|x - \hat{x}(t)|) \le Q(x,t) \le \alpha_2(|x - \hat{x}(t)|), \tag{19}$$

(3') Along any state trajectory $x(t)$*, input* $u(t)$*, observation* $y(t)$ *and noises* $w(t),\ v(t)$ *consistent with the system (3,4)*

$$Q(x(t_2), t_2) \le Q(x(t_1), t_1) + ||w, v||^2_{[t_1,t_2]}$$

Proof: Suppose an observer $\hat{x}(t)$ is convergent, define $Q(x,t)$ as

$$Q(x,t) = \inf\{|z(\tau_1) - \hat{x}(\tau_1)|^2 + ||w, v||^2_{[\tau_1,t]} : 0 \le \tau_1 \le t\} \tag{20}$$

where the infimum is over all $z(\tau),\ w(\tau),\ v(\tau)$ satisfying

$$\frac{d}{d\tau} z(\tau) = f(z(\tau), u(\tau)) + g(z(\tau))w(\tau) \tag{21}$$

$$y(\tau) = h(z(\tau)) + v(\tau) \tag{22}$$

$$z(t) = x. \tag{23}$$

and $u(\tau),\ y(\tau)$ are the control and observation. Clearly $Q(x,t)$ is a causal functional of the the control, observation and estimate $\hat{x}(\tau)$. The latter is a causal functional of the control, observation and initial state estimate $\hat{x}^0$. We claim that

$$\alpha_1(|x - \hat{x}(t)|) \le Q(x,t) \le \alpha_2(|x - \hat{x}(t)|) \tag{24}$$

where α_i are the class K_∞ functions

$$\begin{aligned} \alpha_1(|x - \hat{x}(t)|) &= (\min\{\delta_1(|x - \hat{x}(t)|), \delta_2(|x - \hat{x}(t)|)\})^2 \\ \alpha_2(|x - \hat{x}(t)|) &= |x - \hat{x}(t)|^2 \end{aligned}$$

and δ_i are from the Definition 2 of a convergent observer. The second inequality is obvious, the first can be seen as follows. Suppose it is not true, i.e., there exists an x, τ_1 and $z(\tau),\ w(\tau),\ v(\tau)$ satisfying (21-23) such that

$$|z(\tau_1) - \hat{x}(\tau_1)|^2 + ||w, v||^2_{[\tau_1,t]} < (\min\{\delta_1(|x - \hat{x}(t)|), \delta_2(|x - \hat{x}(t)|)\})^2$$

which implies that

$$\begin{aligned} |z(\tau_1) - \hat{x}(\tau_1)| &< \delta_1(|x - \hat{x}(t)|) \\ ||w, v||_{[\tau_1,t]} &< \delta_2(|x - \hat{x}(t)|) \end{aligned}$$

By property (1) of a convergent observer, this implies the contradiction

$$|x - \hat{x}(t)| < |x - \hat{x}(t)|.$$

Condition (3') follows immediately from the definition of $Q(x,t)$, (20-23).
□

2 Conclusions

We have presented a Lyapunov theorem for convergent observers and a partial converse. The Lyapunov function is of a novel type, it is a causal functional of the past control and observation. Moreover it does not explicitly depend on the current estimate $\hat{x}(t)$ but it determines the estimate as it arg min. In some sense the function $x \mapsto Q(x,t)$ is the state at time t of an infinite dimensional observer. Further refinement of this work is needed to obtain a Lyapunov and converse Lyapunov theory of observers.

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