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To author with
Best wishes and
Thanks for all I
learned from you
Roger.

3

Feedback Linearization

A.J. Krener

ABSTRACT R. W. Brockett is the father of feedback linearization, an important technique for the control of nonlinear systems. He is primarily responsible for recognizing this and fostering its developement. Feedback linearization has also strongly influenced the subsequent development of nonlinear systems theory. It has motivated many later trends in the field as we shall discuss below.

3.1 Introduction

By the early 1970s, the design of controllers for linear systems in state space form had reached a mature level of developement. The linear quadratic Gaussian (LQR) paradigm was well-known and was being applied in a variety of areas. But there was little in the way of systematic design techniques for broad classes of linear systems.

By that time, the importance of the Lie bracket to the analysis of nonlinear systems had been realized by several researchers including R. Hermann, R. W. Brockett, H. Sussmann, V. Jurdjevic, and myself. I remember a conversation I had with Roger during 1974-75 while I was working with him as a postdoc student at Harvard. We were discussing the structure of the brackets associated with a nonlinear system that was affine in the control, a class that includes linear systems. I pointed out that certain Lie brackets vanished if the system is linear. Roger asked the important question, does this characterize the class of linear systems in a coordinate-free way? The answer was yes, as decribed in [45].

But there is another way of transforming a linear system to make it nonlinear, namely, nonlinear state feedback. At the IFAC Congress in Helsinki in 1978, Roger posed the mathematically more difficult and the more applicable question, when is a nonlinear system just a linear system to which nonlinear-state feedback and nonlinear change-of-state coordinates has been applied. He also answered a special case of this question, when the control is scalar and the new control is a constant multiple of the old [11].

Soon afterwards, Korobov [44], Jakuczyk-Respondek [37], Sommer, [65] and Hunt et al. [31, 32, 66] solved the general problem and an industry was born.

Many of the later development in nonlinear systems theory grew out of the feedback linearization point of view. Concepts and techniques such as input-output linearization, zero dynamics, approximate feedback linearization, normal forms and invariants, nonlinear observers with linear error dynamics, nonlinear regulation, nonlinear model matching, backstepping, dynamic inversion, and flatness all have their roots in feedback linearization.

We would like to call the reader's attention to an excellent recent survey [23] by Guardabassi and Savaresi which is complimentary to this survey.

3.2 Linearization of a Smooth Vector Field

Let's start by addressing an apparently simpler question that was first considered by Poincaré. Given a smooth n-dimensional vector field around a critical point, say $x = 0$,

$$\begin{aligned} \dot{x} &= f(x) \\ 0 &= f(0), \end{aligned}$$

find a smooth local change of coordinates

$$\begin{aligned} z &= \phi(x) \\ 0 &= \phi(0) \end{aligned}$$

which transforms it into a linear vector field.

$$\dot{z} = Az.$$

I say "apparently simpler" because, as we shall see in the next section, the corresponding question for a system that is affine in the control is actually easier to answer.

Without loss of generality we can restrict our attention to changes of coordinates which carry $x = 0$ to $z = 0$ and whose Jacobian at this point is the identity, i.e.,

$$z = x + O(x^2)$$

then

$$A = \frac{\partial f}{\partial x}(0).$$

Poincaré's formal solution to this problem was to expand the vector field and the desired change of coordinates in a power series.

$$\begin{aligned} \dot{x} &= Ax + f^{[2]}(x) + O(x^3) \\ z &= x - \phi^{[2]}(x) \end{aligned}$$

where $f^{[2]}$, $\phi^{[2]}$ are n-dimensional vector fields, whose entries are homogeneous polynomials of degree 2 in x. A straightforward calculation yields

$$\dot{z} = Az + f^{[2]}(x) - \left[Ax, \phi^{[2]}(x)\right] + O(x^3)$$

where the Lie bracket of the two vector fields is given by

$$[f(x), g(x)] = \frac{\partial g}{\partial x}(x) f(x) - \frac{\partial f}{\partial x}(x) g(x).$$

Hence $\phi^{[2]}(x)$ must satisfy the so-called homological equation [1]

$$\left[Ax, \phi^{[2]}(x)\right] = f^{[2]}(x).$$

This is a linear equation from the space of quadratic vector fields to the space of quadratic vector fields. The quadratic n-dimensional vector fields form a vector space of dimension $n\binom{n+1}{2}$.

Suppose A is semisimple, i.e., there exist bases of left and right eigenvectors of A,

$$Av^k = \lambda_k v^k \qquad 1 \le k \le n, \tag{1}$$

$$w_i A = \lambda_i w_i \qquad 1 \le i \le n. \tag{2}$$

Define $n\binom{n+1}{2}$ quadratic vector fields

$$\phi^k_{ij}(x) = v^k \; w_i x \; w_j x \qquad 1 \le k \le n, \qquad 1 \le i \le j \le n.$$

These form a basis for the space of quadratic vector fields and are the eigenvectors of $[Ax, \cdot]$,

$$\left[Ax, \phi^k_{ij}(x)\right] = \left(\lambda_i + \lambda_j - \lambda_k\right) \phi^k_{ij}(x).$$

The eigenvalues are $\lambda_i + \lambda_j - \lambda_k$. If none of these expressions are zero, the operator $[Ax, \cdot]$ acting on the space of quadratic vector fields is invertible. If some $\lambda_i + \lambda_j - \lambda_k = 0$ then this is called a resonance, and the homological equation is not solvable for some $f^{[2]}(x)$.

If A is not semisimple the analysis is slightly more complicated involving left and right generalized eigenvectors but the basic result is the same. The homological equation is solvable for every $f^{[2]}(x)$ iff there is no resonance, $\lambda_i + \lambda_j - \lambda_k \neq 0$.

Suppose a change of coordinates exist that linearizes the vector field up to degree r. In the new coordinates, the vector field is of the form

$$\dot{x} = Ax + f^{[r]}(x) + O(x^{r+1}).$$

We seek a change of coordinates of the form

$$z = x - \phi^{[r]}(x)$$

to cancel the degree r terms, i.e., we seek a solution of the r^{th} degree homological equations,

$$\left[Ax, \phi^{[r]}(x)\right] = f^{[r]}(x).$$

As before, the eigenvectors and eigenvalues of $[Ax, \cdot]$ operating on degree r vector fields are given by

$$\phi^k_{i_1 \ldots i_r}(x) = v^k \; w_{i_1} x \ldots w_{i_r} x$$

$$\lambda_{i_1} + \cdots + \lambda_{i_r} - \lambda_k$$

and a resonance occurs if

$$\lambda_{i_1} + \cdots + \lambda_{i_r} - \lambda_k = 0.$$

If there is no resonance then the degree r homological equations are solvable for every $f^{[r]}(x)$.

When it exists the convergence of the formal power series solution is delicate. We refer the reader to Arnol'd [1] for the details.

The simultaneous linearization of a family of vector fields

$$\dot{x} = f^i(x), \qquad i = 1, \ldots, l,$$

around a common critical point

$$0 = f^i(0)$$

has been treated by Hermann [26] and Guillemin-Sternberg [25].

The bifurcations of parametrized vector fields

$$\dot{x} = f(x, \mu)$$

occur at resonances

$$\lambda_{i_1}(\mu) + \cdots + \lambda_{i_r}(\mu) - \lambda_k(\mu) = 0.$$

For example, a Hopf bifurcation occurs at a resonance of degree 3. Consider the dynamics

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \mu & -\nu \\ \nu & \mu \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + c(x_1^2 + x_2^2) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where μ is a parameter. For any value of μ, $x = 0$ is a critical point and the eigenvalues of the Jacobian around this critical point are

$$\lambda_1 = \mu + i\nu, \qquad \lambda_2 = \mu - i\nu.$$

There are two degree 3 resonances at $\mu = 0$,

$$\lambda_1 + \lambda_1 + \lambda_2 - \lambda_1 = 2\mu,$$
$$\lambda_1 + \lambda_2 + \lambda_2 - \lambda_2 = 2\mu.$$

The cubic term in the above dynamics is one of the resonant terms,

$$\phi^1_{112} + \phi^2_{122} = (x_1^2 + x_2^2) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The other resonant term is

$$\phi^1_{112} - \phi^2_{122} = (x_1^2 + x_2^2) \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}.$$

3.3 Linearization of a Smooth Control System by Change-of-State Coordinates

The result [45] that grew out of my conversation with Roger Brockett is the following. Given a system with m-dimensional control u entering affinely

$$\dot{x} = f(x) + g(x)u,$$

when does there exist a smooth local change of coordinates

$$z = \phi(x),$$
$$0 = \phi(0),$$

transforming it to

$$\dot{z} = Az + Bu$$

where

$$A = \frac{\partial f}{\partial x}(0),$$
$$B = g(0).$$

Assuming that A, B is a controllable pair, i.e.,

$$\operatorname{rank}\left[B\ AB \ldots\ A^{n-1}B\right] = n,$$

this is an easier question than that of Poincaré. A necessary and sufficient condition is that

$$\left[\operatorname{ad}^k(f)g^i, \operatorname{ad}^l(f)g^j\right] = 0$$

for $k = 0, \ldots, n-1$, $l = 0, \ldots, n$ where g^i denotes the ith column of g and

$$\operatorname{ad}^0(f)g^i = g^i,$$
$$\operatorname{ad}^k(f)g^i = \left[f, \operatorname{ad}^{k-1}(f)g^i\right].$$

The proof of this theorem is straightforward. Under a change-of-state coordinates, the vector fields and their Lie brackets are transformed by the Jacobian of the coordinate change. Trivially for linear systems

$$\operatorname{ad}^k(Ax)B^i = (-1)^k A^k B,$$
$$\left[\operatorname{ad}^k(Ax)B^i, \operatorname{ad}^l(Ax)B^j\right] = 0.$$

The Lie brackets of the vector fields $f, g^1, \ldots, g^m$ evaluated at some x^0 are the coordinate-free Taylor series coefficients of the control system around x^0. Let me make this more precise [46]. Given a control system and an initial point

$$\dot{x} = f^0(x) + \sum_{j=1}^{m} f^j(x)u_j,$$

$$x(0) = x^0.$$

the Lie jet $\mathcal{LJ}^r(f, x^0)$ to degree r at x^0 is the multi-indexed family of vectors of the form

$$\left[f^{j_1}, \left[\ldots, \left[f^{j_{k-1}}, f^{j_k}\right]\ldots\right]\right](x^0)$$

where $0 \le j_i \le m$, $1 \le k \le r$.

Given another smooth initialized system

$$\dot{z} = g^0(x) + \sum_{j=1}^{m} g^j(z)u_j.$$
$$z(0) = z^0.$$

Suppose that $\mathcal{LJ}^r(f, x^0)$ spans x space then there exists a smooth mapping $z = \phi(x)$ which preserves trajectories to degree r for every bounded control $u(t)$

$$\phi(x(t)) = z(t) + O(t^{r+1})$$

iff there exists a linear map T which carries $\mathcal{LJ}^r(f, x^0)$ to $\mathcal{LJ}^r(g, z^0)$, respecting the indexing to degree r,

$$T\left[f^{j_1}, \left[\ldots, \left[f^{j_{k-1}}, f^{j_k}\right]\ldots\right]\right](x^0) = \left[g^{j_1}, \left[\ldots \left[g^{j_{k-1}}, g^{j_k}\right]\ldots\right]\right](z^0)$$

for any $k \le r$ and any $j_1, \ldots j_k$. The dimensions of x and z need not be the same. This result can be used to construct bilinear and free-nilpotent approximations whose properties are more easily analyzed. In the theory of hypo-elliptic PDOs, there is a similar technique called the Rothschild-Stein Lifting Theorem. There is also a classical theorem for real analytic systems regarding nonlinear maps which preserve trajectories exactly and linear maps which preserve the infinite Lie jets.

3.4 Feedback Linearization

We turn to the question that Roger posed and partially answered at the IFAC Congress in Helsinki in 1978 [11]. Given a system affine in the m-dimensional control

$$\dot{x} = f(x) + g(x)u,$$

find a smooth local change of coordinates and smooth feedback

$$z = \phi(x).$$
$$u = \alpha(x) + \beta(x)v,$$

transforming it to

$$\dot{z} = Az + Bv.$$

Brockett [11] solved this problem under the assumptions that β is constant and the control is a scalar, $m = 1$. The more general question for $\beta(x)$ and arbitrary m was solved in different ways by Korobov [44], Jakubczyk-Respondek [37], Sommer [65] and Hunt et al. [31, 32, 66].

We describe the solution when $m = 1$. If the pair A, B is controllable, then there exist a C such that

$$\begin{aligned} CA^{k-1}B &= 0, \qquad k = 1, \dots, n-1, \\ CA^{n-1}B &= 1. \end{aligned}$$

If the nonlinear system is feedback linearizable then there exists a function $h(x) = C\phi(x)$ such that

$$\begin{aligned} L_{ad^{k-1}(f)g}h &= 0, \qquad k = 1, \dots, n-1, \\ L_{ad^{n-1}(f)g}h &\neq 0, \end{aligned}$$

where the Lie derivative of a function h by a vector field g is given by

$$L_g h = \frac{\partial h}{\partial x} g \,.$$

This is a system of first order PDEs and the solvability conditions are given by the classical Frobenius Theorem, namely, that

$$\{g, \dots, ad^{n-2}(f)g\}$$

is involutive, i.e., its span is closed under a Lie bracket.

For controllable systems this is a necessary and sufficient condition. The controllability condition is that $\{g, \dots, ad^{n-1}(f)g\}$ spans x space.

Suppose $m = 2$. Then the system is said to be controllable with controllability (Kronecker) indices $k_1,\ k_2$ if (possibly after reordering of g^1, g^2)

1. $k_1 \geq k_2 \geq 0$,
2. $k_1 + k_2 = n$,
3. $\{g^1, \dots, ad^{k_1-1}(f)g^1, g^2, \dots ad^{k_2-1}(f)g^2\}$ spans x space,
4. (k_1, k_2) is the smallest such pair in the lexographic ordering.

Such a system is feedback linearizable iff

$$\{g^1, , g^2, \dots, ad^{k_i-2}(f)g^1 . ad^{k_i-2}(f)g^2\}$$

is involutive for $i = 1, 2$. Another way of putting this is that the distribution spanned by the first through k^{th} rows of the following matrix must be involutive for $k = k_i - 1,\ i = 1, 2$. This is equivalent to the distribution spanned by the first through k^{th} rows of the following matrix being invoutive for all $k = 1, \dots, k_1$:

$$\begin{bmatrix} g^1 & g^2 \\ ad(f)g & ad(f)g^2 \\ \vdots & \vdots \\ ad^{k_2-2}(f)g^1 & ad^{k_2-2}(f)g^2 \\ ad^{k_2-1}(f)g^1 & ad^{k_2-1}(f)g^2 \\ \vdots & \\ ad^{k_1-2}(f)g^1 & \\ ad^{k_1-1}(f)g^1 & \end{bmatrix}$$

One might ask if it is possible to use dynamic feedback to linearize a system that is not linearizable by static feedback. Suppose we treat one of the controls u_j as a state and let its derivative be a new control,

$$\dot{u}_j = \bar{u}_j.$$

Can the resulting system be linearized by state feedback and change of state coordinates? Loosely speaking, the effect of adding such an integrator to the j^{th} control is to shift the j^{th} column of the above matrix down by one row. This changes the distribution spanned by the first through k^{th} rows of the above matrix and might make it involutive. A scalar input system $m = 1$ that is linearizable by dynamic-state feedback is also linearizable by static-state feedback. There are multi-input systems $m > 1$ that are dynamically linearizable but not statically linearizable [18, 19].

The generic system is not feedback linearizable but mechanical systems with one actuator for each degree of freedom typically are feedback linearizable. This fact had been used in many applications, e.g., robotics, before Brockett introduced the concept of feedback linearization.

One should not lose sight of the fact that stabilization, model-matching, or some other performance criterion is typically the goal of controller design. Linearization is a means to the goal. We linearize because we know how to meet the performance goal for linear systems.

Even when the system is linearizable, finding the linearizing coordinates and feedback can be a nontrivial task. Mechanical systems are the exception as the linearizing coordinates are usually the generalized positions. Since the $ad^{k-1}(f)g$ for $k = 1, \ldots, n-1$ are characteristic directions of the PDE for h, the general solutions of the ODEs

$$\dot{x} = ad^{k-1}(f)g(x)$$

can be used to construct the solution [10]. The Gardner-Shadwick (GS) algorithm [24] is the most efficient method that is known.

Linearization of discrete time systems was treated by Lee et al. [61] and Jakubczyk [50]. Linearization of discrete time systems around an equilibrium manifold was treated by Barbot et al. [7]. Banaszuk and Hauser have also considered the feedback linearization of the transverse dynamics along a periodic orbit, [2, 3].

3.5 Input-Output Linearization

Feedback linearization as presented above ignores the output of the system but typically one uses the input to control the output. Therefore one wants to linearize the input-output response of the system rather than the dynamics. This was first treated in [35] and [36].

Consider a scalar input, scalar output system of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ y &= h(x). \end{aligned}$$

The relative degree of the system is the number of integrators between the input and the output. To be more precise, the system is of relative degree $r \geq 1$ if for all x of interest,

$$\begin{aligned} L_{ad^j(f)g} h(x) &= 0, \qquad j = 0, \ldots, r-2, \\ L_{ad^{r-1}(f)g} h(x) &\neq 0. \end{aligned}$$

In other words the control appears first in the r^{th} time derivative of the output. Of course a system might not have a well-defined relative degree as the r might vary with x.

Rephrasing the the result of the previous section, a scalar input nonlinear system is feedback linearizable iff there exist an pseudo-output map $h(x)$ such the resulting scalar input, scalar output system has relative degree equal to the state dimension n.

Assuming we have a scalar input, scalar output system with a well defined relative degree $1 \leq r \leq n$. We can define r partial coordinate functions

$$\zeta_i(x) = (L_f)^{i-1} h(x), \qquad i = 1, \ldots, r,$$

and choose $n - r$ functions $\xi_i(x)$, $i = 1, \ldots, n - r$ so that $(\zeta, \ \xi)$ are a full set of coordinates on the state space. Furthermore it is always possible [33] to choose $\xi_i(x)$ so that

$$L_g \xi_i(x) = 0, \qquad i = 1, \ldots, n - r.$$

In these coordinates the system is in the normal form

$$\begin{aligned} y &= \zeta_1, \\ \dot{\zeta}_1 &= \zeta_2, \\ &\vdots \\ \dot{\zeta}_{r-1} &= \zeta_r, \\ \dot{\zeta}_r &= f_r(\zeta, \xi) + g_r(\zeta, \xi)u, \\ \dot{\xi} &= \phi(\zeta, \xi). \end{aligned}$$

The feedback $u = u(\zeta, \xi, v)$ defined by

$$u = \frac{v - f_r(\zeta, \xi)}{g_r(\zeta, \xi)}$$

transforms the system to

$$\begin{aligned} y &= C\zeta, \\ \dot{\zeta} &= A\zeta + Bv, \\ \dot{\xi} &= \phi(\zeta, \xi), \end{aligned}$$

where A, B, and C are the $r \times r$, $r \times 1$, and $1 \times r$ matrices, respectively,

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The system has been transformed into a string of integrators plus additional dynamics that is unobservable from the output.

By suitable choice of additional feedback $v = K\zeta$, one can insure that the poles of $A + BK$ are stable. The stability of the overall system then depends on the stability of the zero dynamics [16, 17],

$$\dot{\xi} = \phi(0, \xi).$$

If this is stable then the overall system will be stable. The zero dynamics is so-called because it it is the dynamics that results from imposing the constraint $y(t) = 0$ on the system. For this to be satisfied the initial value must satisfy $\zeta(0) = 0$ and the control must satisfy

$$u(t) = -\frac{f_r(0, \xi(t))}{g_r(0, \xi(t))}.$$

Similar results hold in the multiple input, multiple output case: see Isidori [33] for the details.

3.6 Approximate Feedback Linearization

Since so few systems are exactly feedback linearizable, Krener [47] introduced the concept of approximate feedback linearization. The goal is to find a smooth local change of coordinates and a smooth feedback

$$\begin{aligned} z &= \phi(x) \\ u &= \alpha(x) + \beta(x)v \end{aligned}$$

transforming the system

$$\dot{x} = f(x) + g(x)u$$

to

$$\dot{z} = Az + Bv + N(x, u)$$

where the nonlinearity $N(x, u)$ is small in some sense. In the design process, the nonlinearity is ignored, the controller design is done on the linear model and then transformed back into a controller for the original system.

There are at least three distinct approaches to approximate feedback linearization,

1. Power Series Method a lá Poincaré.
2. Least Squares Method.
3. Homotopy Method.

The power series approach was taken by Krener and co-workers [48, 51, 52, 55, 59]. See also [40]. It is applicable to systems which may not be affine in the control. The system

$$\dot{x} = f(x, u),$$

the change of coordinates and the feedback are expanded in a power series

$$\begin{aligned}
\dot{x} &= Ax + Bu + f^{[2]}(x, u) + O(x, u)^3, \\
z &= x - \phi^{[2]}(x), \\
v &= u - \alpha^{[2]}(x, u).
\end{aligned}$$

The transformed system is

$$\begin{aligned}
\dot{z} = {} & Az + Bv + f^{[2]}(x, u) \\
& - \left[Ax + Bu, \phi^{[2]}(x)\right] + B\alpha^{[2]}(x, u) \\
& + O(x, u)^3.
\end{aligned}$$

To eliminate the quadratic terms, one seeks a solution of the degree 2 homological equations for $\phi^{[2]}$, $\alpha^{[2]}$

$$\left[Ax + Bu, \phi^{[2]}(x)\right] - B\alpha^{[2]}(x, u) = f^{[2]}(x, u).$$

Unlike before, the degree 2 homological equations are not square but are $n\binom{n+m+1}{2}$ linear equations in $n\binom{n+1}{2} + m\binom{n+m+1}{2}$ unknowns. Almost always the number of unknowns is less than the number of equations. Furthermore, the mapping

$$\left(\phi^{[2]}(x), \alpha^{[2]}(x, u)\right) \mapsto \left[Ax + Bu, \phi^{[2]}(x)\right] - B\alpha^{[2]}(x, u)$$

is less than full rank. Hence only an approximate, e.g., a least squares solution, is possible. Krener has written a MATLAB toolbox to compute term-by-term solutions to the homological equations. The Nonlinear Systems Toolbox is available from

`http://scad.utdallas.edu/scad/software`

The routine `fh2f_h_.m` sequentially computes the least squares solutions of the homological equations to arbitrary degree.

The Least Squares Method was developed by Banaszuk-Swięch-Hauser [6]. Suppose we consider a scalar input system, $m = 1$, affine in the control,

$$\dot{x} = f(x) + g(x)u.$$

The goal is to find an approximate integrating factor β for the one-form ω_0 that is dual to the distribution spanned by

$$\{g, \dots, ad^{n-2}(f)g\}$$

by minimizing the functional

$$I(\beta) = \frac{\|d\beta\omega_0\|^2 + \|\delta\beta\omega_0\|^2}{\|\beta\omega_0\|^2}.$$

To define this functional, one needs a Riemannian metric and its induced volume form. The first term in the functional is a measure of the nonintegrability of $\beta\omega_0$ and the second term is a measure of the nonsmoothness of $\beta\omega_0$. Which Riemannian metric to take is an open question. One can choose it so that

$$\{g, \dots, ad^{n-1}(f)g\}$$

is an orthonormal frame but there is no systems theoretic reason for doing so.

The Homotopy Method is due to Banaszuk and coworkers [4, 64]. A homotopy operator is used to decompose the one-form ω_0 into an exact and antiexact part. The design is based on the linearizable system corresponding to the exact part. As with other methods of approximate feedback linearization, there is a certain arbitrariness in the solution. In this method, it is in the choice of a homotopy operator.

Xu and Hauser [70] have studied the problem of approximate feedback linearization around an equilibrium manifold.

3.7 Normal Forms of Control Systems

In [11], Brockett initiated the study of the feedback invariants of a nonlinear control system

$$\dot{x} = f(x) + g(x)u.$$

These are the quantities that are invariant under change-of-state coordinates and state feedback. In the category of linear sytems under linear changes of state coordinates and linear state feedbacks, the invariants are integers, the familiar Kronecker or controllabilities indices. Brockett generalized this result by giving integer invariants in the nonlinear category. Both the Kronecker indices and Brockett's invariants are the dimensions of certain distributions associated with the nonlinear system.

Kang [42] and Kang and Krener [43, 58] took a different approach and studied the invariants of the quadratic part of a control system

$$\dot{x} = Ax + Bu + f^{[2]}(x) + g^{[1]}(x)u + O(x,u)^3$$

under quadratic changes of coordinates and feedbacks

$$\begin{aligned} z &= x - \phi^{[2]}(x), \\ v &= u - \alpha^{[2]}(x) - \beta^{[1]}(x)u. \end{aligned}$$

Under the assumption that the linear part of the system is controllable, they were able to find a complete set of quadratic invariants and also the quadratic normal forms. The invariants are closely related to the generalized Legendre-Clebsch condition [58].

Suppose the linear part of a scalar input system is controllable. Then by linear change of coordinates and feedback the linear part can be brought into Brunovsky normal form where

$$A = \frac{\partial f}{\partial x}(0) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$B = g(0) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Kang and Krener [43] showed that there exists a quadratic change of coordinates and feedback transforming the system to

$$\dot{x} = Ax + Bu + f^{[2]}(x) + O(x,u)^3$$

where

$$f_i^{[2]}(x) = \begin{cases} \sum_{j=i+2}^{n} a_{ij} x_j^2, & 1 \le i \le n-2, \\ 0, & n-1 \le i \le n. \end{cases}$$

The coefficients a_{ij} are a complete set of invariants. Kang [42] has found the normal forms for higher-degree scalar input systems.

The simplest system with controllable linear part that is not feedback linearizable is

$$\begin{aligned} \dot{x}_1 &= x_2 + x_3^2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= u. \end{aligned}$$

Because it is a scalar input system, it is also not exactly linearizable by dynamic feedback [18]. But it is linearizable to degree 2 by dynamic feedback [43]. We add a feedback and two states,

$$\begin{aligned} u &= w_1 + \alpha^{[2]}(x, w), \\ \dot{w}_1 &= w_2, \\ \dot{w}_2 &= v. \end{aligned}$$

Let $\alpha^{[2]}(x, w) = -2w_1^2 - 2x_3 w_2$ then a change of coordinates linearizes the system through quadratic terms

$$\begin{aligned} z_1 &= x_1, \\ \dot{z}_1 &= z_2 = x_2 + x_3^2, \\ \dot{z}_2 &= z_3 = x_3 + 2x_3 w_1, \\ \dot{z}_3 &= z_4 = w_1 + \alpha^{[2]}(x, w) + 2w_1^2 + 2x_3 w_2 + O(x, w)^3, \\ &= z_4 = w_1, \\ \dot{z}_4 &= z_5 = w_2, \\ \dot{z}_5 &= v. \end{aligned}$$

This technique generalizes to multi-input systems and higher degrees. Every system with controllable linear part is approximately dynamically feedback linearizable to arbitrary degree. But the dimension goes to infinity.

Kang has also studied the normal form of a system with uncontrollable linear part [41]. Just as a parametrized dynamical system bifurcates at a resonance, a parametrized control system bifurcates when there is a loss of linear controllability. Kang has studied the normal forms of such bifurcations where the parameter may or may not be the control [38, 39].

This work has been extended to discrete time systems by Barbot et al. [8].

3.8 Observers with Linearizable Error Dynamics

The dual of linear-state feedback is linear input-output injection. Linear input-output injection is the transformation carrying

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned}$$

into

$$\dot{x} = Ax + Bu + Ly + Mu,$$
$$y = Cx.$$

Linear input-output injection and linear change-of-state coordinates

$$\dot{x} = Ax + Bu + Ly + Mu,$$
$$y = Cx,$$
$$z = Tx,$$
$$\dot{z} = TAT^{-1}x + TLy + TMu,$$

defines a group action on the class of linear systems. Of course, output injection is not physically realizable on the original system but it is realizable on observer error dynamics.

Nonlinear input-output injection is not well-defined independent of the coordinates; input-output injection in one coordinate system does not look like input-output injection in another coordinate system,

$$\dot{x} = f(x, u),$$
$$y = h(x),$$
$$\dot{x} = f(x, u) + \alpha(y, u),$$
$$z = \phi(x),$$
$$\dot{z} = \frac{\partial \phi}{\partial x}(x) f(x) + \frac{\partial \phi}{\partial x}(x)\alpha(y, u).$$

Notice that the injection term in the z coordinates is state dependent and hence is not input-output injection.

If a system

$$\dot{x} = f(x, u),$$
$$y = h(x),$$

can be transformed by nonlinear changes of state and output coordinates

$$z = \phi(x),$$
$$w = \gamma(y),$$

to a linear system with nonlinear input-output injection

$$\dot{z} = Az + Bu + \alpha(y, u),$$
$$w = Cz,$$

then the observer

$$\dot{\hat{z}} = (A + LC)\hat{z} + Bu + \alpha(y, u) - Lw$$

has linear error dynamics

$$\tilde{z} = z - \hat{z},$$
$$\dot{\tilde{z}} = (A + LC)\tilde{z}.$$

If C, A is detectable then $A + LC$ can be made stable.

The case when γ = identity, there are no inputs $m = 0$ and one output $p = 1$,

$$\dot{x} = f(x),$$
$$y = h(x),$$

was solved by Krener-Isidori [49] and Bestle-Zeitz [9] when the pair C, A defined by

$$A = \frac{\partial f}{\partial x}(0),$$
$$C = \frac{\partial h}{\partial x}(0),$$

is observable.

One seeks a change of coordinates $z = \phi(x)$ so that the system is linear up to output injection

$$\dot{z} = Az + \alpha(y),$$
$$y = Cz.$$

If they exist, the z coordinates satisfy the PDEs

$$L_{ad^{n-k}(f)g}(z_j) = \delta_{k,j}$$

where the vector field $g(x)$ is defined by

$$L_g L_f^{k-1} h = \begin{cases} 0, & 1 \leq k < n, \\ 1, & k = n. \end{cases}$$

The solvability conditions for these PDEs are that for $1 \leq k < l \leq n-1$

$$\left[ad^{k-1}(f)g, ad^{l-1}(f)g\right] = 0.$$

The general case with γ, m, p arbitrary was solved by Krener-Respondek [53]. The solution is a three-step process. First, one must set up and solve a linear PDE for $\gamma(y)$. The integrability conditions for this PDE involve the vanishing of a pseudo-curvature [54]. The next two steps are similar to the above. One defines a vector field g^j, $1 \leq j \leq p$ for each output; these define a PDE for the change of coordinates, for which certain integrability conditions must be satisfied. The process is more complicated than feedback linearization and even less likely to be successful, so approximate solutions must be sought which we discuss later in this section. We refer the reader to [53] and related work [63, 67, 68, 69, 71].

To show that the set of systems to which these methods apply is nonempty, we close with an example. Consider that a Van der Pol oscillator

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = a_{21}x_1 + (a_{22} - x_1^2)x_2,$$
$$y = x_1,$$

can be transformed to

$$\begin{aligned}\dot{z}_1 &= z_2 + \alpha_1(y),\\ \dot{z}_2 &= a_{21}z_1 + a_{22}z_2 + \alpha_2(y),\\ y &= z_1,\end{aligned}$$

by the change of coordinates

$$\begin{aligned}z_1 &= x_1,\\ z_2 &= x_2 - \alpha_1(y),\end{aligned}$$

where

$$\begin{aligned}\alpha_1(y) &= \frac{-y^3}{3},\\ \alpha_2(y) &= a_{22}\frac{y^3}{3}.\end{aligned}$$

An observer for the transformed system can easily be constructed,

$$\begin{aligned}\dot{\hat{z}}_1 &= \hat{z}_2 + \alpha_1(y) - k_1(y - \hat{z}_1),\\ \dot{\hat{z}}_2 &= a_{21}\hat{z}_1 + a_{22}\hat{z}_2 + \alpha_2(y) - k_2(y - \hat{z}_1),\end{aligned}$$

which for proper choice of k_1, k_2 has stable linear error dynamics

$$\begin{aligned}\dot{\tilde{z}}_1 &= k_1\tilde{z}_1 + \tilde{z}_2,\\ \dot{\tilde{z}}_2 &= (a_{21} + k_2)\tilde{z}_1 - a_{22}\tilde{z}_2.\end{aligned}$$

Very few systems can be linearized by change-of-state coordinates and input-output injection so Krener and co-workers [48, 51, 52, 55, 59] sought approximate solutions by the power series approach. Again the system and changes of coordintes are expanded in a power series.

$$\begin{aligned}\dot{x} &= Ax + Bu + f^{[2]}(x, u) + O(x, u)^3,\\ y &= Cx + Du + h^{[2]}(x, u) + O(x, u)^3,\\ z &= x - \phi^{[2]}(x).\end{aligned}$$

Input-output injection terms expanded in a power series are added and subtracted to both the dynamics and the output map

$$\begin{aligned}\dot{z} = {} & Az + Bu + \alpha^{[2]}(y, u)\\ & + f^{[2]}(x, u) - \left[Ax + Bu, \phi^{[2]}(x)\right] - \alpha^{[2]}(y, u)\\ & + O(x, u)^3\\ y = {} & Cz + Du + \gamma^{[2]}(y, u)\\ & + h^{[2]}(x, u) + C\phi^{[2]} - \gamma^{[2]}(y, u)\\ & + O(x, u)^3.\end{aligned}$$

The observer is of the form

$$\dot{\hat{z}} = A\hat{z} + Bu + \alpha^{[2]}(y,u) - L\left(y - \left(C\hat{z} + Du + \gamma^{[2]}(y,u)\right)\right)$$

where L is chosen so that $A + LC$ is asymptotically stable. The error dynamics is

$$\begin{aligned}\dot{\tilde{z}} &= (A + LC)\tilde{z} \\ &\quad + f^{[2]}(x,u) - \left[Ax + Bu, \phi^{[2]}(x)\right] - \alpha^{[2]}(y,u) \\ &\quad - L\left(h^{[2]}(x,u) + C\phi^{[2]} - \gamma^{[2]}(y,u)\right) \\ &\quad + O(x,u)^3.\end{aligned}$$

This leads to the degree two homological equations for $\phi^{[2]}$, $\alpha^{[2]}$, and $\gamma^{[2]}$

$$\left[Ax + Bu, \phi^{[2]}(x)\right] + \alpha^{[2]}(y,u) + L\left(C\phi^{[2]} - \gamma^{[2]}(y,u)\right) = f^{[2]}(x,u) - Lh^{[2]}(x,u)$$

Notice that they depend on the observer gain L. These homological equations are $n\binom{n+m+1}{2}$ linear equations in $n\binom{n+1}{2} + (n+p)\binom{p+m+1}{2}$ unknowns. The equations are not of full rank. In particular, the $\gamma^{[2]}(y,u)$ term does not increase the set of $f^{[2]}(x,u)$, $h^{[2]}(x,u)$ for which the equations are solvable, but it does change the higher-degree terms, $O(x,u)^3$. The approach taken in [48, 51, 52, 55, 59] is to seek a least squares solution to the homological equations. Recently, Banaszuk and Sluis have taken a different type of least squares approach [5].

3.9 Nonlinear Regulation and Model Matching

Consider a nonlinear plant

$$\begin{aligned}\dot{x} &= f(x,u,\bar{x},\bar{u}) \\ &= Ax + Bu + F\bar{x} + G\bar{u} + f^{[2]}(x,u,\bar{x},\bar{u}) \\ &\quad + O(x,u,\bar{x},\bar{u})^3, \\ y &= h(x,u,\bar{x},\bar{u}) \\ &= Cx + Du + H\bar{x} + J\bar{u} + h^{[2]}(x,u,\bar{x},\bar{u}) \\ &\quad + O(x,u,\bar{x},\bar{u})^3,\end{aligned}$$

and nonlinear model

$$\begin{aligned}\dot{\bar{x}} &= \bar{f}(\bar{x},\bar{u}) \\ &= \bar{A}\bar{x} + \bar{B}\bar{u} + \bar{f}^{[2]}(\bar{x},\bar{u}) + O(\bar{x},\bar{u})^3 \\ \bar{y} &= \bar{h}(\bar{x},\bar{u}) \\ &= \bar{C}\bar{x} + \bar{D}\bar{u} + \bar{h}^{[2]}(\bar{x},\bar{u}) + O(\bar{x},\bar{u})^3.\end{aligned}$$

The goal of model matching is to use a combination of feedforward and feedback control $u = \alpha(x,\bar{x},\bar{u})$ so that the output of the plant asymptotically tracks that of the model

$$e(t) = y(t) - \bar{y}(t) \longrightarrow 0$$

for every $x(0)$, $\bar{x}(0)$ and every constant $\bar{u}$. The plant should also be internally stable.

The model matching problem is quite broad and in some sense it generalizes the nonlinear problems of feedback linearization, input-output linearization, regulation, disturbance rejection, and gain schelduling. We elaborate on this below.

The solution of the model matching problem is in two steps. The first is to use feedforward from the model state and input to insure exact tracking when the initial conditions of the plant and the model permit this. The linear version of the problem was solved by Francis [22] and its nonlinear generalization is due to Isidori and Byrnes [34]. One seeks $\theta(\bar{x})$, $\lambda(\bar{x}, \bar{u})$, satisfying the Francis-Byrnes-Isidori (FBI) PDE

$$\begin{aligned} f(\theta(\bar{x}), \lambda(\bar{x}, \bar{u}), \bar{x}, \bar{u}) &= \frac{\partial \theta}{\partial \bar{x}}(\bar{x}) \bar{f}(\bar{x}, \bar{u}), \\ h(\theta(\bar{x}), \lambda(\bar{x}, \bar{u}), \bar{x}, \bar{u}) &= \bar{h}(\bar{x}, \bar{u}). \end{aligned}$$

If the FBI PDE is solvable, then the control $u = \lambda(\bar{x}, \bar{u})$ makes $x = \theta(\bar{x})$ an invariant manifold of the combined system consisting of plant and model. And on this manifold, exact tracking occurs, $e = 0$.

One can attempt to solve the FBI equations term by term. Suppose

$$\begin{aligned} \theta(\bar{x}) &= T\bar{x} + \theta^{[2]}(\bar{x}) + O(\bar{x})^3, \\ \lambda(x, \bar{x}, \bar{u}) &= L\bar{x} + M\bar{u} + \lambda^{[2]}(\bar{x}, \bar{u}) + O(\bar{x}, \bar{u})^3. \end{aligned}$$

The linear part of the FBI equations are the Francis equations

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} T & 0 \\ L & M \end{bmatrix} - \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} = - \begin{bmatrix} F & G \\ H & J \end{bmatrix}.$$

These are solvable if there is output zero of the model corresponding to every output zero of the plant [29, 30, 56, 57]. In other words, the model cannot excite those frequencies that are output zeros of the plant.

The higher-degree equations are linear and depend on the solutions of the lower-degree equations. They are solvable if the resonances of the model don't excite the zeros of the plant. The degree 2 equations are

$$\begin{aligned} A\theta^{[2]}(\bar{x}) + B\lambda^{[2]}(\bar{x}, \bar{u}) - \frac{\partial \theta}{\partial \bar{x}}(\bar{x}) \left(\bar{A}\bar{x} + \bar{B}\bar{u}\right) \\ = -f^{[2]}(T\bar{x}, L\bar{x} + M\bar{u}, \bar{x}, \bar{u}) + T\bar{f}^{[2]}(\bar{x}, \bar{u}), \\ C\theta^{[2]}(\bar{x}) + D\lambda^{[2]}(\bar{x}, \bar{u}) \\ = -h^{[2]}(T\bar{x}, L\bar{x} + M\bar{u}, \bar{x}, \bar{u}) + \bar{h}^{[2]}(\bar{x}, \bar{u}). \end{aligned}$$

Now suppose that the FBI equations have been solved. The second step is to use additional feedforward and feedback to insure that the closed loop system converges to the tracking manifold $x = \theta(\bar{x})$ where $e = 0$. This can be achieved locally by linear pole placement techniques, but an alternative approach is to use optimal control methods to achieve a solution [56, 57]. Define transverse coordinates z, v, and e by

$$
\begin{aligned}
z &= x - \theta(\bar{x}) = x - T\bar{x} - \theta^{[2]}(\bar{x}) + O(\bar{x})^3,\\
v &= u - \lambda(\bar{x}, \bar{u}) = u - L\bar{x} - M\bar{u} - \lambda^{[2]}(\bar{x}, \bar{u}) + O(\bar{x}, \bar{u})^3,\\
e &= y - \bar{y}.
\end{aligned}
$$

In these coordinates the plant and model are of the form

$$
\begin{aligned}
\dot{z} &= \tilde{f}(z, v, \bar{x}, \bar{u}) = Ax + Bu + \tilde{f}^{[2]}(z, v, \bar{x}, \bar{u}) + O(z, v, \bar{x}, \bar{u})^3,\\
\dot{\bar{x}} &= \bar{f}(\bar{x}, \bar{u}) = \bar{A}\bar{x} + \bar{B}\bar{u} + \bar{f}^{[2]}(\bar{x}, \bar{u}) + O(\bar{x}, \bar{u})^3,\\
e &= \tilde{h}(z, v, \bar{x}, \bar{u}) = Cz + Dv + \tilde{h}^{[2]}(z, v, \bar{x}, \bar{u}) + O(z, v, \bar{x}, \bar{u})^3,
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{f}(z, v, \bar{x}, \bar{u}) &= f(z + \theta(\bar{x}, \bar{u}), v + \beta(\bar{x}, \bar{u}), \bar{x}, \bar{u}) - f(\theta(\bar{x}, \bar{u}), \beta(\bar{x}, \bar{u}), \bar{x}, \bar{u}),\\
\tilde{h}(z, v, \bar{x}, \bar{u}) &= h(z + \theta(\bar{x}, \bar{u}), v + \beta(\bar{x}, \bar{u}), \bar{x}, \bar{u}) - h(\theta(\bar{x}, \bar{u}), \beta(\bar{x}, \bar{u}), \bar{x}, \bar{u}).
\end{aligned}
$$

A stabilizing feedback can be found by minimizing

$$
\frac{1}{2}\int_0^{\infty} \|e\|^2 + \|v\|^2 dt.
$$

Let $\pi(z, \bar{x}, \bar{u})$ denote the optimal cost and $\kappa(z, \bar{x}, \bar{u})$ the optimal feedback where

$$
\begin{aligned}
\pi(z, \bar{x}, \bar{u}) &= \frac{1}{2}z'Pz + \pi^{[3]}(z, \bar{x}, \bar{u}) + O(z, \bar{x}, \bar{u})^4,\\
\kappa(z, \bar{x}, \bar{u}) &= Kz + \kappa^{[2]}(z, \bar{x}, \bar{u}) + O(z, \bar{x}, \bar{u})^3.
\end{aligned}
$$

Then π, κ satisfy the Hamilton-Jacobi-Bellman (HJB) PDE

$$
\begin{aligned}
0 &= \frac{\partial \pi}{\partial z}(z, \bar{x}, \bar{u})\tilde{f}(z, \kappa(z, \bar{x}, \bar{u}), \bar{x}, \bar{u}) + \frac{\partial \pi}{\partial \bar{x}}(z, \bar{x}, \bar{u})\bar{f}(\bar{x}, \bar{u}) + l(z, \kappa(z, \bar{x}, \bar{u}), \bar{x}, \bar{u}),\\
0 &= \frac{\partial \pi}{\partial z}(z, \bar{x}, \bar{u})\frac{\partial \tilde{f}}{\partial v}(z, \kappa(z, \bar{x}, \bar{u}), \bar{x}, \bar{u}) + \frac{\partial l}{\partial v}(z, \kappa(z, \bar{x}, \bar{u}), \bar{x}, \bar{u}),
\end{aligned}
$$

where

$$
\begin{aligned}
l(z, v, \bar{x}, \bar{u}) &= \frac{1}{2}(\|e\|^2 + \|v\|^2)\\
&= \frac{1}{2}\left(z'Qz + 2z'Sv + v'Rv\right) + l^{[3]}(z, v, \bar{x}, \bar{u}) + O(z, v, \bar{x}, \bar{u})^4.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\tilde{f}(z, v, \bar{x}, \bar{u}) &= O(z, v),\\
\tilde{h}(z, v, \bar{x}, \bar{u}) &= O(z, v),\\
l(z, v, \bar{x}, \bar{u}) &= O(z, v)^2,
\end{aligned}
$$

so we expect that

$$\pi(z,\bar{x},\bar{u}) = O(z,v)^2,$$
$$\kappa(z,\bar{x},\bar{u}) = O(z,v).$$

In particular,

$$\pi(z,\bar{x},\bar{u}) = \frac{1}{2}z'Pz + \pi^{[3]}(z,\bar{x},\bar{u}) + O(z,\bar{x},\bar{u})^4,$$
$$\kappa(z,\bar{x},\bar{u}) = Kz + \kappa^{[2]}(z,\bar{x},\bar{u}) + O(z,\bar{x},\bar{u})^3.$$

The lowest-degree terms in the HJB equations are the familiar Riccati equation and the formula for the optimal linear feedback

$$0 = A'P + PA + Q - (PB+S)R^{-1}(PB+S)',$$
$$K = -R^{-1}(PB+S)'.$$

At each higher-degree $d > 1$, the equations are linear in the unknowns $\pi^{[d+1]}$, $\kappa^{[d]}$ and depend on the lower-order terms of the solution. They are solvable if the linear part of the plant is stabilizable and the model is at least neutrally stable. For example, to find the next terms $\pi^{[3]}(z,\bar{x},\bar{u})$, $\kappa^{[2]}(z,\bar{x},\bar{u})$ one plugs the first two terms of π, κ into HJB equations and collect the next terms (degree 3 from the first HJB equation and degree 2 from the second HJB equation)

$$0 = \frac{\partial \pi^{[3]}}{\partial z}(z,\bar{x},\bar{u})(A+BK)z + \frac{\partial \pi^{[3]}}{\partial \bar{x}}(z,\bar{x},\bar{u})(\bar{A}\bar{x}+\bar{B}\bar{u})$$
$$+ z'P\tilde{f}^{[2]}(z,Kz,\bar{x},\bar{u}) + l^{[3]}(z,Kz,\bar{x},\bar{u}),$$
$$0 = \frac{\partial \pi^{[3]}}{\partial z}(z,\bar{x},\bar{u})B + z'P\frac{\partial \tilde{f}^{[2]}}{\partial v}(z,Kz,\bar{x},\bar{u})$$
$$+ \kappa^{[2]}(z,\bar{x},\bar{u})'R + \frac{\partial l^{[3]}}{\partial v}(z,Kz,\bar{x},\bar{u}).$$

Notice that the first equation involves only $\pi^{[3]}$, the other unknown $\kappa^{[2]}$ does not appear. This equation is solvable if $A+BK$ is asymptotically stable and $\bar{A}$ is at least neutrally stable. Given the solution $\pi^{[3]}$ then we can solve the second equation for $\kappa^{[2]}$

$$\kappa^{[2]}(z,\bar{x},\bar{u}) = -R^{-1}\left(\frac{\partial \pi^{[3]}}{\partial z}(z,\bar{x},\bar{u})B + z'P\frac{\partial \tilde{f}^{[2]}}{\partial v}(z,Kz,\bar{x},\bar{u}) + \frac{\partial l^{[3]}}{\partial v}(z,Kz,\bar{x},\bar{u})\right)'.$$

The higher-degree terms are found in a similar fashion.

Given the solutions of the FBI and HJB equations, the desired feedforward/feedback for model matching is

$$u = \alpha(x,\bar{x},\bar{u}).$$
$$= \kappa(x-\theta(\bar{x}),\bar{x},\bar{u}) + \lambda(\bar{x},\bar{u}).$$

Krener has written a MATLAB toolbox to compute term by term solutions of the FBI and HJB equations. The Nonlinear Systems Toolbox is available from

`http://scad.utdallas.edu/scad/software`

The routines `fbi.m` and `hjb.m` sequentially compute the least square solutions of the FBI and HJB equations to arbitrary degree. The routine `mdl_mtch.m` uses these to solve the model matching problem to arbitrary degree.

As we mentioned above the model matching problem is a generalization of the feedback linearization problem. In the latter, we are given a system

$$\dot{x} = f(x, u) = Ax + Bu + O(x, u)^2$$

and we seek a change-of-state coordinates and state feedback

$$\begin{aligned} x &= \theta(\bar{x}), \\ u &= \lambda(\bar{x}, \bar{u}), \end{aligned}$$

which transforms it into a linear system

$$\dot{\bar{x}} = \bar{f}(\bar{x}, \bar{u}) = A\bar{x} + B\bar{u}.$$

But then θ, λ satisfy the first of the FBI equations

$$f(\theta(\bar{x}), \lambda(\bar{x}, \bar{u})) = \frac{\partial \theta}{\partial \bar{x}}(\bar{x})\bar{f}(\bar{x}, \bar{u}).$$

The model matching problem also generalizes the input-output linearization problem. Suppose we are given a scalar input, scalar output nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ y &= h(x)u, \end{aligned}$$

with well-defined relative degree $r > 0$. Let $\bar{A}, \bar{B}, \bar{C}$ be the $r \times r$, $r \times 1$, and $1 \times r$ matrices

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Define the model to be

$$\begin{aligned} \dot{\bar{x}} &= \bar{f}(\bar{x}, \bar{u}) = \bar{A}\bar{x} + \bar{B}\bar{u}, \\ \bar{y} &= \bar{h}(\bar{x}, \bar{u}) = \bar{C}\bar{x}. \end{aligned}$$

Then the θ, λ that satisfy the FBI equations are a partial change of coordinates $\bar{x} = \xi$ and feedback which transform part of the system into a string of integrators and makes the rest unobservable from the output.

The regulation or servomechanism problem [34] is a particular case of the model matching problem when there is no input to the model, $\bar{u}$ is absent. And the model matching problem can be viewed as a regulation problem by viewing $\bar{u}$ as additional states of the model with trivial dynamics, $\dot{\bar{u}} = 0$.

In the disturbance rejection problem, the model system is a source of disturbances that affect the plant. These disturbances can be static $\bar{u}$ or dynamic $\bar{x}$. The output of the model is taken as $\bar{y} = 0$ so that goal of model matching is to keep the output of the plant at zero in the presence of disturbances. We are of course assuming that the dynamic nature of the disturbances is known and that they are measurable. If this were not the case, then a H_∞ formulation of the problem would be more appropriate.

Finally, the gain scheduling problem is to design a parametric family of linear controllers for a parameter dependent family of linear systems. This can also be viewed as model matching problem where $\bar{x}, \bar{u}$ are dynamic and static parameters. The advantage of the latter point of view is that the control varies continuously with the parameter rather than discretely as is usually the case.

3.10 Backstepping

Recently backstepping [60] has become a popular technique for designing stabilizing control laws for nonlinear systems in strict feedback form,

$$\begin{aligned}
\dot{x}_1 &= x_2 + f_1(x_1) \\
&\vdots \\
\dot{x}_i &= x_{i+1} + f_i(x_1, \dots, x_i) \\
&\vdots \\
\dot{x}_n &= u + f_n(x_1, \dots, x_n).
\end{aligned}$$

Such systems are feedback linearizable by choosing the pseudo-output to be $h(x) = x_1$.

The first step in backstepping is to use x_2 as a pseudo-control for x_1. We seek a relation $x_2 = \alpha_1(x_1)$ to stabilize x_1. Define new partial coordinates

$$\begin{aligned}
z_1 &= x_1, \\
z_2 &= x_2 - \alpha_1(z_1),
\end{aligned}$$

and a partial Lyapunov function

$$V_1 = \frac{1}{2} z_1^2;$$

then

$$\begin{aligned}
\dot{z}_1 &= z_2 + \alpha_1(z_1) + f_1(z_1), \\
\dot{V}_1 &= z_1 \left(z_2 + \alpha_1(z_1) + f_1(x_1)\right).
\end{aligned}$$

If we set

$$\alpha_1(z_1) = -z_1 - f_1(z_1),$$

then

$$\begin{aligned}
\dot{V}_1 &= -z_1^2 + z_1 z_2, \\
\dot{z}_1 &= -z_1 + z_2, \\
\dot{z}_2 &= x_3 + \tilde{f}_2(z_1, z_2), \\
\tilde{f}_2(z_1, z_2) &= f_2(z_1, z_2) - \frac{d}{dt}\alpha_1(z_1).
\end{aligned}$$

Next we seek a relation $x_3 = \alpha_2(z_1, z_2)$ to stabilize z_2. We define a new coordinate and a new partial Lyapunov function

$$\begin{aligned}
z_3 &= x_3 - \alpha_2(z_1, z_2), \\
V_2 &= V_1 + \frac{1}{2} z_2^2.
\end{aligned}$$

Then

$$\begin{aligned}
\dot{z}_2 &= z_3 + \alpha_2(z_1, z_2) + \tilde{f}_2(z_1, z_2), \\
\dot{V}_2 &= -z_1^2 + z_2 \left(z_1 + z_3 + \alpha_2(z_1, z_2) + \tilde{f}_2(z_1, z_2) \right).
\end{aligned}$$

If

$$\alpha_2(z_1, z_2) = -z_1 - z_2 - \tilde{f}_2(z_1, z_2),$$

then

$$\begin{aligned}
\dot{V}_2 &= -z_1^2 - z_2^2 + z_2 z_3, \\
\dot{z}_1 &= -z_1 + z_2, \\
\dot{z}_2 &= -z_1 - z_2 + z_3.
\end{aligned}$$

Assume that by proper choice of α_{i-1},

$$\begin{aligned}
\dot{z}_{i-1} &= -z_{i-2} - z_{i-1} + z_i, \\
V_{i-1} &= \frac{z_1^2}{2} + \cdots + \frac{z_{i-1}^2}{2}, \\
\dot{V}_{i-1} &= -z_1^2 - \cdots - z_{i-1}^2 + z_{i-1} z_i.
\end{aligned}$$

Define

$$\begin{aligned}
z_{i+1} &= x_{i+1} - \alpha_i(z_1, \dots, z_i), \\
V_i &= \frac{z_1^2}{2} + \dots + \frac{z_i^2}{2}.
\end{aligned}$$

Then for some $\tilde{f}_i(z_1, \dots, z_i)$

$$\begin{aligned}
\dot{z}_i &= z_{i+1} + \alpha_i(z_1, \dots, z_i) + \tilde{f}_i(z_1, \dots, z_i) \\
\dot{V}_i &= -z_1^2 - \cdots - z_{i-1}^2 + z_{i-1} z_i + z_i \left(z_{i+1} + \alpha_i(z_1, \dots, z_i) + \tilde{f}_i(z_1, \dots, z_i) \right)
\end{aligned}$$

If

$$\alpha_i(z_1, \ldots, z_i) = -z_{i-1} - z_i - \tilde{f}_i(z_1, \ldots, z_i),$$

then

$$\dot{z}_i = -z_{i-1} - z_i + z_{i+1},$$
$$\dot{V}_i = -z_1^2 - \ldots - z_i^2 + z_i z_{i+1}.$$

At the last step

$$\dot{z_n} = \tilde{f}_n(z_1, \ldots, z_n) + u.$$

Define

$$\begin{aligned} u &= \alpha_n(z_1, \ldots, z_i) \\ &= -z_{n-1} - z_n - \tilde{f}_n(z_1, \ldots, z_n), \\ V_n &= \frac{z_1^2}{2} + \cdots + \frac{z_n^2}{2}. \end{aligned}$$

Then

$$\dot{z_n} = -z_{n-1} - z_n,$$
$$\dot{V}_i = -z_1^2 - \cdots - z_n^2.$$

So asymptotic stability is assured.

Backstepping is feedback linearization on the Lyapunov function; it cancels the nonlinearities in $\dot{V}$. One advantage of backstepping over feedback linearization is that one does not need to cancel benign nonlinearities. For example, consider the system

$$\dot{x}_1 = x_2 - x_1^3.$$
$$\dot{x}_2 = u.$$

If we proceed as above and define

$$\begin{aligned} z_1 &= x_1, \\ z_2 &= x_2 - \alpha_1(z_1), \\ V_1 &= \frac{z_1^2}{2}, \end{aligned}$$

then

$$\dot{V}_1 = z_1(z_2 + \alpha_1(z_1)) - z_1^4.$$

We can take $\alpha_1(z_1) = 0$ because the $-z_1^4$ term partially stabilizes V_1. We define

$$\begin{aligned} V_2 &= \frac{z_1^2}{2} + \frac{z_2^2}{2}, \\ u &= \alpha_2(z_1, z_2) = -z_1 - z_2. \end{aligned}$$

Then

$$\dot{V}_2 = -z_1^4 - z_2^2.$$

Another advantage of backstepping is that because stability is assured by a Lyapunov argument, it can be done robustly and adaptively [60].

3.11 Feedback Linearization and System Inversion

Feedback linearization is sometimes called dynamic inversion [20] for the following reason. To feedback linearize a scalar input system

$$\dot{x} = f(x) + g(x)u,$$

one seeks an output map and feedback

$$y = h(x),$$
$$v = \alpha(x) + \beta(x)u,$$

that transforms the system into a string of integrators

$$\frac{d^n y}{dt^n} = v.$$

If the desired value of the output $y(t)$ is known and it is smooth then the computation of the input $u(t)$ is straightforward,

$$u(t) = \frac{\frac{d^n y}{dt^n}(t) - \alpha(x(t))}{\beta(x(t))}.$$

For systems with m inputs one seek m outputs and a feedback

$$y_i = h_i(x), \qquad i = 1, \ldots, m,$$
$$v = \alpha(x) + \beta(x)u,$$

that transforms the system into m strings of integrators

$$\frac{d^{k_i} y_i}{dt^{k_i}} = v_i, \qquad i = 1, \ldots, m.$$

where $\{k_1, \ldots, k_m\}$ are the Kronecker or controllability indices. Once again if the desired output $y(t)$ is known then the desired input is

$$u(t) = \beta^{-1}(x(t)) \begin{bmatrix} \frac{d^{k_1} y}{dt^{k_1}}(t) - \alpha_{k_1}(x(t)) \\ \vdots \\ \frac{d^{k_m} y}{dt^{k_m}}(t) - \alpha_{k_m}(x(t)) \end{bmatrix}.$$

When the output function h is specified a priori then the problem is one of input-output linearization. The system must have well-defined relative degree to be invertible, and the zero dynamics must be stable. This problem was solved by Hirschorn [27, 28].

Another form of system inversion is possible for differentially flat systems [21]. A system

$$\dot{x} = f(x, u)$$

with m-dimensional input u is differentially flat if there exists an m-dimensional output y of the form

$$y = h(x, u, \dot{u}, \ldots, u^{(p)})$$

such that

$$x = \phi(y, \dot{y}, \ldots, y^{(q)}),$$
$$u = \alpha(y, \dot{y}, \ldots, y^{(q)}).$$

The class of differentially flat systems is slightly larger than the class of feedback linearizable systems [62]. Given a smooth desired output for such a system, it is straightforward to find the desired state trajectory and input.

All of the above approaches to nonlinear inversion are modeled after linear inversion techniques. Roger Brockett, in his characteristically creative fashion, has suggested another approach that depends on a decidedly nonlinear technique. In [12], he considered the system

$$\dot{x} = f^1(x)u_1 + f^2(x)u_2$$

where

$$f^1(x) = \begin{bmatrix} 1 \\ 0 \\ -x_2 \end{bmatrix}, \qquad f^2(x) = \begin{bmatrix} 0 \\ 1 \\ x_1 \end{bmatrix}.$$

Clearly this system is nonlinear and its first-order linear approximation around any x is not controllable because $n = 3$, $m = 2$, and the drift is zero. The system is nonlinearly controllable because

$$\left[f^1(x), f^2(x)\right] = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

All the higher Lie brackets are zero. This is a free-nilpotent system in the following sense. All the brackets up to degree 2 are linearly independent (free), and all brackets of degree 3 or higher are zero (nilpotent). By the theorem quoted above [46] it closely approximates any three-dimensional system with two inputs and no drift. Given any other smooth initialized system

$$\dot{z} = g^1(z)u_1 + g^1(z)u_2,$$
$$z(0) = z^0,$$

there exists a smooth mapping $z = \phi(x)$ such that $\phi(x^0) = z^0$ and ϕ preserves trajectories to degree 2 for every bounded control $u(t)$

$$\phi(x(t)) = z(t) + O(t^3)$$

because $f^1(x^0)$, $f^2(x^0)$, and $\left[f^1, f^2\right](x^0)$ are linearly independent for any x^0.

Consider the problem of steering the system from $x^0 = 0$ to any point $x(t_1) = x^1$ while minimizing

$$\int_0^{t_1} u_1^2(t) + u_2^2(t)\, dt.$$

This is a problem in sub-Riemannian geometry also known as singular or pseudo Riemannian geometry. Loosely speaking, the metric tensor is the inverse of the singular matrix

$$\begin{bmatrix} 1 & 0 & -x_2 \\ 0 & 1 & x_1 \\ -x_2 & x_1 & x_1^2 + x_2^2 \end{bmatrix}$$

In other words we are allowed to travel only in the directions in the span of the vector fields f^1, f^2, which are orthonormal vector fields. Travel in the third direction $\left[f^1, f^2\right]$ is accomplished by alternately going out in the directions f^1, f^2, $-f^1$, $-f^2$ [12]. Brockett [12, 13, 15] has shown that the geometry of such systems is rich and interesting.

We turn now to the problem of inverting such a system [14]. Suppose we are given a desired trajectory $\bar{x}(t)$. What input $u(t)$ should we use so that $|x(t) - \bar{x}(t)|$ is small for $t \in [0, T]$? We assume that $x(0) = \bar{x}(0) = 0$ and that $\bar{x}(t)$ is a low-frequency signal. Brockett's ingenuous solution is to split the control $u(t)$ into low- and high-frequency components. The low-frequency component of $u_j(t)$ is used to track $x_j(t)$ for $j = 1, 2$, and the high-frequency components are used to track $x_3(t)$. Brockett [14] considered a low-pass version of the above system; we will present a different analysis for the system itself. First consider the case where $\bar{x}_1(t) = \bar{x}_2(t) = 0$. Since $\bar{x}_3(t)$ is a low-frequency signal, we can assume that there exists a constant K such that

$$\left|\frac{d^j \bar{x}}{dt^j}(t)\right| \le K, \qquad \text{for } j = 0, 1, 2, 3.$$

First consider the case where $\bar{x}_1(t) = \bar{x}_2(t) = 0$. Let

$$v(t) = \frac{d\bar{x}_3}{dt}(t),$$
$$u_1(t) = \omega^{\frac{1}{2}} v(t) \cos \omega t,$$
$$u_2(t) = \omega^{\frac{1}{2}} \sin \omega t.$$

Then

$$x_1(t) = \frac{v(t)}{\omega^{\frac{1}{2}}} \sin \omega t + \frac{dv}{dt}(t)\frac{\cos \omega t}{\omega^{\frac{3}{2}}} - \int_0^t \frac{d^2 v}{dt^2}(\tau)\frac{\cos \omega \tau}{\omega^{\frac{3}{2}}}\, d\tau.$$

$$x_2(t) = \frac{1 - \cos \omega t}{\omega^{\frac{1}{2}}},$$

$$\dot{x}_3(t) = v(t) - v(t) \cos \omega t + \frac{\sin \omega t}{\omega} \left(\frac{dv(\tau)}{dt} \cos \omega t - \int_0^t \frac{d^2 v(\tau)}{dt^2} \frac{\cos \omega \tau}{\omega^2}\, d\tau \right),$$

so

$$x_1(t) = O\left(\frac{K}{\omega^{\frac{1}{2}}}\right) + O\left(\frac{KT}{\omega^{\frac{3}{2}}}\right),$$

$$x_2(t) = O\left(\frac{K}{\omega^{\frac{1}{2}}}\right),$$

$$x_3(t) = \bar{x}_3(t) + O\left(\frac{K}{\omega^{\frac{1}{2}}}\right) + O\left(\frac{KT}{\omega}\right) + O\left(\frac{KT^2}{\omega}\right).$$

By choosing ω sufficiently large we can insure that $|x(t) - \bar{x}(t)|$ is small.

For general $\bar{x}(t)$ let

$$v(t) = \frac{d\bar{x}_3}{dt}(t) - \frac{d\bar{x}_2}{dt}(t)\bar{x}_1(t) + \frac{d\bar{x}_1(t)}{dt}\bar{x}_2(t),$$

$$u_1(t) = \frac{d\bar{x}_1}{dt}(t) + \omega^{\frac{1}{2}} v(t) \cos \omega t,$$

$$u_2(t) = \frac{d\bar{x}_2}{dt}(t) + \omega^{\frac{1}{2}} \sin \omega t.$$

Then

$$x_1(t) = \bar{x}_1(t) + O\left(\frac{K}{\omega^{\frac{1}{2}}}\right) + O\left(\frac{KT}{\omega^{\frac{1}{2}}}\right),$$

$$x_2(t) = \bar{x}_2(t) + O\left(\frac{K}{\omega^{\frac{1}{2}}}\right),$$

$$x_3(t) = \bar{x}_3(t) + O\left(\frac{K}{\omega^{\frac{1}{2}}}\right) + O\left(\frac{K^2}{\omega^{\frac{1}{2}}}\right) + O\left(\frac{KT}{\omega^{\frac{1}{2}}}\right) + O\left(\frac{K^2 T}{\omega^{\frac{1}{2}}}\right) + O\left(\frac{K^2 T^2}{\omega^{\frac{3}{2}}}\right).$$

Again by choosing ω sufficiently large we can insure that $|x(t) - \bar{x}(t)|$ is small.

3.12 Conclusion

R. W. Brockett originated the concept of feedback linearization in 1978. Over the past two decades this concept and its intellectual descendants has stimulated a very large part of the research work in nonlinear systems theory.

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4

On the

Christoph

ABSTRACT
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4.1 Introd

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