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Fig. 1. Block diagram for nonlinear H_∞ formulation.

H_∞ Tracking Control for a Class of Nonlinear Systems

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Abstract—This paper develops the theory for tracking control using the nonlinear H_∞ control design methodology for a class of nonlinear input affine systems. The authors use a two-step process of first designing the feedforward part of the controller to design for perfect trajectory following and then design the feedback part of the controller using nonlinear H_∞ regulator theory. Results for infinite-time and finite-time horizons are presented.

Index Terms—Control, feedback, nonlinear, robust, tracking.

I. INTRODUCTION

This paper deals with designing tracking control systems formulated in the nonlinear H_∞ setting. The objective in the regulator problem is to drive an unwanted error signal to zero. The objective in the tracking problem is to get a plant output to track a given

model signal. Thus the regulator problem can be seen as the special case of tracking where the signal to be tracked is zero. Conversely, (as shown in [1, Ch. 4]), the tracking problem can be reduced to a regulator problem. Previous work on the tracking problem demanded only that the tracking error tend to zero asymptotically as time tends to infinity [2], [3] or to choose a feedback to minimize a tracking error norm integrated over all time [1], [4], [5]. Our purpose here is to use an H_∞-optimality criterion for judging tracking performance (minimize the worst case tracking error norm over admissible ball of disturbances), by reducing this H_∞-tracking problem to an H_∞-regulator problem, much in the same spirit as was done in [4] and [5] in the non-H_∞ setting.

II. BACKGROUND (NONLINEAR H_∞ CONTROL)

Consider the system

$$\dot{x} = a(x) + b(x)u + g(x)d_1, \qquad a(0) = 0$$
$$y = c(x) + d_2, \qquad c(0) = 0$$
$$z = \begin{bmatrix} h(x) \\ u \end{bmatrix}, \qquad h(0) = 0 \tag{1}$$

where $x = (x_1, \cdots, x_n)$ are local coordinates for a C^∞ state-space manifold $M, u \in R^m$ are the control inputs, $d_1 \in R^r$ and $d_2 \in R^p$ the exogenous inputs consisting of reference and/or disturbance signals, $y \in R^p$ the measured outputs, and $z \in R^s$ outputs to be controlled. System (1) is identified by G. For a full-state measurement case $y = x$. The controller is identified by K. The closed-loop system in Fig. 1 will be denoted by $\Omega(G/K)$.

Definition 1: The closed-loop system $\Omega(G/K)$ is said to have L_2 gain less than or equal to γ for some $\gamma > 0$ if

$$\int_0^T \|z(t)\|^2 \, dt \le \gamma^2 \int_0^T \|w(t)\|^2 \, dt + b(x_0) \tag{2}$$

$\forall T > 0$ and $w(t) \in L_2[0, T]$, where $b(x_0)$ is a positive constant depending on initial condition x_0.

State Feedback H_∞ Control Problem: Find a state feedback controller K: $u = u(x)$ if any, such that the closed-loop system $\Omega(G/K)$ is asymptotically stable and has L_2-gain $\le \gamma$.

Solution [2], [3], [6]–[8]: If there exists a smooth function $V(x) \ge 0$ which satisfies the Hamilton–Jacobi (HJ) inequalities

$$V_x(x)a(x) + \frac{1}{2}V_x(x)\left[\frac{1}{\gamma^2}g(x)g^T(x) - b(x)\right]V_x^T(x)$$
$$+ \frac{1}{2}h^T(x)h(x) \le 0, \qquad V(0) = 0 \tag{3}$$

and we set

$$u_* = -b^T(x)V_x(x) \tag{4}$$

then the closed-loop system $\Omega(G/u_*)$ has gain at most γ. Moreover, if $V(x)$ has a strict local minimum at $x = 0$ and the system

$$\dot{x} = a(x)$$
$$z = \begin{bmatrix} h(x) \\ -b^T(x)V_x^T(x) \end{bmatrix} \tag{5}$$

is zero-state detectable (i.e., $\dot{x} = a(x)$ and $z(xx(t)) \equiv 0$ for $t \geq 0 \Rightarrow \lim_{t\to\infty} x(t) = 0$), then $x = 0$ is a locally asymptotically stable equilibrium of

$$\dot{x} = a(x) - b(x)b^T(x)V_x^T(x). \tag{6}$$

If additionally V has a global strict minimum at $x = 0$ and V is proper (so the inverse image of a compact set under V is again compact), then $x = 0$ is a globally asymptotically stable equilibrium of (6).

For the finite-time horizon problem, where final time T is finite, the solution is given by $u = -b^T(x)V_x(t, x)$, where $V(t, x) \geq 0$ satisfies the following HJ equation:

$$V_t(t, x) + V_x(x)a(x)$$
$$+ \frac{1}{2}V_x(x)\left[\frac{1}{\gamma^2}g(x)g^T(x) - b(x)b^T(x)\right]V_x^T(x)$$
$$+ \frac{1}{2}h^T(x)h(x) = 0, \qquad V(T, x) = V_f(x). \tag{7}$$

The solution for the finite-time can be derived from a min–max differential game perspective [9].

Measurement Feedback H_∞ Control Problem: Find a dynamic feedback controller

$$K: \begin{cases} \dot{\eta} = \kappa(\eta) + \ell(\eta)y \\ u = m(\eta) \end{cases} \tag{8}$$

so that the closed-loop system $\Omega(G/K)$ is asymptotically stable and has L_2-gain $\leq \gamma$.

Solution [2], [3], [6], [8], [10]: A necessary condition for the existence of solutions for which the closed-loop system has a smooth storage function is that there exists a solution $V(x) \geq 0$ of (3) as well as a solution $R(x) \geq 0$ of

$$R_x(x)a(x) + \frac{1}{2\gamma^2}R_x(x)g(x)g^T(x)R_x^T(x) + \frac{1}{2}h^T(x)h(x)$$
$$= \frac{1}{2}\gamma^2 c^T(x)c(x) \leq 0, \qquad R(0) = 0 \tag{9}$$

such that $V(x) \leq R(x)$ for all x.

Conversely, conditions (3) and (9) are sufficient to solve the measurement feedback problem, at least locally. A more complicated version of (9) involving an "information-state" in combination with (3) leads to compensators which solve the problem. However, these compensators are in general infinite-dimensional. This is an ongoing area of research which is beyond the scope of this paper.

III. TRACKING CONTROL

There has been some related work in tracking systems for the linear systems with quadratic performance index (see [1] and references therein). In the approach taken in this paper for tracking control, we use a two-step method for both state feedback as well as measurement feedback problems, as described below. The reference signal to be tracked is given by the output of a known plant, identified by R

$$\dot{x}_m = A(x_m)$$
$$y_d = C(x_m) \tag{10}$$

for $x_m(0)$ initial state. System (10) is completely observable.

A. State-Feedback Problem (Infinite-Horizon)

Step 1: Find a feedforward $u_* = u_*(x, x_m)$ so that:
1) the equilibrium $x = 0$ of

$$\dot{x} = a(x) + b(x)u_*(x, 0) \tag{11}$$

   is exponentially stable;
2) there exists a neighborhood $U \subset X \times X_m$ of $(0, 0)$ such that for each initial condition $(x(0), x_m(0)) \in U$, the solution $(x(t), x_m(t))$ of

$$\dot{x} = a(x) + b(x)u_*(x, x_m)$$
$$\dot{x}_m = A(x_m) \tag{12}$$

   satisfies

$$\lim_{t\to\infty} \{h(\theta(x_m(t))) - (x_m(t))\} = 0. \tag{13}$$

To solve Step 1, following [11], we seek a feedforward $u_* = U_*(x, x_m)$ and an invariant submanifold $x = \theta(x_m)$ of the combined system (12), so that under the closed-loop dynamics the submanifold $x = \theta(x_m)$ is invariant, and the mismatch error

$$e = h(x(t)) - C(x_m(t)) \tag{14}$$

is identically zero on this submanifold. This leads to the Francis–Byrnes–Isidori (FBI) equations [4], [5], [11], [12] for the function $\theta(x_m)$ and $U_*(\theta(x_m), x_m) = \bar{u}_*(x_m)$

$$\frac{\partial\theta}{\partial x_m}(x_m)A(x_m) = a(\theta(x_m)) + b(\theta(x_m))\bar{u}_*(x_m)$$

$$h(x(\theta(x_m)t)) - C(x_m(t)) = 0. \tag{15}$$

Under appropriate assumptions one can prove that solvability of the FBI equation (15) is necessary and sufficient for the solvability of Step 1.

Assumption 1) is made to ensure that the system is asymptotically stable in the absence of a signal to be tracked [11]. In fact, if $A(0) = o, C(0) = 0$ in the model (10) and if $x = a(x), y = h(x)$ is a detectable state output system, then it is redundant.

Once the feedforward and the invariant manifold have been identified, the next step is to use a feedback law to drive the system to this invariant manifold in an optimal fashion. The new feature in this paper is to use an H_∞ formulation for the optimality criterion for this step.

To formulate this step, we consider the combined system

$$\dot{x} = a(x) + b(x)u + g(x)d_1$$
$$\dot{x}_m = A(x_m) \tag{16}$$

and introduce the change of variable

$$\xi = x - \theta(x_m)$$
$$x_m = x_m$$
$$v = u - \bar{u}_*(x_m)$$
$$d_1 = d_1. \tag{17}$$

The result is

$$\dot{\xi} = F(\xi, x_m) + B(\xi, x_m)v + G(\xi, x_m)d_1$$
$$\dot{x}_m = A(x_m) \tag{18}$$

where

$$F(\xi, x_m) = a(\xi + \theta(x_m)) - \frac{\partial\theta}{\partial x_m}(x_m)A(x_m)$$
$$+ b(\xi + \theta(x_m))\bar{u}_*(x_m)$$
$$B(\xi, x_m) = b(\xi + \theta(x_m))$$
$$G(\xi, x_m) = g(\xi + \theta(x_m)). \tag{19}$$

For the error term, we use

$$\hat{z} = \begin{bmatrix} h(\xi + \theta(x_m)) - C(x_m) \\ v \end{bmatrix}. \quad (20)$$

The second step of our servo problem (state feedback H_∞ formulation) then is as follows.

Step 2: Find a feedback $v_* = v_*(\xi, x_m)$ so that the solution $(\xi(t), x_m(t))$ of (18) should satisfy

$$\int_0^T \|\hat{z}(t)\|^2 \, dt \le \gamma^2 \int_0^T \|d_1(t)\|^2 \, dt + b(\xi(0), x_m(0)) \quad (21)$$

or more explicitly

$$\int_0^T \{\|h(\xi + \theta(x_m)) - C(x_m)\|^2 + \|v\|^2\} \, dt)$$

$$\le \gamma^2 \int_0^T \|d_1(t)\|^2 \, dt + b(\xi(0), x_m(0)) \quad (22)$$

for all disturbances $w(t)$ and initial conditions $(\xi(0), x_m(0))$ sufficiently close to the origin, for all $T < \infty$. Note that if $\xi(0) = 0$ and $w(t) \equiv 0$, then we may take $v(t) \equiv 0$ to attain perfect tracking.

The problem stated here is in the form of a standard state feedback H_∞ problem, so we can simply quote the solution from [7] (for example) applied to the present setting.

Theorem 3.1: Assume that a feedforward $u_* = u_*(x, x_m)$ and an invariant manifold $x = \theta(x_m)$ for the associated closed-loop dynamics has been found as required in Step 1 of the servo problem. Suppose we can find a smooth function $V(\xi, x_m)$ with

$$0 \le V(\xi, x_m) \quad (23)$$

so that

$$V_\xi(\xi, x_m)F(\xi, x_m) + V_{x_m}(\xi, x_m)A(x_m)$$
$$+ \frac{1}{2}V_\xi(\xi, x_m)\left[\frac{1}{\gamma^2}G(\xi, x_m)G^T(\xi, x_m)\right.$$
$$\left. - B(\xi, x_m)B^T(\xi, x_m)\right]V_\xi^T(\xi, x_m)$$
$$+ \frac{1}{2}\|h(\xi + x_m) - C(x_m)\|^2 \le 0. \quad (24)$$

Then the closed-loop system has gain at most γ, i.e., (21) holds over all paths of the system (18). If V is proper with respect to the ξ-variable [i.e., $V(\xi, x_m) \to \infty$ when $\|\xi\| \to \infty$] and if the system (18) is detectable in the sense that $\hat{z}(t) \equiv 0$ for system (18) with $v(t) \equiv 0, w(t) \equiv 0$ implies that $\lim_{t\to\infty} \xi(t) = 0$, then in addition $\lim_{t\to\infty} \xi(t) = 0$ for any initial condition $(\xi(0), x_m(0))$.

Similar results hold for the measurement feedback version of the problem. Again, we are given the plant and the model (16). In this case the full state is not directly available for measurement, but only $y(t)$ is. The solution of this problem is also given in two steps as follows.

B. Measurement Feedback Problem (Infinite-Horizon)

Step 1: Find a controller of the form

$$\dot{x}_k = a_k(x_k) + b_k(x_k)y$$
$$u = c_k(x_k, x_m) \quad (25)$$

so that: 1) the equilibrium $(x, x_k) =)0, 0)$ of

$$\dot{x} = a(x) + b(x)c_k(x_k, x_m)$$
$$\dot{x}_k = a_k(x_k) + b_k(x_k)c(x) \quad (26)$$

is exponentially stable and 2) there exists a neighborhood $U \subset X \times X_k \times X_m$ of $(0, 0, 0)$ such that for each initial condition $(x(0), x_k(0)), x_m(0)) \in U$, the solution $(x(t), x_k(t), x_m(t))$ of

$$\dot{x} = a(x) + b(x)c_k(x_k, x_m)$$
$$\dot{x}_k = a_k(x_k) + b_k(x_k)c(x)$$
$$\dot{x}_m = A(x_m) \quad (27)$$

satisfies

$$\lim_{t\to\infty} \{h(\theta(x(t))) - C(x_m(t))\} = 0. \quad (28)$$

To solve Step 1) of this error-feedback servo problem, we follow [11], [12] and look for a compensator $(a_k(x_k), b_k(x_k), c_k(x_k, x_m))$ and a manifold $\{x = \theta(x_m), x_k = \sigma(x_m)\}$ which is invariant under the closed-loop dynamics

$$\dot{x} = a(x) + b(xc(x_k, x_m)$$
$$\dot{x}_k = a_k(x_k) + b_k(x_k)c(x)$$
$$\dot{x}_m = A(x_m) \quad (29)$$

such that the mismatch error $h(x(t)) - C(x_m(t))$ vanishes identically on this submanifold. This leads to the FBI equations [4], [5], [11], [12]

$$\frac{\partial\theta}{\partial x_m}(x_m)A(x_m) = a(\theta(x_m)) + b(\theta(x_m))\bar{u}_*(x_m)$$
$$\frac{\partial\sigma}{\partial x_m}(x_m)A(x_m) = a_k(\sigma(x_m)) + b_k(\sigma(x_m))y_*(x_m$$
$$h(\theta(x_m)) - C(x_m) = 0 \quad (30)$$

where $\bar{u}_*(x_m) = c_k(\sigma(x_m), x_m)$ and $y_*(x_m) = c(\theta(x_m))$. To formulate a measurement-feedback H_∞ problem for Step 2) of the error-feedback servo problem, we again introduce a change of variables. With implementation of the error feedforward compensator from Step 1) and now allowing the disturbances d_1 and d_2 to be nonzero, we arrive at the composite system

$$\dot{x} = a(x) + b(x)u + g(x)d_1$$
$$\dot{x}_k = a_k(x_k) + b_k(x_k)c(x) + b_k(x_k)d_2$$
$$\dot{x}_m = A(x_m)$$
$$y = c(x) + d_2 \quad (31)$$

with desired error term to be specified later. We impose the change of variables

$$\xi_1 = x - \theta(x_m)$$
$$\xi_2 = x_k - \sigma(x_m)$$
$$x_m = x_m$$
$$v = u - u_*(x_m)$$
$$d_i = d_i, \qquad i = 1, 2. \quad (32)$$

System (31) expressed with this change of variables has the form

$$\begin{aligned}
\dot{\xi}_1 &= F_1(\xi_1, x_m) + B(\xi_1, x_m)v + G_1(\xi_1, x_m)d_1 \\
\dot{\xi}_2 &= F_2(\xi_1, \xi_2, x_m) + G_2(\xi_2, x_m)d_2 \\
\dot{x}_m &= A(x_m) \\
y &= \begin{bmatrix} c(\xi_1, x_m) + d_2 \\ x_m \end{bmatrix}
\end{aligned} \tag{33}$$

where

$$\begin{aligned}
F_1(\xi_1, x_m) &= a(\xi_1 + \theta(x_m)) - \frac{\partial \theta}{\partial x_m}(x_m)A(x_m) \\
&\quad + b(\xi_1 + \theta(x_m))u_*(x_m) \\
F_2(\xi_1, \xi_2, x_m) &= a_k(\xi_2 + \sigma(x_m)) \\
&\quad + b_k(\xi_2 + \sigma(x_m))c(\xi_1 + \sigma(x_m)) \\
&\quad - \frac{\partial \sigma}{\partial x_m}(x_m)A(x_m) \\
B(\xi_1, x_m) &= b(\xi_1 + \theta(x_m)), c(\xi_1, x_m) = c(\xi_1 + \theta(x_m)) \\
G_1(\xi_1, x_m) &= g(\xi_1 + \theta(x_m)) \\
G_2(\xi_2, x_m) &= b_k(\xi_2 + \sigma(x_m)).
\end{aligned} \tag{34}$$

Step 2: Construct a dynamic compensator of the form (25) so that the closed-loop system (25) and (33) is exponentially stable and has L_2 gain $\leq \gamma$, i.e.,

$$\begin{aligned}
\int_0^T &\{ \|h(\xi_1(t) + \theta(x_m(t))) - C(x_m)\|^2 + \|v(t)\|^2 \, dt \\
&\leq \gamma^2 \int_0^T (\|d_1(t)\|^2 \\
&\quad + \|d_2(t)\|^2) \, dt + [b(\xi_1(0), \xi_2(0), x_m(0))]^2
\end{aligned} \tag{35}$$

for all disturbances d_1, d_2 and initial conditions $(\xi_1(0), \xi_1(0), x_m(0))$ sufficiently close to the origin, for all $T < \infty$. This fits in the framework of an H_∞-measurement feedback problem. The results listed in Section II do not apply directly since the noise term d_2 affects only one component of the measurement y in (34). Nevertheless, application of the results of [7], [8] for general nonlinear systems leads to the existence of a solution $V(\xi_1, \xi_2, x_m) \geq 0$ and $R(\xi_1, \xi_2, x_m) \geq 0$ of the following two HJ inequalities such that $V(x) \leq R(x)$ for all x as a necessary condition for existence of a solution of the measurement H_∞-control problem

$$\begin{aligned}
V_{\xi_1}(x)&F_1(\xi_1, x_m) + V_{\xi_2}(x)F_2(x) + V_{x_m}(x)A(x_m) \\
&+ \frac{1}{2}V_{\xi_1}(x)\left[\frac{1}{\gamma^2}G_1(\xi_1, x_m)G_1^T(\xi_1, x_m)\right. \\
&\quad \left. - B(\xi_1, x_m)B^T(\xi_1, x_m)\right]V_{\xi_1}^T(x) \\
&+ \frac{1}{2\gamma^2}V_{\xi_2}(x)G_2(\xi_2, x_m)G_2^T(\xi_2, x_m)V_{\xi_2}^T(x) \\
&+ \frac{1}{2}\|h(\xi_1 + \theta(x_m)) - C(x_m)\|^2 \leq 0, \qquad V(0,0,0) = 0
\end{aligned} \tag{36}$$

$$\begin{aligned}
R_{\xi_1}(\xi_1, \xi_2, 0)&F_1(\xi_1, 0) + R_{x_m}(\xi_1, \xi_2, 0)A(0) \\
&+ R_{\xi_2}(\xi_1, \xi_2, 0)[F_2(\xi_1, \xi_2, 0) - G_2(\xi_2, 0)c(\xi_1, 0)] \\
&+ \frac{1}{2\gamma^2}R_{\xi_1}(\xi_1, \xi_2, 0)G_1(\xi_1 0)G_1^T(\xi_1, 0)R_{\xi_1}^T(\xi_1, \xi_2, 0) \\
&- \frac{1}{2}\gamma^2\|c(\xi_1, 0)\|^2 \\
&+ \frac{1}{2}\|h(\xi_1, +\theta(0)) - C(0)\|^2 \leq 0, \qquad R(0,0,0) = 0.
\end{aligned} \tag{37}$$

C. Finite-Horizon Cases

For the finite-horizon cases of the above, two stability is not an issue. The solution criteria are the same except that we need a solution $V(t, \xi, x_m)$ to the time-varying HJ inequalities

$$\begin{aligned}
V_t(t, \xi, x_m) &+ V_\xi(t, \xi, x_m)F(t, \xi, x_m) \\
&+ V_{x_m})(t, \xi, x_m)A(x_m) \\
&+ \frac{1}{2}V_\xi(t, \xi, x_m)\left[\frac{1}{\gamma^2}G_1(t, \xi, x_m)G_1^T(t, \xi, x_m)\right. \\
&\quad \left. - B(t, \xi, x_m)B^T(t, \xi, x_m)\right]V_\xi^T(t, \xi, x_m) \\
&\frac{1}{2}\|h(\xi + \theta(x_m)) - C(x_m)\|^2 \leq 0
\end{aligned} \tag{38}$$

subject to:
1) $0 \leq V(t, \xi, x_m), V(t, \xi, x_m) \leq b(\xi, x_m)$;
2) $V(t, 0, 0) = 0$;
3) $V(T, \xi, x_m) = V_f(\xi)$;

where V_f is the final cost function for the state feedback case.

The time-varying HJ equations for the time-varying measurement feedback problem are the same as those for the time-invariant case with similar modifications.

IV. CONCLUSIONS

The solution for the nonlinear tracking problem formulated in an H_∞ setting has been presented. This paper showed the relationship between the regulator problem and the tracking problem and how the solution of the latter is related to that of the former in an H_∞ formulation. The solution reduces the problem to solving FBI equations and HJ inequalities. These in turn can be solved (at least locally) by working with power series expansions (see [5]). If one considers the linear terms of the FBI equations (15), one obtains the Francis equation. The higher degree terms of the FBI equations satisfy a set of linear equations depending on the lower order solutions. The routine, fbi.m, in the Nonlinear Systems Toolbox [13] will solve these equations exactly to arbitrary degree when this is possible and in a least squares sense when it is not. The Nonlinear Systems Toolbox is a MATLAB toolbox for the design of nonlinear controllers. If the Hamilton-Jacobi-Bellman inequality (24) is required to hold with equality, then one can attempt to solve this term-by-term. The lowest order terms must satisfy a Riccati equation. The higher degree terms satisfy linear equations depending on the lower order solutions. The routine hji.m in the Nonlinear Systems Toolbox will solve these equations exactly to arbitrary degree when this is possible. We plan to discuss these matters in more details elsewhere.

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Controllability of a Planar Body with Unilateral Thrusters

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Abstract—This paper investigates the minimal number of unilateral thrusters required for different versions of nonlinear controllability of a planar rigid body. For one to three unilateral thrusters, one gets a new property with each additional thruster: one thruster suffices for small-time accessibility on the body's state space $TSE(2)$; two thrusters suffice for global controllability on $TSE(2)$; and three thrusters suffice for small-time local controllability at zero velocity states.

Index Terms—Controllability, gas jets, hovercraft, spacecraft, unilateral constraints.

I. INTRODUCTION

In this paper we study the minimal number of unilateral thrusters required for different versions of nonlinear controllability of a planar rigid body. The dynamics can be viewed as a simple model of a planar spacecraft or hovercraft, also studied by Manikonda and Krishnaprasad [7] and Lewis and Murray [4]. The configuration space of the body is $C = SE(2)$, the set of planar positions and orientations, and its state space is the tangent bundle TC. The configuration of the planar body is q and its state is $(q, \dot{q})$.

We place the following restrictions on the thrusters.

1) Each thruster provides a line of force fixed in the body frame.
2) Each thruster is *unilateral*. A pair of opposing thrusters is counted as two thrusters.
3) Each thruster has only two states, off or on, with thrust magnitudes zero or one.
4) Only one thruster may be on at a time.

With these restrictions on the thrusters, we can choose thruster configurations verifying the following properties (which will be made formal later).

One Thruster: The planar body is *small-time accessible*. For any $(q, \dot{q})$ and any neighborhood V of $(q, \dot{q})$, the body can reach a full-dimensional subset of TC without leaving V.

Two Thrusters: The planar body is *controllable*. It can reach any $(q_1, \dot{q}_1)$ from any other $(q_0, \dot{q}_0)$ in finite time.

Three Thrusters: The planar body is *small-time locally controllable* at zero velocity states.

These properties are tight—a planar body with one thruster can never be controllable, and a planar body with two thrusters can never be small-time locally controllable. These properties are also tight if we relax restrictions 3) and 4) on the thrusters, allowing simultaneous use of multiple thrusters with thrust values in $[0, 1]$.

II. DEFINITIONS

A coordinate frame $\mathcal{F}_B$ is attached to the center of mass of the planar body $\mathcal{B}$, and its configuration in an inertial frame $\mathcal{F}_W$ is given by $q = (x_w, y_w, \theta_w)^T$. The state of $\mathcal{B}$ is written $(q, \dot{q}) \in TC$. We define the zero velocity section Z as the three-dimensional space of zero velocity states $(q, 0)$.

The control system is written

$$(\dot{q}, \ddot{q}) = X_0(q, \dot{q}) + \sum_{i=1}^{n} u_i X_i(q, \dot{q}) \qquad (1)$$

$(u_1, \cdots, u_n)^T = u \in U = \{0, (1, 0, \cdots, 0, 0)^T, (0, 1, \cdots, 0, 0)^T, \cdots, (0, 0, \cdots, 1, 0)^T, (0, 0, \cdots, 0, 1)^T\}$, where $X_0(q, \dot{q}) = (\dot{x}_w, \dot{y}_w, \dot{\theta}_w, 0, 0, 0)^T$ is the drift vector field, u_i is the thrust applied at the ith thruster, and $X_i(q, \dot{q})$ is the corresponding control vector field. The body $\mathcal{B}$ has n thrusters. Only one thruster can be on at a time, and the thrust is unit. A feasible trajectory for $\mathcal{B}$ is a solution of (1) for a control function $u(t) \in U$ for all t.

To simplify the equations of motion, we choose the unit mass to be the mass of $\mathcal{B}$ and the unit distance to be the radius of gyration of inertia of $\mathcal{B}$. Unit time is chosen to make the thrust magnitude unit. The control vector field X_i can then be written $(0, 0, 0, f_{xi} \cos \theta_w - f_{yi} \sin \theta_w, f_{xi} \sin \theta_w + f_{yi} \cos \theta_w, \tau_i)^T$, where (f_{xi}, f_{yi}) is the unit thruster force expressed in the frame $\mathcal{F}_B$ ($f_{xi}^2 + f_{yi}^2 = 1$) and τ_i is the torque about the center of mass. We will write $f_i = (f_{xi}, f_{yi}, \tau_i)^T$, where (f_{xi}, f_{yi}) is the linear component of f_i.

Modifying notation from Nijmeijer and van der Schaft [8], we define $R^V(q_0, \dot{q}_0, T)$ to be the reachable set from $(q_0, \dot{q}_0)$ at time $T > 0$ by feasible trajectories remaining in the neighborhood V of $(q_0, \dot{q}_0)$ at times $t \in [0, T]$. Define $R^V(q_0, \dot{q}_0, \leq T) = \cup_{0 \leq t \leq T} R^V(q_0, \dot{q}_0, t)$. Then the system (1) (or simply the planar rigid body $\mathcal{B}$) is *small-time accessible* (or *locally accessible*) *from* $(q_0, \dot{q}_0)$ if $R^V(q_0, \dot{q}_0, \leq T)$ contains a nonempty open set of TC for any neighborhood V of $(q_0, \dot{q}_0)$ and all $T > 0$. $\mathcal{B}$ is *small-time locally controllable from* $(q_0, \dot{q}_0)$ if $R^V(q_0, \dot{q}_0, \leq T)$ contains a neighborhood of $(q_0, \dot{q}_0)$ for any neighborhood V and all $T > 0$. $\mathcal{B}$ is *controllable from* $(q_0, \dot{q}_0)$ if, for any $(q_1, \dot{q}_1) \in TC$, there exists a finite time T such that $(q_1, \dot{q}_1) \in R^{TC}(q_0, \dot{q}_0, T)$. The phrase "from $(q_0, \dot{q}_0)$" can be eliminated from each of these definitions if the condition applies at all $(q_0, \dot{q}_0)$.

III. PREVIOUS WORK

Partial controllability results for the planar rigid body with thrusters have been reported previously by Manikonda and Krishnaprasad [7] and Lewis and Murray [4]. Manikonda and Krishnaprasad [7]