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LPV Control of Two Dimensional Wing Flutter

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1 Introduction

We utilize a standard linear model for the control of a thin airfoil in subsonic flow [BAH, D, L]. The airfoil is modeled by a two dimensional section with three degrees of freedom; plunge, pitch angle and flap angle. This is a six dimensional linear system with states; plunge, pitch angle, flap angle and their rates. The system has three inputs; the lift and moment generated by the air flowing over the wing and the torque applied at the flap hinge. This torque consists of two parts, the torque generated by the air flow and the external torque that can be applied by a motor. The latter is the control.

The fluid is assumed to be incompressible and irrotational so that the response of the flow to a sinusoidal motion in the degrees of freedom is linear and can be computed by complex variables techniques. The result is an irrational and improper transfer matrix from the degrees of freedom to the lift, moment and hinge torque. This transfer matrix involves the well-known Theodoresen function [KS]. We take a standard second order rational approximation to the Theodoresen function to get an improper rational transfer matrix which is connected in feedback with the proper transfer matrix from the lift, moment and hinge torque to plunge, pitch angle, flap angle. The complete model is a proper linear system which can be realized by an eight dimensional state space model. The one dimensional control is the additional torque that can be applied at the hinge by a motor.

This linear model is parameterized by the free stream velocity. As the velocity is increased, a pair of poles moves from the left to the right half of the complex plane resulting in flutter. Flutter is caused by a coupling of the pitch and plunge modes. As the air foil plunges down, it is pitched down and as it plunges up it is pitched up so that energy is extracted from the air and the system loses stability.

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We only study a linear model or more precisely, a family of linear models parameterized by the air speed. The nonlinear system appears to undergo a Hopf bifurcation at the flutter speed [M]. The linear model descibing the effect of lift, moment and hinge torque on pitch, plunge and flap angle is probably adequate to capture this bifurcation. But the linear model describing the effect of pitch, plunge and flap angle on lift, moment and hinge torque is not. Unfortunately, the known nonlinear models of the fluid are infinite dimensional.

The goal is to use feedback to stabilize the air foil at or above its flutter speed. We consider several standard control strategies. The simplest is to assume that all states are measurable and to design a stabilizing state feedback using the Linear Quadratic Regulator theory (LQR). A more realistic approach is to assume that only some or all of the six physical states are measurable and to use dynamic state feedback based on the Linear Quadratic Gaussian approach (LQG). Lau [L] explored the stability and robustness of these and contrasted them with those of suboptimal H_∞ controllers. All of these approaches involve solving one or more algebraic Riccati equations.

One major difficulty with all these approaches is that they must be schelduled on a parameter, the freestream air speed. The usual approach is to choose several parameter values, i.e. freestream speeds, and design a controller at each one by solving one or more algebraic Riccati equations. Near a parameter value, one uses the appropriate controller but between parameter values one must interpolate between controllers in some fashion, e. g., piecewise linear. This requires the solution of many algebraic Riccati equations.

We take an alternative approach and approximately solve the parametrized Riccati equations themselves. This is done by expanding the system and the controllers in a power series in the parameter, the current speed minus the flutter speed. Then each pa-

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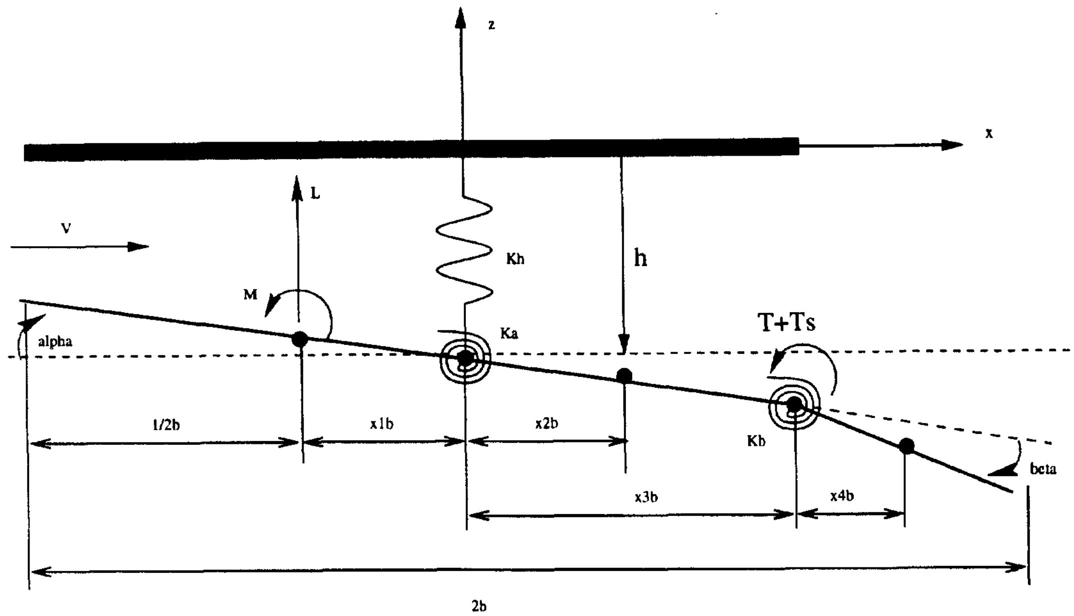

Figure 1: Basic Model

rameterized Riccati equation reduces to a unparameterized Riccati equation for the lowest degree terms plus a sequence of Sylvester equations for the higher terms. MATLAB code has been developed to do this to any degree [K1]. We show that the resulting parametrized controllers closely approximate the locally optimal LQR, LQG and H_∞ ones over a large parameter range [L].

2 Model of Wing Section

The model is the interconnection of two submodels. One submodel descibes how the aerodynamic forces affect the motions of the wing section. This is a state space model derived from classial mechanics. The other submodel describes how the motion of the wing section affects the the aerodynamic forces. This model is realized in the frequency domain by solving an unsteady Laplace PDE and then it is approximated by a state space model.

The first submodel describes the motion of the wing section given the aerodynamic forces acting on it, Figure 1. The model of our section will consist of two pieces, a main section and a flap. By convention, we assume that the entire wing has a length of $2b$. The elastic axis, where the springs are attached as seen in the diagram, will serve as our reference point. We denote the point one-fourth the total span from the left end of the wing as the quarter chord point. The quarter chord point is also at a distance $x_1 b$ from the elastic axis. The center of gravity of the wing is $x_2 b$ to the right of the elastic axis. Located $x_3 b$ to the right of the elastic axis is the hinge which is the connection between the main section and its tail. Finally, the center of gravity of the flap is $x_4 b$ to the right of the hinge. The wing is situated in a fluid with a free-stream velocity, V. Several springs will model the effects of nearby attachments to the wing. A linear spring K_h and a torsional spring K_α are mounted at the elastic axis. Another torsional spring K_β is mounted on the hinge.

The total lift L, defined positive upward, and the pitching moment M, measured counterclockwise, are calculated at the quarter chord point. The flap torque consists of a part due to the fluid, T, and a part due to the actuator, T_s. They are calculated counterclockwise at the flap hinge. Corresponding to these forces are the three degrees of freedom. The plunge, h, is the vertical movement of the wing along the z-axis measured positive down. The pitch angle (or the angle of attack), α, is the angle measure clockwise from the unperturbed axis (dotted line in Figure 1). And the flap angle, β, is measured clockwise with respect to the main section.

From Newton's Laws we obtain the equations of motion

$$\begin{bmatrix} -L \\ -M_{.25} \\ -(T+T_s) \end{bmatrix} = \tag{2.1}$$

$$\begin{bmatrix} mb & S_\alpha & S_\beta \\ mx_1 b^2 + S_\alpha b & I_\alpha & I_\beta + S_\beta(x_1 + x_3)b \\ S_\beta b & I_\beta + S_\beta x_3 b & I_\beta \end{bmatrix} \begin{bmatrix} \frac{\ddot{h}}{b} \\ \ddot{\alpha} \\ \ddot{\beta} \end{bmatrix}$$

$$+\begin{bmatrix} K_h b & 0 & 0 \\ K_h x_2 b^2 & K_\alpha & 0 \\ 0 & 0 & K_\beta \end{bmatrix} \begin{bmatrix} \frac{h}{b} \\ \alpha \\ \beta \end{bmatrix}$$

where m_1, m_2 are the masses of the main section and flap, I_1, I_2 are the moments of inertia of the main section and the flap and

$$\begin{aligned} m &= m_1 + m_2 \\ S_\alpha &= (m_1 x_2 + m_2 x_3 + m_2 x_4) b \\ S_\beta &= m_2 x_4 b \\ I_\alpha &= m_1 x_2 (x_1 + x_2) b^2 + I_1 \\ &+ m_2 (x_1 + x_3 + x_4)(x_3 + x_4) b^2 + I_2 \\ I_\beta &= m_2 x_4^2 b^2 + I_2 \end{aligned}$$

3 Unsteady Aerodynamic Loads

Küssner and Schwarz [KS], derive the unsteady aerodynamic loads L, M, and T due to sinusoidal motions in the three degrees of freedom in the frequency domain. We note that all the terms in these equations are related to the free-stream velocity, V. As the air moves past the wing, the wing pushes the air around it. This will cause the wing to experience a force pushing down on it. If the air was stagnant, then the interaction would disappear after the system reaches equilibrium. But since the air is moving past the wing at velocity V, or in the nondimensional form, $U = V/\omega_\alpha b$, where ω_α is the frequency of the uncoupled pitch oscillation in a vacuum, the wing will keep pushing on new air. More energy is needed each time the wing accelerates previously unperturbed air. Therefore, the faster the air moves, the more energy is transferred between the wing and the surrounding air.

Each of the loads has essentially two parts. One is the circulatory term involving the Theodorsen function, $C(k)$ and the other is the non-circulatory term without $C(k)$. The Theodorsen function, $C(k)$, depending on the reduced frequency k, describes a circulatory lift build-up after a periodic change in incidence The reduced frequency is defined as $k = \omega b/U = -isb/U$ where $\omega = is$ is the frequency of oscillation of the degree of freedom.

Note that the moment equation lacks the circulatory term because we have chosen to take the moment at the quarter chord point, where the $C(k)$ term conveniently cancels out.

$$\begin{aligned} \frac{L(s)}{\pi\rho V^2 b} &= \left(\frac{2C(k)sb}{V} + \frac{s^2b^2}{V^2}\right)\bar{h} \\ &+ \left(2C(k) + \frac{2C(k)sb}{V} + \frac{sb}{V} + \frac{s^2b^2}{2V^2}\right)\alpha \\ &+ \frac{1}{\pi}\left(2C(k)\Phi_1 + \frac{2C(k)sb}{2V}\Phi_2 + \frac{sb}{V}\Phi_3 + \frac{s^2b^2}{2V^2}\Phi_4\right)\beta \\ \frac{M(s)}{\pi\rho V^2 b} &= \frac{s^2b^2}{2V^2}\bar{h} + \left(\frac{sb}{V} + \frac{3s^2b^2}{8V^2}\right)\alpha \\ &+ \frac{1}{\pi}\left(\Phi_5 + \frac{sb}{2V}\Phi_6 + \frac{s^2b^2}{4V^2}\Phi_7\right)\beta \\ \frac{T(s)}{\rho V^2 b} &= \left(\frac{sb}{V}C(k)\Phi_8 + \frac{s^2b^2}{2V^2}\Phi_4\right)\bar{h} \\ &+ \left(C(k)\Phi_8 + \frac{sb}{V}C(k)\Phi_8 + \frac{sb}{2V}\Phi_9 + \frac{s^2b^2}{4V^2}\Phi_7\right)\alpha \\ &+ \frac{1}{\pi}\left(C(k)\Phi_1\Phi_8 + \frac{sb}{2V}C(k)\Phi_2\Phi_8\right. \\ &\left.+\Phi_{10} + \frac{sb}{2V}\Phi_{11} + \frac{s^2b^2}{4V^2}\Phi_{12})\right)\beta \end{aligned} \tag{3.2}$$

Here $\bar{h} = h/b$ and Φ_i are constants which depend on the location of the flap hinge [KS].

We replace the Theodorsen function $C(k)$ in the above equations with the Jones' approximation in the scaled frequency p

$$C(p) = 1 + \frac{\zeta_1 p}{p + \nu_1 U} + \frac{\zeta_2 p}{p + \nu_2 U}$$

where $ik = i\omega b/U = sb/U = p\omega_\alpha b/U$ and $\zeta_1 = 0.165$, $\zeta_2 = 0.335$, $\nu_1 = 0.041$, $\nu_2 = 0.32$.

We define the aeroelastic lag states

$$\begin{aligned} \dot{G}_\alpha &= -\nu_1 U G_\alpha + \left[\nu_1 \dot{h} + (\nu_1 - 1)\dot{\alpha} + \left(\tfrac{1}{2\pi}\nu_1\Phi_2 - \tfrac{1}{\pi}\Phi_1\right)\dot{\beta}\right] \\ \dot{G}_\beta &= -\nu_2 U G_\beta + \left[\nu_2 \dot{h} + (\nu_2 - 1)\dot{\alpha} + \left(\tfrac{1}{2\pi}\nu_2\Phi_2 - \tfrac{1}{\pi}\Phi_1\right)\dot{\beta}\right] \end{aligned} \tag{3.3}$$

4 LPV Model

By combining (2.1,3.2,3.3) we obtain a family of linear state space model parametrized by the free-stream air speed U of the form

$$\dot{x} = Ax + Bu$$

where the eight dimensional state is

$$x = [h/b \;\; \alpha \;\; \beta \;\; \dot{h}/b \;\; \dot{\alpha} \;\; \dot{\beta} \;\; G_\alpha \;\; G_\beta]^T$$

the one dimensional control is $u = T_s$ and the matrix A varies with U [L]. Note that this is a different x than that in Figure 1.

The eigenvalues of A change with the free stream velocity. At low velocities all of the eigenvalues are in the open left-half plane but as the speed increases, the system looses stability as a pair of eigenvalues moves across the imaginary axis to the right-half plane. The speed at which this happens is called the flutter speed. Table 1 shows the eigenvalues for a typical section as the speed varies where the flutter speed has been normalized to $V_f = 1.00$.

Stable $V = 0.95$	Marginally Stable $V_f = 1.00$	Unstable $V = 1.05$
- 0.05 + 1.21 i	-3.78 10^{-7} + 1.12 i	+ 0.07 + 1.06 i
- 0.05 - 1.21 i	-3.78 10^{-7} - 1.12 i	+ 0.07 - 1.06 i
- 0.10	- 0.09	- 0.09
- 0.35 + 0.73 i	- 0.46 + 0.79 i	- 0.61 + 0.84
- 0.35 - 0.73 i	- 0.46 - 0.79 i	- 0.61 - 0.84 i
- 0.52	- 0.47	- 0.42
- 0.57 + 3.19 i	- 0.60 + 3.21 i	- 0.63 + 3.23 i
- 0.57 - 3.19 i	- 0.60 - 3.21 i	- 0.63 - 3.23 i

Table 1: Open Loop Eigenvalues at Different Speeds

5 Control

We consider several standard approaches to linear control. All start with a model of the form

$$\begin{aligned} \dot{x} &= Ax + Bu + Gw \\ y &= Cx + v \\ z &= \begin{bmatrix} Hx \\ u \end{bmatrix} \end{aligned} \tag{5.4}$$

where x is the state, u is the control, y is the observed signal, z is the signal to be controlled, w is the driving noise and v is the observation noise. Throughout the paper we take the simplest forms to speed the exposition while illustrating the basic approach. More complcated models can often be reduced to these forms.

The Linear Quadratic Regulator approach assumes that there are no noises, $w = 0$, $v = 0$, and the state is fully observable, $C = I$. One finds the optimal feedback $u = Kx$ which minimizes some criterion such as

$$\int_0^\infty |z|^2 \, dt \tag{5.5}$$

The optimal cost given that $x(0) = x$ is $x^T P x$ where P satisfies the algebraic Riccati equation

$$PA + A^T P + H^T H - PBB^T P = 0 \tag{5.6}$$

If (A, B) is stabilizable and (H, A) is observable, then there is a unique positive definite solution to this equation. The optimal feedback gain is

$$K = -B^T P \tag{5.7}$$

and this stabilizes the system

Of course we have a family of linear models so we must solve a family of algebraic Riccati equations (5.6) for the optimal costs P and feedbacks K depending on the parameter, U. These feedbacks do make the closed loop sytem more stable as can be seen by comparing Table 1 and Table 2 where $H = I$.

$V = 0.95$	$V = 1.00$	$V = 1.05$
- 0.10 + 1.26 i	- 0.11 + 1.19 i	- 0.13 + 1.11 i
- 0.10 - 1.26 i	- 0.11 - 1.19 i	- 0.13 - 1.11 i
- 0.13	- 0.14	- 0.15
- 0.42 + 0.77 i	- 0.49 + 0.82 i	- 0.58 + 0.86 i
- 0.42 - 0.77 i	- 0.49 - 0.82 i	- 0.58 - 0.86 i
- 0.67	- 0.68	- 0.68
- 1.23	- 1.22	- 1.20
- 161.00	- 161.00	- 161.00

Table 2: Closed Loop Eigenvalues at Different Speeds

If the full state is not available for measurement and the measurements are corrupted by noise then one must estimate the state. The Linear Quadratic Gaussian approach is to use a Kalman filter to estimate the state. This is derived by assuming that the noises w and v in (5.4) are standard white Gaussian noises. Assuming that the measurement process has gone on long enough to reach steady state then the optimal estimate $\hat{x}(t)$ of $x(t)$ is given by

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \tag{5.8}$$

where the filter gain is $L = QC$ and Q is a solution of the algebraic Riccati equation

$$AQ + QA^T + GG^T - QC^T CQ = 0 \tag{5.9}$$

If (H, A) is detectable and (A, G) is controllable, then there is a unique positive definite solution to this equation. The feedback

$$u = K\hat{x} \tag{5.10}$$

minimizes the expected long term average cost

$$E\left\{ \lim_{n \to \infty} \frac{1}{T} \int_0^T |z|^2 \, dt \right\} \tag{5.11}$$

The closed loop system consisting of plant (5.4) and a controller, for example, the LQG controller (5.8, 5.10) defines a mapping from the noises w, v to the controlled variable z. As shown by Doyle, the LQG design can fail to be robust. A small perturbation could cause the closed-loop system to become unstable. The H_∞ controller is designed to minimize the operator norm of the closed loop mapping from noises w, v to controlled variable z. The $\mathcal{L}_2$ norms are used to measure the sizes of w, v and z. If $T_{wv,z}(s)$ denotes the transfer function from w, v to z, then the optimal H_∞ controller minimizes

$$\|T_{wv,z}\|_\infty = \sup_\omega \bar{\sigma}(T_{wv.z}(i\omega)) \tag{5.12}$$

where $\bar{\sigma}$ denotes the maximum singular value.

This is a difficult problem to solve directly so instead one sets a desired attenuation level γ and seeks a controller to achieve this,

$$\|T_{wv,z}\|_\infty \leq \gamma \tag{5.13}$$

This is equivalent to finding a controller such that

$$\inf_{w,v,x(0)=0} \int_0^\infty \gamma^2|w|^2 + \gamma^2|v|^2 - |z|^2 \; dt \geq 0$$

As with LQG design, this problem can be solved in two steps.

First we assume that the state is fully observable, $C = I$, and there is no observation noise, $v = 0$. We seek a feedback $u = Kx$ which

$$\sup_K \inf_{w,v,x(0)=x} \int_0^\infty \gamma^2 \,|w|^2 - |z|^2 \; dt \tag{5.14}$$

It can be shown that this is a quadratic form $x^T Px$ where P staisfies the algebraic Riccati equation

$$PA + A^T P + H^T H - P\left(BB^T - \tfrac{1}{\gamma^2}GG^T\right)P = 0 \tag{5.15}$$

If this equation has a positive definite solution and (H, A) is observable then the feedback (5.7) stabilizes the system. The worst case noise for this control law is

$$w = \tfrac{1}{\gamma^2}G^T Px \tag{5.16}$$

We define $\bar{A} = A + \frac{1}{\gamma^2}G^T P$, this is the open loop dynamics assuming the worst case noise.

The second step is to construct an H_∞ estimator of the state from the noisy observations. We follow Tadmor [T], see also [K2]. The dynamics of the estimate is given by

$$\dot{\hat{x}} = \bar{A}\hat{x} + Bu + L(y - C\hat{x}) \tag{5.17}$$

where

$$L = QC^T \tag{5.18}$$

and Q is a solution of the algebraic Riccati equation

$$\bar{A}Q + Q\bar{A}^T + GG^T - Q\left(C^T C - \tfrac{1}{\gamma^2}K^T K\right)Q = 0 \tag{5.19}$$

If (A, G) is controllable and Q is positive definite then the controller (5.17, 5.10) stabilizes the system and achieves the attenuation level γ, (5.13).

6 LPV Control

The LQG controller requires the solutions of the two algebaic Riccati equations (5.6) and (5.9). The H_∞ controller requires the solutions of the two algebaic Riccati equations (5.15) and (5.19). Since the system (5.4) depends on a parameter, the free stream air speed U, so does each of these Riccati equations. One approach to designing a controller that operates over a range of speeds is to choose a set of representative operating points, i. e., parameter values, and then solve the appropriate pair Riccati equations at each of these parameter values. Near a parameter value one uses the appropriate controller but between parameter values one must interpolate between controllers in some fashion. This can be difficult to implement and for high accuracy would require many parameter values. We propose an alternative approach which is to solve the parametrized algebaic Riccati equations in a series form. The expansion is in terms of powers of the deviation of the parameter from some set point. This requires solving a Riccati equation and a sequence of Sylvester equations. The latter are much easier to solve than the former as they are linear rather than quadratic equations.

We start by considering the parameter dependent version of the LQR problem. Suppose the model (5.4) is expanded in powers of a parameter U,

$$\begin{aligned} \dot{x} &= f(x,u,U) = A(U)x + B(U)u \\ &= \left(A_{[0]} + A_{[1]}U + A_{[2]}U^2 + \ldots\right)x \\ &+ \left(B_{[0]} + B_{[1]}U + B_{[2]}U^2 + \ldots\right)u \\ z &= \begin{bmatrix} H(U)x \\ u \end{bmatrix} = \begin{bmatrix} \left(H_{[0]} + H_{[1]}U + H_{[2]}U^2 + \ldots\right)x \\ u \end{bmatrix} \end{aligned} \tag{6.20}$$

We seek a controller that minimizes the cost function (5.5). This is equivalent to solving the Hamilton-Jacobi-Bellman equations

$$\begin{aligned} 0 &= \frac{\partial\pi}{\partial x}(x,U)f(x,\kappa(x,U),U) + l(x,\kappa(x,U),U) \\ 0 &= \frac{\partial\pi}{\partial x}(x,U)\frac{\partial f}{\partial u}(x,\kappa(x,U),U) + \frac{\partial l}{\partial u}(x,\kappa(x,U),U) \end{aligned} \tag{6.21}$$

for the optimal cost $\pi(x, U)$ and the optimal feedback $\kappa(x, U)$ where $l(x, u, U) = |z|^2 = |Hx|^2 + |u|^2$.

It is not hard to see that the optimal cost π is quadratic in x,

$$\pi(x,U) = \tfrac{1}{2}x^T P(U)x, \tag{6.22}$$

the optimal feedback is linear in x,

$$\kappa(x,U) = K(U)x \tag{6.23}$$

and they satisfy the parametrized forms of (5.6) and (5.7).

We expand P, K in powers of U

$$\begin{aligned} P(U) &= P_{[0]} + P_{[1]}U + P_{[2]}U^2 + \ldots \\ K(U) &= K_{[0]} + K_{[1]}U + K_{[2]}U^2 + \ldots \end{aligned} \tag{6.24}$$

We plug these into (5.6), (5.7) and extract the terms that are constant in U to obtain

$$\begin{aligned} 0 &= A_{[0]}^T P_{[0]} + P_{[0]} A_{[0]} + H_{[0]} H_{[0]}^T - P_{[0]} B_{[0]} B_{[0]}^T P_{[0]} \\ K_{[0]} &= -B_{[0]} P_{[0]} \end{aligned} \quad (6.25)$$

These are solved for $P_{[0]}$, $K_{[0]}$. We plug these into (5.6), (5.7) and extract the terms that are linear in U to obtain

$$\begin{aligned} 0 &= A_{[1]}^T P_{[0]} + P_{[0]}^T A_{[1]} + A_{[0]}^T P_{[1]} + P_{[1]}^T A_{[0]} \\ &+ (HH^T)_{[1]} - P_{[0]}(BB^T)_{[1]} P_{[0]} \\ &- P_{[0]}(BB^T)_{[0]} P_{[1]} - P_{[1]}(BB^T)_{[0]} P_{[0]} \\ K_{[1]} &= -B_{[1]}^T P_{[0]} - B_{[0]}^T P_{[1]} \end{aligned} \quad (6.26)$$

The first equation reduces to

$$\begin{aligned} 0 &= (A+BK)_{[0]}^T P_{[1]} + P_{[1]}(A+BK)_{[0]} \\ &+ (HH^T)_{[1]} + P_{[0]}(BB^T)_{[1]} P_{[0]} \end{aligned} \quad (6.27)$$

This is in the form of a Sylvester (or Lyapunov) equation,

$$AX + XA = C \quad (6.28)$$

where A and C are known and X is the unknown that needs to be found. This is a linear equation for $P_{[1]}$. This process can be repeated to obtain higher degree controllers. Each step requires the solution of another Sylvester equation.

The other parametrized algebraic Riccati equations can be solve in a similar manner. The software package Nonlinear Systems Toolbox [K1] solves this and more complicated problems to any degree.

We computed the gains at 60 operating points over the range, flutter speed $\pm 5\%$, by solving 60 algebraic Riccati equations (5.6). We also computed the LPV controller to third degree in $U = V - V_f$ by solving 1 algebraic Riccati equation and 3 Sylvester equations. Figure 2 shows the relative error in the gains between these solutions. It required the solution of 60 algebraic Riccati equations to guarantee that the piecewise linear interpolated gains where within 1% of the true solution throughout the range, flutter speed $\pm 5\%$. The third degree LPV controller achieves the same goal and only requires the solution of 1 algebraic Riccati equation 3 Sylvester equations. Moreover, the feedback is easier to impliment because it is a simple cubic polynomial in U rather than a piecewise linear funcion.

For more details regarding the stability and robustnss of the LQR, LQG and H_∞ controllers we refer the reader to [L].

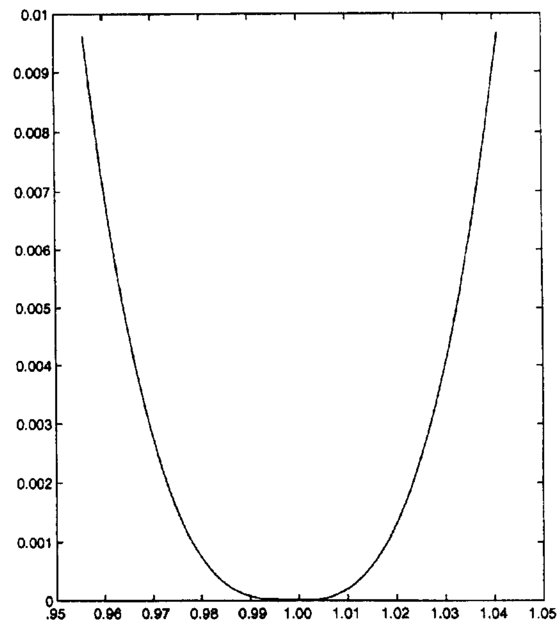

Figure 2: Relative Error between LPV and LQR controllers, $V_f = 1.00$

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