Precursors of Bifurcations

Arthur J. Krener [1]
Department of Mathematics and
Institute of Theoretical Dynamics
University of California
Davis, CA 95616-8633
ajkrener@ucdavis.edu

Abstract

Consider a smooth (at least C^4) dynamical system

$$\dot{x} = f(x, \mu) \tag{0.1}$$

depending on a parameter μ. Suppose that the system is operating around asymptotically stable equilibrium point x_e, but at some critical value of the parameter μ_c the system undergoes a bifurcation where the equilibrium point becomes unstable or vanishes. One would like to estimate how close to the critical parameter value the system is operating when the parameter is not directly measurable. To accomplish this we assume that the system is experiencing a small harmonic noise $w(t)$

$$\dot{x} = f(x, \mu) + \epsilon w(t) \tag{0.2}$$

whose frequency ω is known but whose magnitude ϵ is unknown. An example would be rotating machinery where one would expect the period of the dominant noise to be the same as the period of rotation. By comparing the first, second and third harmonics we will be able to estimate how close μ is to its critical value.

Keywords: Identification of bifurcations, normal forms, Moore-Greitzer Compressor Model

1 Center Manifolds and Normal Forms

We assume the system and variables x, μ are such that $x_e = 0$ is an equilibrium point of the system (0.1) for all small μ. We assume that for $\mu < 0$, the linearized dynamics

$$\dot{x} = A(\mu)x \tag{1.3}$$

$$A = \frac{\partial f}{\partial x}(0, \mu) \tag{1.4}$$

is asymptotically stable and for $\mu > 0$ it is unstable. At $\mu = 0$ one or two eigenvalues of A cross the imaginary axis with positive rate with respect to μ and a bifurcation occurs. The first step in analyzing this system is to compute its center manifold. We start by making a linear change of coordintes to separate the asymptotically stable and critically stable parts of $A(0)$. After this change of coordinates, the linearized system (1.3) at $\mu = 0$ becomes

$$\begin{bmatrix} \dot{x}_s \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A_s & 0 \\ 0 & A_c \end{bmatrix} \begin{bmatrix} x_s \\ x_c \end{bmatrix} \tag{1.5}$$

and the nonlinear system becomes

$$\begin{aligned} \begin{bmatrix} \dot{x}_s \\ \dot{x}_c \end{bmatrix} &= \begin{bmatrix} A_s & 0 \\ 0 & A_c \end{bmatrix} \begin{bmatrix} x_s \\ x_c \end{bmatrix} \\ &+ \begin{bmatrix} f_s^{[2]}(x_s, x_c, \mu) + f_s^{[3]}(x_s, x_c, \mu) + \ldots \\ f_c^{[2]}(x_s, x_c, \mu) + f_c^{[3]}(x_s, x_c, \mu) + \ldots \end{bmatrix} \end{aligned} \tag{1.6}$$

where the spectrum of A_s is in the left half plane and the spectrum of A_c is on the imaginary axis. Later we will restrict our attention to the cases where the dimension of x_c is one or two but, for now, it is arbitrary, n_c. The vector fields $f_s^{[d]}(x_s, x_c, \mu), f_c^{[d]}(x_s, x_c, \mu)$ are homogeneous polynomials of degree d in x_s, x_c, μ. We seek a change of coordinates in the asymptotically stable variables

$$\begin{aligned} z_s &= x_s - \phi_s(x_c, \mu) \\ &= x_s - \phi_s^{[2]}(x_c, \mu) - \phi_s^{[3]}(x_c, \mu) + \ldots \end{aligned}$$

so that $\dot{z}_s = 0$ when $z_s = 0$. Then $x_s = \phi_s(x_c, \mu)$ is a submanifold of x_s, x_c, μ space which invariant under the nonlinear system dynamics (0.1) and the additional parameter dynamics

$$\dot{\mu} = 0 \tag{1.7}$$

This submanifold, called the center manifold [2], is tangent at the origin to the critically stable subspace, $x_s = 0$, of the linearized system (1.5, 1.7). The center manifold can be computed term by term to any degree of approximation assuming that the original system is smooth enough [1]. For example the degree two and three approximations satisfy

$$0 = A_s \phi_s^{[2]}(x_c, \mu) - \frac{\partial \phi_s^{[2]}}{\partial x_c}(x_c, \mu) A_c x_c + f_s^{[2]}(0, x_c, \mu)$$

[1] Research supported in part by AFOSR-49620-95-1-0409 and by NSF 9970998.

$$
\begin{aligned}
0 &= A_s\phi_s^{[3]}(x_c,\mu) - \frac{\partial \phi_s^{[3]}}{\partial x_c}(x_c,\mu)A_c x_c \\
&+ \left(f_s^{[2]}(\phi_s^{[2]}(x_c,\mu),x_c,\mu)\right)^{[3]} + f_s^{[3]}(0,x_c,\mu) \\
&- \frac{\partial \phi_s^{[2]}}{\partial x_c}(x_c,\mu) f_c^{[2]}(0,x_c,\mu)
\end{aligned}
$$

where $(\ )^{[3]}$ denotes the cubic part. These linear equations are always solvable because of the assumptions on the eigenvalues of A_s, A_c. After $\phi_s^{[2]}$, $\phi_s^{[3]}$ have been computed, we calculate the approximate dynamics on the center manifold,

$$
\dot{x}_c = A_c x_c + \bar{f}_c^{[2]}(x_c,\mu) + \bar{f}_c^{[3]}(x_c,\mu) + \ldots
$$

where

$$
\begin{aligned}
\bar{f}_c^{[2]}(x_c,\mu) &= f_c^{[2]}(0,x_c,\mu) \\
\bar{f}_c^{[3]}(x_c,\mu) &= f_c^{[3]}(0,x_c,\mu) + \left(f_c^{[2]}(\phi_s^{[2]}(x_c,\mu),x_c,\mu)\right)^{[3]}
\end{aligned}
$$

Notice that we did not need to know $\phi_s^{[d]}(x_c,\mu)$ to calculate $\bar{f}_c^{[d]}(x_c,\mu)$.

The next step is to bring this into normal form by a change of x_c coordinates,

$$
z_c = x_c - \phi_c(x_c,\mu) = x_s - \phi_c^{[2]}(x_c,\mu) - \phi_c^{[3]}(x_c,\mu)
$$

to cancel as many of the terms of degrees two and three as possible. The transformed dynamics is

$$
\dot{z}_c = A_c z_c + g_c^{[2]}(x_c,\mu) + g_c^{[3]}(x_c,\mu) + \ldots
$$

where

$$
\begin{aligned}
g_c^{[2]}(x_c,\mu) &= A_c\phi_c^{[2]}(x_c,\mu) - \frac{\partial \phi_c^{[2]}}{\partial x_c}(x_c,\mu)A_c x_c \\
&\quad + \bar{f}_c^{[2]}(x_c,\mu) \\
g_c^{[3]}(x_c,\mu) &= A_c\phi_c^{[3]}(x_c,\mu) - \frac{\partial \phi_c^{[3]}}{\partial x_c}(x_c,\mu)A_c x_c \\
&\quad + \bar{f}_c^{[3]}(x_c,\mu) - \frac{\partial \phi_c^{[2]}}{\partial x_c}(x_c,\mu)\bar{f}_c^{[2]}(x_c,\mu)
\end{aligned}
$$

The spectrum of the linear mapping

$$
\phi_c^{[2]}(x_c,\mu) \mapsto A_c\phi_c^{[2]}(x_c,\mu) - \frac{\partial \phi_c^{[2]}}{\partial x_c}(x_c,\mu)A_c x_c
$$

consists of all combinations of the form $\lambda_k - \lambda_i - \lambda_j$, $0 \le i \le j \le n_c$, $1 \le k \le n_c$ where $\lambda_1, \ldots, \lambda_{n_c}$ are eigenvalues of A_c and $\lambda_0 = 0$ is the eigenvalue of the parameter dynamics (1.7). The eigenvalues need not be distinct. Hence if none of these combinations are zero, then there exists a $\phi_c^{[2]}(x_c,\mu)$ so that $g_c^{[2]}(x_c,\mu) = 0$. If a combination is zero then there is said to be a degree two resonance and it may not be possible to cancel all of $g_c^{[2]}$. Suppose A_c is diagonal with entries $\lambda_1, \lambda_2, \ldots$.

If $\lambda_k - \lambda_i - \lambda_j = 0$ then the coefficient of $e_c^k x_{ci} x_{cj}$ in $g_c^{[2]}$ cannot be changed by $\phi_c^{[2]}$ where x_{ci} denotes the i^{th} coordinate and e_c^k denotes k^{th} unit column vector in x_c space. If $\lambda_k - \lambda_i = 0$ then the coefficient of $e_c^k x_{ci}\mu$ in $g_c^{[2]}$ cannot be changed and if $\lambda_k = 0$ then the coefficient of $e_c^k \mu^2$ in $g_c^{[2]}$ cannot be changed. We choose $\phi_c^{[2]}$ to cancel the rest of $g_c^{[2]}$.

Now suppose that x_c is one dimensional so A_c must be 0. We let λ denote the eigenvalue of A and ν denote the eigenvalue of the parameter dynamics (1.7), of course both are 0. There are three resonances of degree two, $\lambda - 2\lambda = \lambda - \lambda = \lambda = 0$ so the normal form is

$$
\dot{z}_c = b_1 z_c^2 + b_2 z_c\mu + b_3\mu^2 + \ldots
$$

Ae have assumed that $x = 0$ is a critical point for all μ so $b_3 = 0$. We have also assumed that the linearized system (1.3) is stable for $\mu < 0$ and the eigenvalue crosses the imaginary axis with positive rate with respect to μ so $b_2 > 0$, by redefining μ, without loss of generality $b_2 = 1$. Finally we can scale z_c and obtain the normal form of a transcritical bifurcation

$$
\dot{z}_c = \mu z_c + z_c^2 + \ldots \tag{1.8}
$$

A pair of equilibria $z_c = 0, z_c \approx -\mu$ exist for all small μ, the smaller one is stable and the larger one is unstable.

Suppose that x_c is two dimensional and

$$
A_c = \begin{bmatrix} 0 & -\lambda \\ \lambda & 0 \end{bmatrix} \tag{1.9}
$$

The case where $\lambda = 0$ is quite complicated and because of space limitations, we do not consider it. We also omit the case where

$$
A_c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
$$

We refer the reader to [3]. When $\lambda \neq 0$, then there are two resonances, $i\lambda - i\lambda = 0$ and $(-i\lambda) - (-i\lambda) = 0$. The normal form of degree two is

$$
\begin{bmatrix} \dot{z}_{c1} \\ \dot{z}_{c2} \end{bmatrix} = \lambda \begin{bmatrix} -z_{c2} \\ z_{c1} \end{bmatrix} + \mu \begin{bmatrix} 1 & -b \\ b & 1 \end{bmatrix} \begin{bmatrix} z_{c1} \\ z_{c2} \end{bmatrix} + \ldots \tag{1.10}
$$

after rescaling μ. For $\mu < 0$ we have a stable focus which bifurcates into an unstable focus for $\mu > 0$. The stability at $\mu = 0$ depends on the higher degree terms as we discuss below.

Now suppose that the quadratic terms in z_c can be canceled in the dynamics on the center manifold, hence the only quadratic terms are those which are bilinear in z_c and μ. There never are terms in μ alone because $x = 0$ is assumed to be an equilibrium for all μ. Bilinear terms must be present if the eigenvalues of $A(\mu)$ that cross the imaginary axis are to do so at a positive rate.

The terms quadratic in x vanish if the system (0.1) has an odd symmetry $f(x,\mu) = -f(-x,\mu)$ so then there are no terms even in z_c in the center manifold dynamics. If the center manifold is two dimensional and $\lambda \neq 0$ in (1.9) then as we have seen (1.10) the terms quadratic in z_c can be canceled.

The spectrum of the linear mapping

$$\phi_c^{[3]}(x_c,\mu) \mapsto A_c\phi_c^{[3]}(x_c,\mu) - \frac{\partial\phi_c^{[3]}}{\partial x_c}(x_c,\mu)A_c x_c$$

consists of all combinations of the form $\lambda_l - \lambda_i - \lambda_j - \lambda_k$, $0 \leq \lambda_i \leq \lambda_j \leq \lambda_k \leq n_c$, $1 \leq \lambda_l \leq n_c$.

Suppose that x_c is one dimensional so $A_c = 0$. There are four degree three resonances so the normal form is

$$\dot{z}_c = \mu z_c + b_1 z_c^3 + b_2 z_c^2\mu + b_3 z_c\mu^2 + b_4\mu^3 + \ldots$$

but $b_4 = 0$ as before. For the moment assume that $b_1 = \pm 1, b_2 = b_3 = 0$ and the higher terms are zero then we obtain the normal form of a pitchfork bifurcation,

$$\dot{z}_c = z_c(\mu \pm z_c^2) \tag{1.11}$$

If $b_1 = 1$ then for small $\mu < 0$ there is one stable equilibrium, $z_c = 0$, and two unstable equilibria, $z_c = \pm\sqrt{-\mu}$. For small $\mu > 0$ there is one unstable equilibrium, $z_c = 0$. This is called a subcritical pitchfork bifurcation.

If $b_1 = -1$ then for small $\mu < 0$ there is one stable equilibrium, $z_c = 0$. For small $\mu > 0$ there is one unstable equilibrium, $z_c = 0$ and two stable equilibria, $z_c = \pm\sqrt{\mu}$. This is called a supercritical pitchfork bifurcation.

For general $b_1 \neq 0$ and $b_2,\ b_3$ and the higher terms not necessarily zero, the basic picture stays the same except that the shapes of the outer prongs of the pitchfork are changed. They are approximately the real roots of

$$b_1 z_c^2 + b_2\mu z_c + b_3\mu^2 + \mu = 0$$

as a function of μ. Notice that for small μ the roots are real where $b_1\mu \leq 0$ as before. If $b_1 = 0$ then higher degree terms must be considered.

Now consider the case where z_c is two dimensional and (1.9) holds. As before the case where $\lambda = 0$ is quite complicated and will not be treated here. If $\lambda \neq 0$ then the degree three normal form extending (1.10) is

$$\begin{aligned}\begin{bmatrix} \dot{z}_{c1} \\ \dot{z}_{c2} \end{bmatrix} &= (\mu + b_1 r^2 + b_1\mu^2)\begin{bmatrix} z_{c1} \\ z_{c2} \end{bmatrix} \\ &\quad +(\lambda + b_1\mu + b_4 r^2 + b_5\mu^2)\begin{bmatrix} -z_{c2} \\ z_{c1} \end{bmatrix} + \ldots\end{aligned} \tag{1.12}$$

where $r^2 = z_{c1}^2 + z_{c2}^2$.

For the moment assume that $b_1 = \pm 1,\ b_2 = b_3 = b_4 = b_5 = 0$ and the higher terms are zero then we obtain the normal form of a Hopf bifurcation,

$$\begin{bmatrix} \dot{z}_{c1} \\ \dot{z}_{c2} \end{bmatrix} = \lambda\begin{bmatrix} -z_{c2} \\ z_{c1} \end{bmatrix} + (\mu \pm r^2)\begin{bmatrix} z_{c1} \\ z_{c2} \end{bmatrix}$$

If $b_1 = 1$ then for small $\mu < 0$ there is a stable equilibrium, $z_c = 0$, and an unstable limit cycle of radius, $r = \sqrt{-\mu}$. For small $\mu > 0$ there is one unstable equilibrium, $z_c = 0$. This is called a subcritical Hopf bifurcation.

If $b_1 = -1$ then for small $\mu < 0$ there is one stable equilibrium, $z_c = 0$. For small $\mu > 0$ there is an unstable equilibrium, $z_c = 0$ and a stable limit cycle of radius $z_c = \pm\sqrt{\mu}$. This is called a supercritical Hopf bifurcation.

In the general case, the picture stays the same as long as $b_1 \neq 0$. Of course the shape and period of the limit cycle is changed. If $b_1 = 0$ then higher degree terms must be considered.

An important bifurcation, the fold or saddle node, does not immediately fit into the above context. It's normal form is

$$\dot{x} = \mu + z^2$$

For $\mu < 0$ there is a stable equilibrium $x_e = -\sqrt{-\mu}$ and an unstable equilibrium $x_e = \sqrt{-\mu}$ which coalesce at $\mu = 0$. There are no equilibria for $\mu > 0$. This does not fit into the above picture because we assumed that $x_e = 0$ is a equilibrium for all small μ. Moreover given such a system it is easy to estimate the parameter μ if the equilibrium is known.

Consider a parameter dependent change of coordinates that freezes the stable equilibrium at 0,

$$z = x + |\mu|^{1/2}$$

Of course this is not smooth in μ at $\mu = 0$ but we are only interested in descibing the system before bifurcation has occured, $\mu < 0$, and determining how close the parameter is to bifurcating value. For $\mu < 0$ the new system is

$$\dot{z} = -2|\mu|^{1/2}z + z^2$$

which we recognize as a reparametrized transcritical bifurcation (1.8). So given a system with a stable equilibrium for $\mu < 0$ that vanishes in a saddle node bifurcation at $\mu = 0$, we can treat it as a transcritical for purposes of detecting nearness to bifurcation.

2 Bifurcations

We assume the system is in a near normal form of a transcritical bifurcation

$$\dot{x} = \mu x + bx^2 + \epsilon w(t) + \dots$$

and it is being perturbed by a noise $w(t)$ of known frequency ω and unknown small intensity ϵ. Assuming $\mu < 0$ so that the system is stable, we expect the energy in the system at the fundamental frequency ω will be proportional to ϵ, at the second harmonic frequency, 2ω, it will be proportional to ϵ^2 and so on. If we consider the quadratic term in the dynamics as an exogenous input then the transfer function from x^2 to x is

$$T(s) = b/(s - \mu)$$

We can estimate this at 2ω, $\hat{T}(2\omega i)$ as the cross power spectral density at 2ω between x and x^2. Notice that this does not involve the unknown noise intensity ϵ. Since the frequecy ω is known, this yields the estimate

$$\frac{|\mu|}{|b|} \approx \sqrt{\frac{1}{|\hat{T}(2\omega i)|^2} - \frac{4\omega^2}{b^2}}$$

The expression $|\mu|/|b|$ is a better measure of how close the system is to instability than $|\mu|$ alone as the unstable equilibrium is at $x \approx -\mu/b$. Notice that this depends only on the amplitude of $T(2\omega i)$, a less reliable estimate could be obtained using the phase of $T(2\omega i)$ as we shall show in an example in the next section.

Next we assume that the system is in a near normal form of a pitchfork bifurcation

$$\dot{x} = \mu x + b_1 x^3 + b_2 x^2 \mu + b_3 x \mu^2 + \epsilon w(t) + \dots$$

and as before it is being perturbed by a noise $w(t)$ of known frequency ω and unknown small intensity ϵ. Assuming $\mu < 0$ so that the system is stable, we expect the energy in the system at the fundamental frequency ω will be proportional to ϵ, that at the second harmonic frequency, 2ω, will be proportional to $\epsilon^2\mu$, at the third harmonic frequency, 3ω, will be proportional to ϵ^3 and so on. The cross power spectral density of $x(t)$ with $x^3(t)$ at frequency 3ω should approximately be the transfer function $T_1(s) = b_1/(s - (\mu + b_3\mu^2))$ at $s = 3\omega i$. If we assume that μ is small enough so that $b_3\mu^2$ can be neglected, this gives us an estimate of $|\mu|/|b_1|$ as before. This is a good measure of nearness to instability because to leading order the bifurcating equilibria are at $x \approx \pm\sqrt{|\mu/b_1|}$ to leading order in μ. The sign of b_1 and the criticality of the pitchfork bifurcation can be estimated from the fact that if $\pm b_1 > 0$ then $\arg(T_1(3\omega i) \to \mp\pi/2$ as $\mu \to 0$.

The normal form of a Hopf bifurcation (1.12) in polar coordinates closely resembles the pitchfork

$$\begin{aligned} \dot{r} &= \mu r b_1 r^3 + b_2 r \mu^2 + \dots \\ \dot{\theta} &= \lambda + b_3\mu + b_4 r^2 + b_5\mu^2 + \dots \end{aligned}$$

and the analysis is similar.

3 Example

We close with an example that shows how the above techniques can be used even when the system is not in normal form We consider the Moore Greitzer 3 dimensional model [5] for an axial flow compressor with a smooth compressor characteristic $\Psi_c(\phi)$ around stable operating point $\Psi = \Psi_e, \Phi = \Phi_e, A = 0$. In displacement coordinates $\Psi_d = \Psi - \Psi_e, \Phi_d = \Phi - \Phi_e, A$, the dynamics is

$$\begin{aligned} \dot{\Psi}_d &= \alpha_{-1}\left[\Phi_d + \Phi_e - (K_T + u)\sqrt{\Psi_e + \Psi_d}\right] \\ \dot{\Phi}_d &= \alpha_0\left[D^1\Psi_c\,\Phi_d - \Psi_d + D^2\Psi_c\left[\frac{{\Phi_d}^2}{2} + \frac{A^2}{4}\right]\right. \\ &\quad \left. + D^3\Psi_c\left[\frac{{\Phi_d}^3}{6} + \frac{A^2\Phi_d}{4}\right]\right] + \dots \\ \dot{A} &= \alpha_1\left[D^1\Psi_c\,A + D^2\Psi_c\,A\Phi_d\right. \\ &\quad \left. + D^3\Psi_c\left[\frac{A{\Phi_d}^2}{2} + \frac{A^3}{8}\right]\right] + \dots \end{aligned}$$

where

Ψ	=	annulus averaged pressure rise
Φ	=	annulus averaged mass flow
A	=	amplitude of first mode stall
B	=	Greitzer stability parameter
K_T	=	throttle parameter
l_c	=	length of inlet duct in wheel radians
m	=	Moore outlet duct parameter
μ	=	inertia of gas in blade passage

$$\begin{aligned} \alpha_{-1} &= \frac{1}{4B^2 l_c} \\ \alpha_0 &= \frac{1}{l_c} \\ \alpha_1 &= \frac{1}{m + \mu} \end{aligned}$$

and $D^j\Psi_c$ is the j^{th} derivative of the compressor characteristic Ψ_c evaluated at the operating point. The parameter is K_T which fixes equilbrium point Ψ_e, Φ_e, A_e. For K_T sufficiently large, the system is stable but at a critical value of K_T the system looses stability in a pitchfork bifurcation which is usually supercritical. We refer the reader to McCaughan [4] for an excellent analysis of the dynamics. McCaughan uses $R = A^2$ rather than A as a coordinate thereby transforming a pitchfork bifurcation to a transcritical one.

The bifurcation takes place at the top of the compressor characteristic where $D^1\Psi_c = 0$. Because the compressor characteristic changes with age and operating temperatures, it is not known how close to bifurcation (called rotating stall) the machine is operating. Early warning of impending stall would be extremely helpful for safe operation. Because of it rotary nature, there is considerable noise at this frequency.

The transfer function from $A\Phi_d$ to A is

$$T(s) = \frac{\alpha_1 D^2 \Psi_c}{s - \alpha_1 D^1 \Psi_c}$$

If $D^2 \Psi_c < 0$ then the amplitude at $s = 2\omega_e i$ is

$$|T(2\omega_e i)| = \frac{-\alpha_1 D^2 \Psi_c}{\sqrt{4\omega_e^2 + \alpha_1^2 D^1 {\Psi_c}^2}}$$

where ω_e is the frequency of the compressor revolutions (one rev).

This changes with the set point Φ_e. The derivative with respect to Φ_e is

$$\frac{-\alpha_1 D^3 \Psi_c(4\omega_e^2 + \alpha_1^2 D^1 {\Psi_c}^2) + \alpha_1^3 D^1 \Psi_c D^2 {\Psi_c}^2}{(4\omega_e^2 + \alpha_1^2 D^1 {\Psi_c}^2)^{3/2}}$$

At the stall point where $D^1 \Psi_c = 0$ this reduces to

$$\frac{-\alpha_1 D^3 \Psi_c}{2\omega_e}$$

Hence the change in the gain depends on the skewness of the comprssor characteristic.

Right Skew	$D^3 \Psi_c > 0$	$D^1 \lvert T(2\omega_e i)\rvert < 0$
Left Skew	$D^3 \Psi_c < 0$	$D^1 \lvert T(2\omega_e i)\rvert > 0$

As a right skewed compressor goes into stall, the amplitude gain at $2\omega_e$ from the product of A and Φ to A is increasing. As a left skewed compressor goes into stall, the amplitude gain is decreasing.
(This assumes Ψ_c is smooth thru stall.)

The phase of the transfer function at $s = 2\omega_e i$ is

$$\arctan\left(\frac{2\omega_e}{\alpha_1 D^1 \Psi_c}\right)$$

which is in $(\frac{\pi}{2}, \pi)$ since $D^1 \Psi_c < 0$.

The derivative with respect to Φ_e is

$$\frac{-2\omega_e \alpha_1 D^2 \Psi_c}{4\omega_e^2 + \alpha_1^2 D^1 {\Psi_c}^2}$$

This is positive under the reasonable assumption that $D^2 \Psi_c < 0$. Hence as the compressor stalls the phase decreases to $\frac{\pi}{2}$.

We now describe the behaviour of these quantities on Wright Patterson 4 stage data supplied by Hoying. The rig consists of 4 stages of a core compressor. The stages are labeled 2-5. There were eight sensors, which can be resolve into the 0-3 circumferential Fourier modes, labeled $\Phi, A_1, \Theta_1, A_2, \Theta_2, A_3, \Theta_3$. The low speed runs ($< 85\%$) were very noisy and therefore ignored. The high speed runs were taken with a sample rate $1000hz$ and an analog cut off frequency of $400hz$ to avoid aliasing. For the 100% speed run, twice the rotor frequency $2\omega_e$ is higher than the Nyquist rate of $500hz$ so only a 90% speed run, $efc6207\_st5\_90\%$, was processed. A window of 2048 data points was used and the cross spectral density was computed using the MATLAB Signal Processing Toolbox. Then the window was shifted a step of 256 data points and the cross spectral density recomputed. One step of 256 data points is approximately 64 rotor revolutions.

Figure 1. shows the amplitude gain as the compressor stalls along with a raw sensor signal. Figure 2. shows the phase as the compressor stalls along with a raw sensor signal. The rise in amplitude indicates the impending stall 400 rotor revs early. The fall off in amplitude immediately prior to stall is because of the increasing noise in this regime. The phase is less conclusive.

4 Discussion

The above approach of estimating nearness to a bifurcation works when the driving noise is predominantly of one frequency and the system is in near normal form. Suitably modified it will also work when the noise consists several discrete frequencies by taking resonances into account. If the noise has a continuous spectrum then it will not work.

References

[1] J. Carr, *Applications of Centre Manifold Theory.* Springer-Verlag, New York, 1981.

[2] A. Kelley, The stable, center-stable, center, center-unstable and unstable manifolds. J. Diff. Eqns., 3: 546–570, 1967.

[3] Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory.* Springer-Verlag, New York, 1998.

[4] F. E. McCaughan. *Bifurcation analysis of axial flow compressor stability.* SIAM J. Appl. Math., **20**(1990), 1232-1253.

[5] F. K. Moore and E. M. Greitzer. A theory of post stall transients in axial compression systems: Part I-Development of the equations. Trans. ASME **108**,(1986), 68-76.

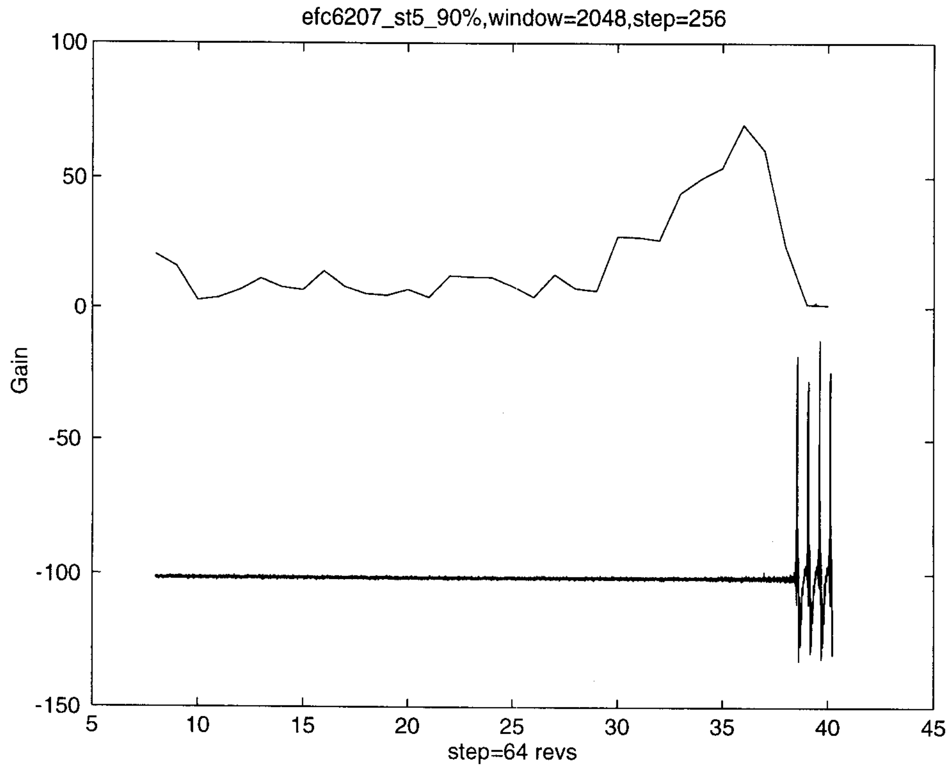

Figure 1:

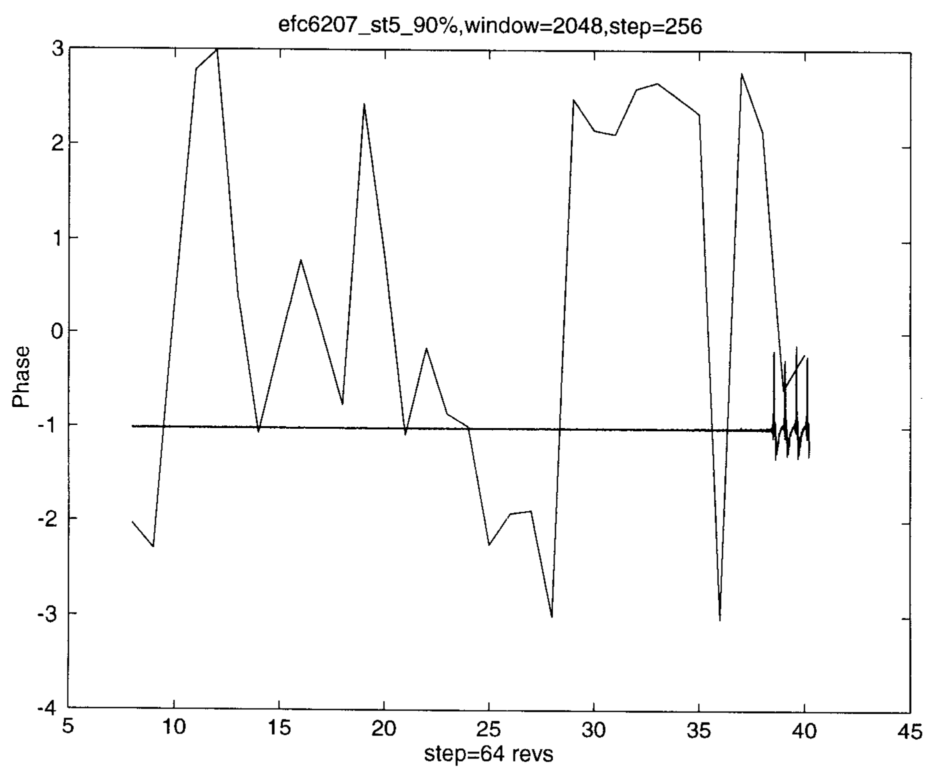

Figure 2: