QUADRATIC AND CUBIC NORMAL FORMS OF DISCRETE TIME NONLINEAR CONTROL SYSTEMS

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Abstract— We present a normal form for the quadratic and cubic terms of a nonlinear discrete time control system around a stationary point under change of coordinates and feedback.

Key Words— Discrete time nonlinear control system; Normal form

1 Introduction

The theory of normal forms of nonlinear difference equations goes back to Poincaré (Arnold, 1983), (Guckenheimer and Holmes, 1983), (Wiggins, 1990). Briefly it is as follows. Consider a smooth (C^4) n dimensional difference equation expanded in a power series around a fixed point, say $x = 0,\ f(0) = 0$,

$$\begin{aligned} x' = f(x) = Ax + f^{[2]}(x) \\ +f^{[3]}(x) + O(x)^4 \end{aligned} \qquad (1.1)$$

where $x'(t) = x(t+1)$ and $f^{[d]}(x)$ denotes an n dimensional vector field that is a homogeneous polynomial of degree d in x. The group of smooth changes of coordinates

$$\begin{aligned} z = \phi(x) = Tx - \phi^{[2]}(x) \\ -\phi^{[3]}(x) + O(x)^4 \end{aligned} \qquad (1.2)$$

acts on this difference equation and transforms it into

$$\begin{aligned} z' &= \tilde{f}(z) = \phi(f(\phi^{-1}(z))) \\ &= \tilde{A}z + \tilde{f}^{[2]}(z) + \tilde{f}^{[3]}(z) + O(z)^4 \end{aligned}$$

If we consider the effect of the transformation term by term, it simplifies considerably. The linear part is just

$$\tilde{A} = TAT^{-1}$$

so we can always transform A into Jordan form.

If we assume that A is originally in Jordan form then we can take $T = I$ and use $\phi^{[2]}(x)$ to transform the quadratic term to

$$\tilde{f}^{[2]}(z) = f^{[2]}(z) - < Az, \phi^{[2]}(z) >$$

where

$$< Az, \phi^{[2]}(z) >= \phi^{[2]}(Az) - A\phi^{[2]}(z)$$

The mapping

$$\phi^{[2]}(z) \mapsto < Az, \phi^{[2]}(z) >$$

is a linear operator on the vector space $\{\phi^{[2]}(z)\}$ of n-dimensional quadratic vector fields and a simple argument shows that its eigenvalues are of the form $\lambda_j\lambda_k - \lambda_i$ where λ_i are the eigenvalues of A. We assume that A is diagonal then the eigenvectors of the above mapping are $\mathbf{e}^i z_j z_k$ where $\mathbf{e}^k$ is the k^{th} unit vector. Therefore we can cancel the part of $f^{[2]}(x)$ that is in the span of the eigenvectors corresponding to nonzero eigenvalues and reduce the quadratic part to

$$\tilde{f}^{[2]}(z) = \sum_{\lambda_i = \lambda_j\lambda_k} c_i^{jk} \mathbf{e}^i\ z_j\ z_k \qquad (1.3)$$

This is the quadratic normal form of Poincaré.

In a similar fashion we can use $\phi^{[3]}(z)$ to reduce $\tilde{f}^{[3]}(z)$ to the cubic normal form of Poincaré.

$$\tilde{f}^{[3]}(z) = \sum_{\lambda_i = \lambda_j\lambda_k\lambda_l} c_i^{jkl} \mathbf{e}^i\ z_j\ z_k\ z_l \qquad (1.4)$$

(Kang and Krener, 1992) developed a quadratic normal form for continuous time nonlinear systems whose linear part is controllable. This was extended to discrete time systems by (Barbot et al., 1997). These authors considered a larger group of transformations to bring the system to normal form, including state feedback as well as change of state coordinates. (Kang, 1998a), (Kang, 1998b) also developed a quadratic normal form for continuous time nonlinear systems whose linear part may have uncontrollable modes.

In this paper we will develop quadratic and cubic normal forms for discrete time nonlinear control systems of the form

$$\begin{aligned} x' = f(x,u) = Ax + Bu + f^{[2]}(x,u) \\ +f^{[3]}(x,u) + O(x,u)^4 \end{aligned} \qquad (1.5)$$

where $x,\ u$ are of dimensions $n,\ 1$. We do not assume that the linear part of the system is controllable. Moreover our quadratic normal form differs from that of (Barbot et al., 1997) for linearly controllable systems.

2 Quadratic Normal Forms

Consider a smooth (C^3) system of the form (1.5) under the action of linear and quadratic change of state coordinates and state feedback

$$z = \phi(x) = Tx - \phi^{[2]}(x) \quad (2.1)$$
$$v = \alpha(x,u) = Kx + Lu - \alpha^{[2]}(x,u) \quad (2.2)$$

where T, L are invertible.

It is well known that there exists a linear change of coordinates and a linear feedback that transforms the system into the linear normal form

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u + \begin{bmatrix} f_1^{[2]}(x_1,x_2,u) \\ f_2^{[2]}(x_1,x_2,u) \end{bmatrix} + O(x_1,x_2,u)^3 \quad (2.3)$$

where x_1, x_2 are n_1, n_2 dimensional, $n_1 + n_2 = n$, A_1 is in Jordan form, A_2, B_2 are in controller (Brunovsky) form,

$$A_2 = \begin{bmatrix} 0 & 1 & \dots & 0 \\ & \ddots & \ddots & \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The following result generalizes that of (Li, 1999).

Theorem 2.1 *Assume that A_1 is diagonal. There exist a quadratic change of coordinates and a quadratic feedback*

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} \phi_1^{[2]}(x_1,x_2) \\ \phi_2^{[2]}(x_1,x_2) \end{bmatrix}$$
$$v = u - \alpha^{[2]}(x_1,x_2,u)$$

which transforms the system (2.3) into the quadratic normal form

$$\begin{bmatrix} z_1' \\ z_2' \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} v + \begin{bmatrix} \bar f_1^{[2;0]}(z_1;z_2,v) \\ 0 \end{bmatrix} + \begin{bmatrix} \bar f_1^{[1;1]}(z_1;z_2,v) \\ 0 \end{bmatrix} + \begin{bmatrix} \bar f_1^{[0;2]}(z_1;z_2,v) \\ \bar f_2^{[0;2]}(z_1;z_2,v) \end{bmatrix} + O(z_1,z_2,v)^3 \quad (2.4)$$

where $\bar f_1^{[2;0]}(z_1,z_2,v)$ is in the quadratic normal form of Poincaré,

$$\bar f_1^{[2;0]} = \sum_{\lambda_i = \lambda_j \lambda_k} \beta_i^{jk} \, \mathbf{e}_1^i \, z_{1,j} \, z_{1,k} \quad (2.5)$$

and the other terms are as follows.

$$\bar f_1^{[1;1]} = \sum_{\lambda_i=0} \sum_{\lambda_j=0} \sum_{k=1}^{n_2+1} \gamma_i^{jk} \, \mathbf{e}_1^i \, z_{1,j} \, z_{2,k} + \sum_{\lambda_i\neq 0} \sum_{\lambda_j \neq 0} \gamma_i^{j1} \, \mathbf{e}_1^i \, z_{1,j} \, z_{2,1} \quad (2.6)$$

$$\bar f_1^{[0;2]} = \sum_{\lambda_i \neq 0} \sum_{k=1}^{n_2+1} \delta_i^{1k} \, \mathbf{e}_1^i \, z_{2,1} \, z_{2,k} \quad (2.7)$$

$$\bar f_2^{[0;2]} = \sum_{i=1}^{n_2-1} \sum_{k=i+2}^{n_2+1} \epsilon_i^j \, \mathbf{e}_2^i \, z_{2,1} \, z_{2,k} \quad (2.8)$$

We use the notation $\bar f_i^{[d_1;d_2]}(z_1;z_2,v)$ to denote a polynomial vector field homogeneous of degree d_1 in z_1 and homogeneous of degree d_2 in z_2, v. For notational convenience we have defined $z_{2,n_2+1} = v$.

Proof: The proof is tedious so we only sketch the details. We can expand the change of coordinates and feedback as follows,

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} \phi_1^{[2;0]}(x_1;x_2) \\ \phi_2^{[2;0]}(x_1;x_2) \end{bmatrix} - \begin{bmatrix} \phi_1^{[1;1]}(x_1;x_2) \\ \phi_2^{[1;1]}(x_1;x_2) \end{bmatrix} - \begin{bmatrix} \phi_1^{[0;2]}(x_1;x_2) \\ \phi_2^{[0;2]}(x_1;x_2) \end{bmatrix}$$

$$v = u - \alpha^{[2;0]}(x_1;x_2,u) - \alpha^{[1;1]}(x_1;x_2,u) - \alpha^{[0;2]}(x_1;x_2,u)$$

The effect of this on the quadratic part of the dynamics is

$$\begin{aligned} \bar f_i^{[d_1;d_2]}(z_1;z_2,v) = {} & f_i^{[d_1;d_2]}(z_1;z_2,v) \\ & -\phi_i^{[d_1;d_2]}(A_1 z_1; A_2 z_2, v) \\ & +A_i \phi_i^{[d_1;d_2]}(z_1;z_2,v) \\ & -B_i \alpha^{[d_1;d_2]}(z_1;z_2,v) \end{aligned}$$

where $B_1 = 0$, so the proof splits into six cases, $i = 1,2;\ d_1 = 0,1,2;\ d_2 = 2 - d_1$.

$\bar f_2^{[0;2]}(z_1;z_2,v)$

There are two basic operations, *pull up* and *push down*, which are used to transform the system into normal form. Consider a part of the dynamics

$$\begin{aligned} z_{2,i-1}' &= z_{2,i} + \dots \\ z_{2,i}' &= z_{2,i+1} + c z_{2,j} z_{2,k} + \dots \\ z_{2,i+1}' &= z_{2,i+2} + \dots \end{aligned}$$

where $1 < i \le n_2$, $1 \le j \le k \le n_2 + 1$, recall $z_{2,n+1} = v$.

If $1 < j$ we can *pull up* the quadratic term by defining

$$\bar z_{2,i} = z_{2,i} - c z_{2,j-1} z_{2,k-1}$$

then the dynamics becomes

$$\begin{aligned} z_{2,i-1}' &= \bar z_{2,i} + c z_{2,j-1} z_{2,k-1} + \dots \\ \bar z_{2,i}' &= z_{2,i+1} + \dots \\ z_{2,i+1}' &= z_{2,i+2} + \dots \end{aligned}$$

and all the other quadratic terms remain the same. Notice that if $i = 1$, we can still pull up and the term disappears. By pulling up all quadratic terms until $j = 1$, we obtain

$$z'_{2,i} = z_{2,i+1} + cz_{2,1}z_{2,k} + \dots \tag{2.9}$$

The other operation on the dynamics is *push down*. If $k \leq n_2$ define

$$\bar{z}_{2,i+1} = z_{2,i+1} + cz_{2,j}z_{2,k}$$

yielding

$$\begin{aligned} z'_{2,i-1} &= z_{2,i} + \dots \\ z'_{2,i} &= \bar{z}_{2,i+1} + \dots \\ \bar{z}'_{2,i+1} &= z_{2,i+2} + cz_{2,j+1}z_{2,k+1} + \dots \end{aligned}$$

and all the other quadratic terms remain unchanged. Notice that if $i + 1 = n_2$ then we can absorb the quadratic term into the control. From (2.9) we push down every term where $k \leq i + 1$. These terms can be pushed all the way down and absorbed in the control. The result is (2.8).

$\bar{f}_2^{[1;1]}(z_1; z_2, v)$

The two basic operations, pull up and push down, are slightly different. Consider a part of the dynamics

$$\begin{aligned} z'_{2,i-1} &= z_{2,i} + \dots \\ z'_{2,i} &= z_{2,i+1} + cz_{1,j}z_{2,k} + \dots \\ z'_{2,i+1} &= z_{2,i+2} + \dots \end{aligned}$$

where $1 \leq i \leq n_2$, $1 \leq j \leq n_1$, $1 \leq k \leq n_2 + 1$.

If $\lambda_j \neq 0$ and $1 < k$ we can pull up the quadratic term by defining

$$\bar{z}_{2,i} = z_{2,i} - \frac{c}{\lambda_j} z_{1,j}z_{2,k-1}$$

then the dynamics becomes

$$\begin{aligned} z'_{2,i-1} &= \bar{z}_{2,i} + \frac{c}{\lambda_j} z_{1,j}z_{2,k-1} + \dots \\ \bar{z}'_{2,i} &= z_{2,i+1} + \dots \\ z'_{2,i+1} &= z_{2,i+2} + \dots \end{aligned}$$

and all the other quadratic terms remain the same. Again if $i = 1$, we can still pull up and the term disappears. So by pulling up all quadratic terms where $\lambda_j \neq 0$ until $k = 1$, we obtain

$$z'_{2,i} = z_{2,i+1} + cz_{1,j}z_{2,1} + \dots$$

Pushing down eliminates this term and any term with $\lambda_j = 0$. Define

$$\bar{z}_{2,i+1} = z_{2,i+1} + cz_{1,j}z_{2,k}$$

yielding

$$\begin{aligned} z'_{2,i-1} &= z_{2,i} + \dots \\ z'_{2,i} &= \bar{z}_{2,i+1} + \dots \\ \bar{z}'_{2,i+1} &= z_{2,i+2} + c\lambda_j z_{1,j}z'_{2,k} + \dots \end{aligned}$$

and all the other quadratic terms remain unchanged. If $\lambda_j = 0$ then the term drops out. If $\lambda_j \neq 0$ then by the above, we can assume that $k = 1$ and so the term can be pushed down repeatedly until it is absorbed in the control. The result is $\bar{f}_2^{[1;1]}(z_1; z_2, v) = 0$.

$\bar{f}_2^{[2;0]}(z_1; z_2, v)$

Consider a part of the dynamics

$$\begin{aligned} z'_{2,i-1} &= z_{2,i} + \dots \\ z'_{2,i} &= z_{2,i+1} + cz_{1,j}z_{1,k} + \dots \\ z'_{2,i+1} &= z_{2,i+2} + \dots \end{aligned}$$

where $1 \leq i \leq n_2$, $1 \leq j \leq k \leq n_1$.

Pushing down one or more times eliminates this term. Define

$$\bar{z}_{2,i+1} = z_{2,i+1} + cz_{1,j}z_{1,k}$$

yielding

$$\begin{aligned} z'_{2,i-1} &= z_{2,i} + \dots \\ z'_{2,i} &= \bar{z}_{2,i+1} + \dots \\ \bar{z}'_{2,i+1} &= z_{2,i+2} + c\lambda_j\lambda_k z_{1,j}z_{1,k} + \dots \end{aligned}$$

and all the other quadratic terms remain unchanged. If $\lambda_j\lambda_k = 0$ then the term drops out. Otherwise the term can be pushed down repeatedly until it is absorbed in the control. The result is $\bar{f}_2^{[2;0]}(z_1; z_2, v) = 0$.

$\bar{f}_1^{[2;0]}(z_1; z_2, v)$

This is just the quadratic normal form of Poincaré and the proof can be found in (Arnold, 1983), (Guckenheimer and Holmes, 1983) or (Wiggins, 1990).

$\bar{f}_1^{[1;1]}(z_1; z_2, v)$

Consider a part of the dynamics

$$z'_{1,i} = \lambda_i z_{1,i} + cz_{1,j}z_{2,k} + \dots$$

where $1 \leq i \leq n_1$, $1 \leq j \leq n_1$, $1 \leq k \leq n_2$.

If $\lambda_j \neq 0$ and $k > 1$ then we can pull up by defining

$$\bar{z}_{1,i} = z_{1,i} - \frac{c}{\lambda_j} z_{1,j}z_{2,k-1}$$

then

$$\bar{z}'_{1,i} = \lambda_i \bar{z}_{1,i} + \frac{c\lambda_i}{\lambda_j} z_{1,j}z_{2,k-1} + \dots$$

If $\lambda_i = 0$ then the term disappears otherwise we can continue to pull up until $k = 1$.

If $\lambda_i \neq 0$ then we can push down by defining

$$\bar{z}_{1,i} = z_{1,i} + \frac{c}{\lambda_i} z_{1,j}z_{2,k}$$

then

$$\bar{z}'_{1,i} = \lambda_i \bar{z}_{1,i} + \frac{c\lambda_j}{\lambda_i} z_{1,j}z'_{2,k} + \dots$$

If $\lambda_j = 0$ then the term disappears.

If $\lambda_i = \lambda_j = 0$ then we can't pull up or push down. The result is (2.6).

$\tilde{f}_1^{[0;2]}(z_1; z_2, v)$
Consider a part of the dynamics

$$z'_{1,i} = \lambda_i z_{1,i} + c z_{2,j} z_{2,k} + \dots$$

where $1 \le i \le n_1$, $1 \le j \le k \le n_2$.

If $j > 1$ then we can pull up by defining

$$\bar{z}_{1,i} = z_{1,i} - c z_{2,j-1} z_{2,k-1}$$

then

$$\bar{z}'_{1,i} = \lambda_i \bar{z}_{1,i} + c\lambda_i z_{2,j-1} z_{2,k-1} + \dots$$

If $\lambda_i = 0$ then the term disappears otherwise we can continue to pull up until $j = 1$. The result is (2.7).

3 Cubic Normal Forms

Theorem 3.1 *Consider a smooth (C^4) system in the quadratic normal form described above*

$$\begin{aligned}\begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u \\ &+ \begin{bmatrix} f_1^{[2;0]}(x_1; x_2, u) \\ 0 \end{bmatrix} + \begin{bmatrix} f_1^{[1;1]}(x_1; x_2, u) \\ 0 \end{bmatrix} \\ &+ \begin{bmatrix} f_1^{[0;2]}(x_1; x_2, u) \\ f_2^{[0;2]}(x_1; x_2, u) \end{bmatrix} + \begin{bmatrix} f_1^{[3]}(x_1; x_2, u) \\ f_2^{[3]}(x_1; x_2, u) \end{bmatrix} \\ &+ O(x_1, x_2, u)^4 \end{aligned} \quad (3.1)$$

where A_1 is diagonal. There exist a cubic change of coordinates and a cubic feedback

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} \phi_1^{[3]}(x_1, x_2) \\ \phi_2^{[3]}(x_1, x_2) \end{bmatrix}$$

$$v = u - \alpha^{[3]}(x_1, x_2, u)$$

which transforms the system (2.3) into the cubic normal form

$$\begin{aligned}\begin{bmatrix} z'_1 \\ z'_2 \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} v \\ &+ \begin{bmatrix} f_1^{[2;0]}(z_1; z_2, v) \\ 0 \end{bmatrix} + \begin{bmatrix} f_1^{[1;1]}(z_1; z_2, v) \\ 0 \end{bmatrix} \\ &+ \begin{bmatrix} f_1^{[0;2]}(z_1; z_2, v) \\ f_2^{[0;2]}(z_1; z_2, v) \end{bmatrix} + \begin{bmatrix} \tilde{f}_1^{[3;0]}(z_1; z_2, v) \\ 0 \end{bmatrix} \\ &+ \begin{bmatrix} \tilde{f}_1^{[2;1]}(z_1; z_2, v) \\ 0 \end{bmatrix} + \begin{bmatrix} \tilde{f}_1^{[1;2]}(z_1; z_2, v) \\ \tilde{f}_2^{[1;2]}(z_1; z_2, v) \end{bmatrix} \\ &+ \begin{bmatrix} \tilde{f}_1^{[0;3]}(z_1; z_2, v) \\ \tilde{f}_2^{[0;3]}(z_1; z_2, v) \end{bmatrix} + O(z_1, z_2, v)^4 \end{aligned} \quad (3.2)$$

where $\tilde{f}_1^{[3;0]}(z_1, z_2, v)$ is in the cubic normal form of Poincaré,

$$\tilde{f}_1^{[3;0]}(z_1; z_2, v) = \sum_{\lambda_i = \lambda_j \lambda_k \lambda_l} \beta_i^{jkl}\, \mathbf{e}_1^i\, z_{1,j}\, z_{1,k}\, z_{1,l} \quad (3.3)$$

and the other terms are as follows.

$$\begin{aligned}\tilde{f}_1^{[2;1]} &= \sum_{\lambda_i=0} \sum_{\lambda_j\lambda_k=0} \sum_{l=1}^{n_2+1} \gamma_i^{jkl}\, \mathbf{e}_1^i\, z_{1,j}\, z_{1,k}\, z_{2,l} \\ &+ \sum_{\lambda_i\neq 0} \sum_{\lambda_j\lambda_k\neq 0} \gamma_i^{j1}\, \mathbf{e}_1^i\, z_{1,j}\, z_{1,k}\, z_{2,1} \end{aligned} \quad (3.4)$$

$$\begin{aligned}\tilde{f}_1^{[1;2]} &= \sum_{\lambda_i=0} \sum_{\lambda_j=0} \sum_{k=1}^{n_2+1} \sum_{l=k}^{n_2+1} \delta_i^{jkl}\, \mathbf{e}_1^i\, z_{1,j}\, z_{2,k}\, z_{2,l} \\ &+ \sum_{\lambda_i\neq 0} \sum_{\lambda_j\neq 0} \sum_{k=1}^{n_2+1} \delta_i^{jk1}\, \mathbf{e}_1^i\, z_{1,j}\, z_{2,1}\, z_{2,k} \end{aligned} \quad (3.5)$$

$$\tilde{f}_1^{[0;3]} = \sum_{\lambda_i\neq 0} \sum_{k=1}^{n_2+1} \epsilon_i^{1kl}\, \mathbf{e}_1^i\, z_{2,1}\, z_{2,k}\, z_{2,l} \quad (3.6)$$

$$\tilde{f}_2^{[1;2]} = \sum_{i=1}^{n_2-1} \sum_{\lambda_j\neq 0} \sum_{l=i+2}^{n_2+1} \zeta_i^{j1l}\, \mathbf{e}_2^i\, z_{1,j}\, z_{2,1}\, z_{2,l} \quad (3.7)$$

$$\tilde{f}_2^{[0;3]} = \sum_{i=1}^{n_2-1} \sum_{l=i+2}^{n_2+1} \sum_{k=1}^{l} \eta_i^{1kl}\, \mathbf{e}_2^i\, z_{2,1}\, z_{2,k}\, z_{2,l} \quad (3.8)$$

Proof: The proof is tedious so we only sketch the details. As before, the proof splits into cases, this time eight cases, $i = 1, 2;\ d_1 = 0, 1, 2, 3;\ d_2 = 3 - d_1$.

$\tilde{f}_2^{[0;3]}(z_1; z_2, v)$
We use the two basic operations, pull up and push down, again. Consider a part of the dynamics

$$\begin{aligned} z'_{2,i-1} &= z_{2,i} + \dots \\ z'_{2,i} &= z_{2,i+1} + c z_{2,j} z_{2,k} z_{2,l} + \dots \\ z'_{2,i+1} &= z_{2,i+2} + \dots \end{aligned}$$

where $1 < i \le n_2$, $1 \le j \le k \le l \le n_2 + 1$, recall $z_{2,n+1} = v$.

If $1 < j$ we can *pull up* the cubic term by defining

$$\bar{z}_{2,i} = z_{2,i} - c z_{2,j-1} z_{2,k-1} z_{2,l-1}$$

then the dynamics becomes

$$\begin{aligned} z'_{2,i-1} &= \bar{z}_{2,i} + c z_{2,j-1} z_{2,k-1} z_{2,l-1} + \dots \\ \bar{z}'_{2,i} &= z_{2,i+1} + \dots \\ z'_{2,i+1} &= z_{2,i+2} + \dots \end{aligned}$$

and all the other cubic terms remain the same. Notice that if $i = 1$, we can still pull up and the term disappears. By pulling up all cubic terms until $j = 1$, we obtain that

$$z'_{2,i} = z_{2,i+1} + cz_{2,1}z_{2,k}z_{2,l} + \dots \qquad (3.9)$$

The other operation the dynamics is *push down*. If $l \leq n_2$ define

$$\bar{z}_{2,i+1} = z_{2,i+1} + cz_{2,j}z_{2,k}z_{2,l}$$

yielding

$$\begin{aligned} z'_{2,i-1} &= z_{2,i} + \dots \\ z'_{2,i} &= \bar{z}_{2,i+1} + \dots \\ \bar{z}'_{2,i+1} &= z_{2,i+2} + cz_{2,j+1}z_{2,k+1}z_{2,l+1} + \dots \end{aligned}$$

and all the other cubic terms remain unchanged. Notice that if $i + 1 = n_2$ then we can absorb the cubic term into the control. From (3.9) we push down every term where $l \leq i + 1$. These terms can be pushed all the way down and absorbed in the control. The result is (3.8).

$\bar{f}_2^{[1;2]}(z_1; z_2, v)$
The two basic operations, pull up and push down, are slightly different. Consider a part of the dynamics

$$\begin{aligned} z'_{2,i-1} &= z_{2,i} + \dots \\ z'_{2,i} &= z_{2,i+1} + cz_{1,j}z_{2,k}z_{2,l} + \dots \\ z'_{2,i+1} &= z_{2,i+2} + \dots \end{aligned}$$

where $1 \leq i \leq n_2$, $1 \leq j \leq n_1$, $1 \leq k \leq l \leq n_2 + 1$.

If $\lambda_j \neq 0$ and $1 < k$ we can pull up the cubic term by defining

$$\bar{z}_{2,i} = z_{2,i} - \frac{c}{\lambda_j} z_{1,j}z_{2,k-1}z_{2,l-1}$$

then the dynamics becomes

$$\begin{aligned} z'_{2,i-1} &= \bar{z}_{2,i} + \frac{c}{\lambda_j} z_{1,j}z_{2,k-1}z_{2,l-1} + \dots \\ \bar{z}'_{2,i} &= z_{2,i+1} + \dots \\ z'_{2,i+1} &= z_{2,i+2} + \dots \end{aligned}$$

and all the other cubic terms remain the same. Again if $i = 1$ we can still pull up and the term disappears. So by pulling up all cubic terms where $\lambda_j \neq 0$ until $k = 1$, we obtain

$$z'_{2,i} = z_{2,i+1} + cz_{1,j}z_{2,1}z_{2,l} + \dots$$

We can also push down by defining

$$\bar{z}_{2,i+1} = z_{2,i+1} + cz_{1,j}z_{2,k}z_{2,l}$$

yielding

$$\begin{aligned} z'_{2,i-1} &= z_{2,i} + \dots \\ z'_{2,i} &= \bar{z}_{2,i+1} + \dots \\ \bar{z}'_{2,i+1} &= z_{2,i+2} + c\lambda_j z_{1,j}z'_{2,k}z'_{2,l} + \dots \end{aligned}$$

and all the other cubic terms remain unchanged. If $\lambda_j = 0$ then the term drops out. If $\lambda_j \neq 0$ then by the above, we can assume that $k = 1$ and so if $l \leq i + 1$ then the term can be pushed down repeatedly and absorbed in the control. The result is (3.7)

$\bar{f}_2^{[2;1]}(z_1; z_2, v)$
Consider a part of the dynamics

$$\begin{aligned} z'_{2,i-1} &= z_{2,i} + \dots \\ z'_{2,i} &= z_{2,i+1} + cz_{1,j}z_{1,k}z_{2,l} + \dots \\ z'_{2,i+1} &= z_{2,i+2} + \dots \end{aligned}$$

where $1 \leq i \leq n_2$, $1 \leq j \leq k \leq n_1$, $1 \leq l \leq n_2 + 1$.

If $\lambda_j\lambda_k \neq 0$ and $1 < l$ we can pull up the cubic term by defining

$$\bar{z}_{2,i} = z_{2,i} - \frac{c}{\lambda_j\lambda_k} z_{1,j}z_{1,k}z_{2,l-1}$$

then the dynamics becomes

$$\begin{aligned} z'_{2,i-1} &= \bar{z}_{2,i} + \frac{c}{\lambda_j\lambda_k} z_{1,j}z_{1,k}z_{2,l-1} + \dots \\ \bar{z}'_{2,i} &= z_{2,i+1} + \dots \\ z'_{2,i+1} &= z_{2,i+2} + \dots \end{aligned}$$

and all the other cubic terms remain the same. Again if $i = 1$ we can still pull up and the term disappears. So by pulling up all cubic terms where $\lambda_j\lambda_k \neq 0$ until $l = 1$, we obtain

$$z'_{2,i} = z_{2,i+1} + cz_{1,j}z_{1,k}z_{2,1} + \dots$$

Pushing down eliminates this term and any term with $\lambda_j\lambda_k = 0$. Define

$$\bar{z}_{2,i+1} = z_{2,i+1} + cz_{1,j}z_{1,k}z_{2,l}$$

yielding

$$\begin{aligned} z'_{2,i-1} &= z_{2,i} + \dots \\ z'_{2,i} &= \bar{z}_{2,i+1} + \dots \\ \bar{z}'_{2,i+1} &= z_{2,i+2} + c\lambda_j\lambda_k z_{1,j}z_{1,k}z'_{2,l} + \dots \end{aligned}$$

and all the other cubic terms remain unchanged. If $\lambda_j\lambda_k = 0$ then the term drops out. If $\lambda_j\lambda_k \neq 0$ then by the above, we can assume that $l = 1$ and so the term can be pushed down repeatedly until it is absorbed in the control. The result is $\bar{f}_2^{[2;1]}(z_1; z_2, v) = 0$.

$\bar{f}_2^{[3;0]}(z_1; z_2, v)$
Consider a part of the dynamics

$$\begin{aligned} z'_{2,i-1} &= z_{2,i} + \dots \\ z'_{2,i} &= z_{2,i+1} + cz_{1,j}z_{1,k}z_{1,l} + \dots \\ z'_{2,i+1} &= z_{2,i+2} + \dots \end{aligned}$$

where $1 \leq i \leq n_2$, $1 \leq j \leq k \leq l \leq n_1$.

Pushing down one or more times eliminates this term. Define

$$\bar{z}_{2,i+1} = z_{2,i+1} + cz_{1,j}z_{1,k}z_{1,l}$$

yielding

$$\begin{aligned} z'_{2,i-1} &= z_{2,i} + \dots \\ z'_{2,i} &= \bar{z}_{2,i+1} + \dots \\ \bar{z}'_{2,i+1} &= z_{2,i+2} + c\lambda_j\lambda_k\lambda_l z_{1,j}z_{1,k}z_{1,l} + \dots \end{aligned}$$

and all the other cubic terms remain unchanged. If $\lambda_j\lambda_k\lambda_l = 0$ then the term drops out. Otherwise the term can be pushed down repeatedly until it is absorbed in the control. The result is $\tilde{f}_2^{[3;0]}(z_1; z_2, v) = 0$.

$\tilde{f}_1^{[3;0]}(z_1; z_2, v)$
This is just the cubic normal form of Poincaré and the proof can be found in (Arnold, 1983), (Guckenheimer and Holmes, 1983) and (Wiggins, 1990).

$\tilde{f}_1^{[2;1]}(z_1; z_2, v)$
Consider a part of the dynamics

$$z'_{1,i} = \lambda_i z_{1,i} + cz_{1,j}z_{1,k}z_{2,l} + \dots$$

where $1 \le i \le n_1$, $1 \le j \le k \le n_1$, $1 \le l \le n_2$.

If $\lambda_j\lambda_k \neq 0$ and $l > 1$, we can pull up by defining

$$\bar{z}_{1,i} = z_{1,i} - \frac{c}{\lambda_j\lambda_k} z_{1,j}z_{1,k}z_{2,l-1}$$

so that

$$\bar{z}'_{1,i} = \lambda_i\bar{z}_{1,i} + \frac{c\lambda_i}{\lambda_j\lambda_k} z_{1,j}z_{1,k}z_{2,l-1} + \dots$$

If $\lambda_i = 0$ then the term disappears otherwise we can continue to pull up until $l = 1$.

If $\lambda_i \neq 0$ and $\lambda_j\lambda_k = 0$ then the term disappears by pushing down,

$$\bar{z}_{1,i} = z_{1,i} + \frac{c}{\lambda_i} z_{1,j}z_{1,k}z_{2,l}$$

so that

$$\begin{aligned} \bar{z}'_{1,i} &= \lambda_i\bar{z}_{1,i} + \frac{c\lambda_j\lambda_k}{\lambda_i} z_{1,j}z_{1,k}z'_{2,l} + \dots \\ &= \lambda_i\bar{z}_{1,i} \end{aligned}$$

If $\lambda_i = \lambda_j\lambda_k = 0$ then we can't pull up or push down. The result is (3.4).

$\tilde{f}_1^{[1;2]}(z_1; z_2, v)$
Consider a part of the dynamics

$$z'_{1,i} = \lambda_i z_{1,i} + cz_{1,j}z_{2,k}z_{2,l} + \dots$$

where $1 \le i \le n_1$, $1 \le j \le n_1$, $1 \le k \le l \le n_2$.

If $\lambda_j \neq 0$ and $k > 1$ then we can pull up by defining

$$\bar{z}_{1,i} = z_{1,i} - \frac{c}{\lambda_j} z_{1,j}z_{2,k-1}z_{2,l-1}$$

then

$$\bar{z}'_{1,i} = \lambda_i\bar{z}_{1,i} + \frac{c\lambda_i}{\lambda_j} z_{1,j}z_{2,k-1}z_{2,l-1} + \dots$$

If $\lambda_i = 0$ then the term disappears otherwise we can continue to pull up until $k = 1$.

If $\lambda_i \neq 0$ and $\lambda_j = 0$ then the term disappears by pushing down,

$$\bar{z}_{1,i} = z_{1,i} + \frac{c}{\lambda_i} z_{1,j}z_{2,k}z_{2,l}$$

then

$$\begin{aligned} \bar{z}'_{1,i} &= \lambda_i\bar{z}_{1,i} + \frac{c\lambda_j}{\lambda_i} z_{1,j}z'_{1,k}z'_{2,l} + \dots \\ &= \lambda_i\bar{z}_{1,i} \end{aligned}$$

The result is (3.5).

$\tilde{f}_1^{[0;3]}(z_1; z_2, v)$
Consider a part of the dynamics

$$z'_{1,i} = \lambda_i z_{1,i} + cz_{2,j}z_{2,k}z_{2,l} + \dots$$

where $1 \le i \le n_1$, $1 \le j \le k \le l \le n_2$.

If $j > 1$, we can pull up by defining

$$\bar{z}_{1,i} = z_{1,i} - cz_{2,j-1}z_{2,k-1}z_{2,l-1}$$

then

$$\bar{z}'_{1,i} = \lambda_i\bar{z}_{1,i} + c\lambda_i z_{2,j-1}z_{2,k-1}z_{2,l-1} + \dots$$

If $\lambda_i = 0$ then the term disappears otherwise we can continue to pull up until $j = 1$. The result is (3.6).

4 Conclusion

We have developed a theory of quadratic and cubic normal forms for discrete time control systems. To avoid notational difficulties, we have restricted our attention to scalar input systems whose uncontrollable part is diagonalizable. But the basic operations of pull up and push down extend to more general systems. Because of limitations of space, we have not shown the uniqueness of the normal forms but we believe they are unique based on results found in (Li, 1999). The development of a normal form is the first step in the analysis of the local behaviour of control systems. In particular, it is essential in the study of the possible bifurcations of control systems which we are now engaged.

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