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THE LOCAL SOLVABILITY OF A HAMILTON–JACOBI–BELLMAN PDE AROUND A NONHYPERBOLIC CRITICAL POINT*

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Abstract. We show the existence of a local solution to a Hamilton–Jacobi–Bellman (HJB) PDE around a critical point where the corresponding Hamiltonian ODE is not hyperbolic, i.e., it has eigenvalues on the imaginary axis. Such problems arise in nonlinear regulation, disturbance rejection, gain scheduling, and linear parameter varying control. The proof is based on an extension of the center manifold theorem due to Aulbach, Flockerzi, and Knobloch. The method is easily extended to the Hamilton–Jacobi–Isaacs (HJI) PDE. Software is available on the web to compute local approximtate solutions of HJB and HJI PDEs.

Key words. parametrized optimal control, nonlinear regulation, nonlinear disturbance rejection, gain scheduling, linear parameter varying control, H_∞ regulation

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1. Introduction.

Consider a smooth optimal control problem of minimizing

$$\frac{1}{2}\int_0^\infty \|e\|^2 + \|u\|^2 dt \tag{1.1}$$

subject to

$$\begin{array}{rcl} \dot{x} &=& f(x,u) = Ax + Bu + O(x,u)^2, \\ e &=& h(x,u) = Cx + Du + O(x,u)^2. \end{array} \tag{1.2}$$

The optimal cost $\pi(x)$ and the optimal feedback $\kappa(x)$ satisfy the Hamiliton–Jacobi–Bellman (HJB) PDE

$$0 = \frac{\partial \pi}{\partial x}(x) f(x,\kappa(x)) + l(x,\kappa(x)), \tag{1.3}$$

$$\kappa(x) = \operatorname{argmin}_u \left\{ \frac{\partial \pi}{\partial x}(x) f(x,u) + l(x,u) \right\}, \tag{1.4}$$

where

$$\begin{aligned} l(x,u) &= \frac{1}{2}(\|e\|^2 + \|u\|^2) \\ &= \frac{1}{2}\left(x'Qx + 2x'Su + u'Ru\right) + O(x,u)^3 \end{aligned} \tag{1.5}$$

and $Q = C'C,\ S = C'D,\ R = I + D'D$.

The HJB PDE may not admit a smooth global solution but under suitable conditions there does exist a viscosity solution. We refer the reader to [10], [11] for details. It is well known [22] that the HJB PDE admits a smooth solution locally around

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$x = 0$ under suitable conditions. We briefly review these conditions and the method of proof.

Consider first the linear quadratic part of the above problem, minimizing

$$\frac{1}{2}\int_0^\infty (x'Qx + 2x'Su + u'Ru) \tag{1.6}$$

subject to

$$\dot{x} \quad = \quad Ax + Bu. \tag{1.7}$$

If they exist, the optimal cost is quadratic, $\frac{1}{2}x'Px$, and the optimal feedback is linear, $u = Kx$. Moreover, P satisfies the algebraic Riccati equation

$$0 = A'P + PA + Q - (PB + S)R^{-1}(PB + S)'$$

and

$$K = -R^{-1}(PB + S)'.$$

It is well known [3] that if the pair A, B is stabilizable and the pair C, A is detectable, then there is a unique nonnegative definite solution to the algebraic Riccati equation and the resulting feedback is asymptotically stabilizing. If C, A is observable, then P is positive definite.

The $2n$ dimensional Hamiltonian system associated with this problem is linear,

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda}' \end{bmatrix} = \mathbf{H} \begin{bmatrix} x \\ \lambda' \end{bmatrix}, \tag{1.8}$$

where

$$\mathbf{H} = \begin{bmatrix} A - BR^{-1}S' & -BR^{-1}B' \\ -Q + SR^{-1}S' & -A' + SR^{-1}B' \end{bmatrix}. \tag{1.9}$$

If A, B is stabilizable and C, A is detectable, then this system is hyperbolic, i.e., none of the eigenvalues of $\mathbf{H}$ lie on the imaginary axis. Since $\mathbf{H}$ is Hamiltonian, this implies that n eigenvalues lie in the open left half plane and n eigenvalues lie in the open right half plane. In fact, the n dimensional stable subspace of $\mathbf{H}$ is the graph of the gradient of the unique nonnegative definite solution to the algebraic Riccati equation,

$$\lambda = x'P.$$

In other words, the stable subspace is the span of the columns of

$$\begin{bmatrix} I \\ P \end{bmatrix}. \tag{1.10}$$

We return to the nonlinear problem (1.1), (1.2). The associated Hamiltonian system is nonlinear,

$$\begin{aligned} \dot{x}' &= \frac{\partial H}{\partial \lambda}(\lambda, x, \kappa(\lambda, x)), \\ \dot{\lambda} &= -\frac{\partial H}{\partial x}(\lambda, x, \kappa(\lambda, x)), \end{aligned} \tag{1.11}$$

where the Hamiltonian is

$$
\begin{aligned}
H(\lambda, x, u) &= \lambda f(x,u) + l(x,u) \\
&= \lambda\,(Ax + Bu) \\
&\quad + \tfrac{1}{2}\,(x'Qx + 2x'Su + u'Ru) \\
&\quad + O(\lambda, x, u)^3
\end{aligned} \tag{1.12}
$$

and the optimal control as determined by the Pontryagin maximum principle satisfies

$$
u = \kappa(\lambda, x) = \mathrm{argmin}_u H(\lambda, x, u). \tag{1.13}
$$

The linearization of this system (1.11) around the origin is the linear Hamiltonian system above (1.8). Hence if A, B is stabilizable and C, A is detectable, then there is an n dimensional local stable manifold around the origin [15]. Moreover, this submanifold is the graph of the gradient of the optimal cost,

$$
\lambda = \frac{\partial \pi}{\partial x}(x).
$$

Hence the HJB PDE (1.3), (1.4) is locally solvable. The details can be found in Lukes [22].

In this paper we show that the HJB PDE is locally solvable in certain situations where the linear part of the system is not stabilizable or not detectable. Such systems arise naturally in the problems of nonlinear regulation, disturbance rejection, gain scheduling, and linear parameter varying control. In these problems there tends to be certain modes of the linearized system at the origin which are neutrally stable, uncontrollable, and/or unobservable. But fortunately these modes tend to be sufficiently separated from the others or can be made so by feedforward from the exosystem state so that an extension of the stable manifold theorem can be used to prove the local solvability of the HJB PDE. We proved this extension, which we call the stable and partial center manifold theorem, only to learn that a similar result had already been shown by Aulbach, Flockerzi, and Knobloch [6] and Aulbach and Flockerzi [7]. Since their result is not well known and may not be readily available, we include our proof. We also prove an additional result that the Taylor series of the stable and partial center manifold can be computed term-by-term. This justifies the term-by-term solution of the HJB PDE in these situations in the spirit of Al'brecht [2].

The rest of the paper is organized as follows. In the next section we introduce the problems of nonlinear regulation, disturbance rejection, gain scheduling, and linear parameter varying control and discuss when they can be transformed so that a local solution of the HJB equation exists. In section 4 we state and prove two theorems, the stable and partial center manifold theorem and a theorem on its term-by-term development. In section 5 we show how these theorems can be used to prove the local solvability of the HJB PDE and to construct approximate solutions. In the last section we discuss the local solvability of HJI PDEs and how they arise in H_∞ extensions of the above problems.

2. Nonlinear regulation and related problems.

Consider a smooth nonlinear plant

$$
\begin{aligned}
\dot{x} &= f(x, u, \bar{x}) \\
&= Ax + Bu + F\bar{x} \\
&\quad + f^{[2]}(x, u, \bar{x}) + O(x, u, \bar{x})^3, \\
e &= h(x, u, \bar{x}) \\
&= Cx + Du + H\bar{x} \\
&\quad + h^{[2]}(x, u, \bar{x}) + O(x, u, \bar{x})^3
\end{aligned} \tag{2.1}
$$

which is perturbed by a smooth nonlinear exosystem

$$\begin{aligned} \dot{\bar{x}} &= \bar{f}(\bar{x}) \\ &= \bar{A}\bar{x} + \bar{f}^{[2]}(\bar{x}) + O(\bar{x})^3, \end{aligned} \tag{2.2}$$

where superscript $[d]$ denotes terms composed of homogeneous polynomials of degree d. The dimensions of $x, u, \bar{x}, e$ are $n, m, \bar{n}, p$, respectively.

The goal of regulation is to use a combination of feedforward and feedback control $u = \alpha(x, \bar{x})$ so that the output of the plant asymptotically goes to 0,

$$e(t) \longrightarrow 0$$

for every $x(0),\ \bar{x}(0)$. The plant should also be internally stable.

The exosystem could be a system whose output we wish the plant to track (regulation), a noise source whose disturbances we wish the plant to reject (disturbance rejection), or static and/or dynamic parameters to be used for scheduling the controller of the plant (gain scheduling).

A linear parameter varying (LPV) system,

$$\begin{aligned} \dot{x} &= A(\bar{x})x + B(\bar{x})u \\ &= \left(A^{[0]} + A^{[1]}(\bar{x}) + \cdots\right) x + \left(B^{[0]} + B^{[1]}(\bar{x}) + \cdots\right) u, \\ e &= C(\bar{x})x + D(\bar{x})u \\ &= \left(C^{[0]} + C^{[1]}(\bar{x}) + \cdots\right) x + \left(D^{[0]} + D^{[1]}(\bar{x}) + \cdots\right) u, \end{aligned}$$

falls into the last category.

We make the reasonable assumptions that the linear part of the plant is stabilizable and detectable when $\bar{x} = 0$ and the linear part of the exosystem is stable. Most plants are designed to be linearly stabilizable and detectable. If the exosystem was unstable then it would probably be impossible to overcome its effect on the plant. The combined system (2.1), (2.2) is not linearly stabilizable because we have no control over the stable modes of the exosystem and some of these might not be linearly detectable.

The solution of the regulator problem is in two steps. The first is to use feedforward from the exosystem state to insure exact tracking when the initial conditions of the plant and the exosystem permit this. We are assuming that the state of the exosystem is available for measurement. The more general case, when it is not measurable, was treated in [14] and [8]. Even when the state of the exosystem is not measurable, one must find the feedforward control law that would insure exact tracking if it were measurable. We discuss only the case when $x,\ \bar{x}$ are measurable.

The linear version of the problem was solved by Francis [12] and its nonlinear generalization is due to Isidori and Byrnes [14]. One seeks $\theta(\bar{x}),\ \beta(\bar{x})$ satisfying the Francis–Byrnes–Isidori (FBI) PDE

$$\begin{aligned} f(\theta(\bar{x}), \beta(\bar{x}), \bar{x}) &= \frac{\partial \theta}{\partial \bar{x}}(\bar{x})\bar{f}(\bar{x}), \\ h(\theta(\bar{x}), \beta(\bar{x}), \bar{x}) &= 0. \end{aligned} \tag{2.3}$$

If the FBI PDE is solvable, then the control $u = \beta(\bar{x})$ makes $x = \theta(\bar{x})$ an invariant manifold of the combined system consisting of plant and exosystem. On this manifold, exact tracking occurs, $e = 0$.

One can attempt to solve the FBI equations term-by-term. Suppose

$$\theta(\bar{x}) = T\bar{x} + \theta^{[2]}(\bar{x}) + O(\bar{x})^3,$$
$$\beta(\bar{x}) = L\bar{x} + \beta^{[2]}(\bar{x}) + O(\bar{x})^3.$$

The linear part of the FBI equations are the Francis equations

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} T \\ L \end{bmatrix} - \begin{bmatrix} T \\ 0 \end{bmatrix} \bar{A} = - \begin{bmatrix} F \\ H \end{bmatrix}.$$

These equations are solvable for any F, H iff no output zero of the plant is a pole of the exosystem [13, 16, 17]. In other words, the exosystem should not excite those frequencies that the plant cannot produce.

The output zeros of the plant are those complex numbers s for which there exist complex n and p row vectors ξ and ζ such that

$$\begin{bmatrix} \xi & \zeta \end{bmatrix} \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

There may be a finite or infinite number of output zeros. For example, if $m = p$, then there are either n zeros or every s is a zero. The poles $\lambda_1, \ldots, \lambda_{\bar{n}}$ of the exosystem are the eigenvalues of $\bar{A}$.

If there is a resonance between a pole and zero, the equations will still be solvable for some F, H. The solvability depends on the direction ξ, ζ of the zero and the eigenvector of the pole.

The higher degree equations are linear and depend on the solutions of the lower degree equations. They are solvable for arbitrary higher degree terms iff the harmonics of the exosystem don't resonate with the zeros of the plant [13, 16, 17].

For example, the degree two equations are

$$\begin{aligned} &A\theta^{[2]}(\bar{x}) + B\beta^{[2]}(\bar{x}) - \frac{\partial \theta}{\partial \bar{x}}(\bar{x}) \left(\bar{A}\bar{x}\right) \\ &\qquad = -f^{[2]}(T\bar{x}, L\bar{x}, \bar{x}) + T\bar{f}^{[2]}(\bar{x}), \\ &C\theta^{[2]}(\bar{x}) + D\beta^{[2]}(\bar{x}) \\ &\qquad = -h^{[2]}(T\bar{x}, L\bar{x}, \bar{x}). \end{aligned}$$

These are solvable for arbitrary $f^{[2]}, h^{[2]}$ iff no output zero of the plant equals the sum of two poles of the exosystem, $\lambda_k \neq s_i + s_j$. If there is a resonance, they are solvable for some $f^{[2]}, h^{[2]}$. Matlab-based software is available on the web to compute the solution of the FBI PDE to any degree using the function `fbi.m` of the Nonlinear Systems Toolbox [20].

Now suppose that the FBI equations have been solved. The second step is to use additional feedforward and feedback to insure that the closed loop system converges to the tracking manifold $x = \theta(\bar{x})$ where $e = 0$. This can be achieved locally by linear pole placement techniques [14], but an alternative approach is to use optimal control methods to achieve a nonlinear solution [16, 17]. Define transverse coordinates z, v by

$$\begin{aligned} z &= x - \theta(\bar{x}) = x - T\bar{x} - \theta^{[2]}(\bar{x}) + O(\bar{x})^3, \\ v &= u - \beta(\bar{x}) = u - L\bar{x} - \beta^{[2]}(\bar{x}) + O(\bar{x})^3. \end{aligned} \tag{2.4}$$

In these coordinates the plant and exosystem are of the form

$$\begin{aligned} \dot{z} &= \tilde{f}(z,v,\bar{x}) = Ax + Bu + \tilde{f}^{[2]}(z,v,\bar{x}) + O(z,v,\bar{x})^3, \\ \dot{\bar{x}} &= \bar{f}(\bar{x}) = \bar{A}\bar{x} + \bar{f}^{[2]}(\bar{x}) + O(\bar{x})^3, \\ e &= \tilde{h}(z,v,\bar{x}) = Cz + Dv + \tilde{h}^{[2]}(z,v,\bar{x}) + O(z,v,\bar{x})^3, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} \tilde{f}(z,v,\bar{x}) &= f(z+\theta(\bar{x}), v+\beta(\bar{x}), \bar{x}) - f(\theta(\bar{x}), \beta(\bar{x}), \bar{x}), \\ \tilde{h}(z,v,\bar{x}) &= h(z+\theta(\bar{x}), v+\beta(\bar{x}), \bar{x}). \end{aligned} \tag{2.6}$$

Notice that the linear part of the z dynamics and the linear part of the output are unaffected by $\bar{x}$. Recall we have assumed that the linear part of the plant is stabilizable and detectable and the linear part of the exosystem is neutrally stable. Furthermore,

$$\begin{aligned} \tilde{f}(0,0,\bar{x}) &= f(\theta(\bar{x}), \beta(\bar{x}), \bar{x}) - f(\theta(\bar{x}), \beta(\bar{x}), \bar{x})) = 0, \\ \tilde{h}(0,0,\bar{x}) &= h(\theta(\bar{x}), \beta(\bar{x}), \bar{x}) = 0. \end{aligned} \tag{2.7}$$

A stabilizing feedback can be found by minimizing

$$\tfrac{1}{2}\int_0^\infty \|e\|^2 + \|v\|^2 dt \tag{2.8}$$

subject to the dynamics (2.5). Other cost criterions l can be used as long as they satisfy (2.11). In particular a cost criterion like $x'Qx + u'Ru$ should not be used as then (2.11) will not hold. Intuitively, one should not cost the part of the state and the control that are necessary to achieve exact tracking. Even in the linear case, there is considerable confusion on this point, e.g., [3].

Let $\pi(z,\bar{x})$ denote the optimal cost and $\gamma(z,\bar{x})$ the optimal feedback; then $\pi,\ \gamma$ satisfy the HJB PDE

$$\begin{aligned} 0 &= \frac{\partial \pi}{\partial z}(z,\bar{x})\tilde{f}(z,\gamma(z,\bar{x}),\bar{x}) + \frac{\partial \pi}{\partial \bar{x}}(z,\bar{x})\bar{f}(\bar{x}) \\ &\quad + l(z,\gamma(z,\bar{x}),\bar{x}), \\ 0 &= \frac{\partial \pi}{\partial z}(z,\bar{x})\frac{\partial \tilde{f}}{\partial v}(z,\gamma(z,\bar{x}),\bar{x}) + \frac{\partial l}{\partial v}(z,\gamma(z,\bar{x}),\bar{x}), \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} l(z,v,\bar{x}) &= \tfrac{1}{2}(\|e\|^2 + \|v\|^2) \\ &= \tfrac{1}{2}\left(z'Qz + 2z'Sv + v'Rv\right) \\ &\quad + l^{[3]}(z,v,\bar{x}) + O(z,v,\bar{x})^4 \end{aligned} \tag{2.10}$$

for the matrices $Q = C'C,\ S = C'D,\ R = I + D'D$, and some cubic polynomial $l^{[3]}$.

By generalizing Al'brecht's method [2], we can solve the HJB PDE term-by-term [16]. Since

$$\begin{aligned} \tilde{f}(z,v,\bar{x}) &= O(z,v), \\ \tilde{h}(z,v,\bar{x}) &= O(z,v), \\ l(z,v,\bar{x}) &= O(z,v)^2, \end{aligned} \tag{2.11}$$

we expect that

$$\begin{array}{rcl} \pi(z,\bar{x}) & = & O(z)^2, \\ \gamma(z,\bar{x}) & = & O(z). \end{array} \tag{2.12}$$

In particular, we expect that

$$\pi(z,\bar{x}) = \frac{1}{2}z'Pz + \pi^{[3]}(z,\bar{x}) + O(z,\bar{x})^4,$$
$$\gamma(z,\bar{x}) = Kz + \gamma^{[2]}(z,\bar{x}) + O(z,\bar{x})^3.$$

The lowest degree terms in the HJB equations are the familiar Riccati equation and the formula for the optimal linear feedback

$$\begin{array}{rcl} 0 & = & A'P + PA + Q - (PB+S)R^{-1}(PB+S)', \\ K & = & -R^{-1}(PB+S)'. \end{array} \tag{2.13}$$

At each higher degree $d > 1$, the equations are linear in the unknowns $\pi^{[d+1]}$, $\gamma^{[d]}$ and depend on the lower order terms of the solution. They are solvable if the linear part of the plant is stabilizable and the linear part of the exosystem is stable. For example, to find the next terms $\pi^{[3]}(z,\bar{x})$, $\gamma^{[2]}(z,\bar{x})$, one plugs the first two terms of π, γ into HJB equations and collects the next terms (degree 3 from the first HJB equation and degree 2 from the second HJB equation)

$$\begin{aligned} 0 = {} & \frac{\partial \pi^{[3]}}{\partial z}(z,\bar{x})(A+BK)z + \frac{\partial \pi^{[3]}}{\partial \bar{x}}(z,\bar{x})(\bar{A}\bar{x}) \\ & + z'P\tilde{f}^{[2]}(z,Kz,\bar{x}) + l^{[3]}(z,Kz,\bar{x}), \\ 0 = {} & \frac{\partial \pi^{[3]}}{\partial z}(z,\bar{x})B + z'P\frac{\partial \tilde{f}^{[2]}}{\partial v}(z,Kz,\bar{x}) \\ & + \gamma^{[2]}(z,\bar{x})'R + \frac{\partial l^{[3]}}{\partial v}(z,Kz,\bar{x}). \end{aligned} \tag{2.14}$$

Notice that the first equation involves only $\pi^{[3]}$, the other unknown $\gamma^{[2]}$ does not appear. This equation is solvable if $A+BK$ is asymptotically stable and $\bar{A}$ is stable. This follows from the fact that the mapping

$$\pi^{[3]}(z,\bar{x}) \mapsto \frac{\partial \pi^{[3]}}{\partial z}(z,\bar{x})(A+BK)z + \frac{\partial \pi^{[3]}}{\partial \bar{x}}(z,\bar{x})(\bar{A}\bar{x})$$

is a linear operator on cubic polynomials. It is not hard to see that its eigenvalues are the sum of three eigenvalues of $A+BK$ and $\bar{A}$. The operator restricts to a linear operator on the subspace of $\pi^{[3]}(z,\bar{x})$ satisfying (2.12), where its eigenvalues are the sum of three eigenvalues of $A+BK$ or the sum of two eigenvalues of $A+BK$ and one eigenvalue of $\bar{A}$. Since the eigenvalues of $A+BK$ are in the open left half plane and those of $\bar{A}$ are in the closed left half plane, the restricted operator is invertible and the first equation of (2.14) is always solvable. We discuss this further in the proof of Theorem 4.2.

Given the solution $\pi^{[3]}$, we can then solve the second equation for $\gamma^{[2]}$

$$\gamma^{[2]}(z,\bar{x}) = -R^{-1}\left(\frac{\partial \pi^{[3]}}{\partial z}(z,\bar{x})B + z'P\frac{\partial \tilde{f}^{[2]}}{\partial v}(z,Kz,\bar{x}) + \frac{\partial l^{[3]}}{\partial v}(z,Kz,\bar{x})\right)'.$$

The higher degree terms are found in a similar fashion. Matlab-based software is available on the web to compute the solution of the HJB PDE to any degree using the function `hjb.m` in the Nonlinear Systems Toolbox [20]. If one wants to solve the FBI PDE and then the HJB PDE in the transverse coordinates, use the function `mdl_mtch.m`.

Given the solutions of the FBI and HJB equations, the desired feedforward/feedback is

$$\begin{aligned} u &= \alpha(x, \bar{x}) \\ &= \beta(\bar{x}) + \gamma(x - \theta(\bar{x}), \bar{x}). \end{aligned}$$

Of course the above discussion is formal. We shall show using results from [6], [7] that the HJB PDE (2.9) is locally solvable. Furthermore, its Taylor series expansion can be computed term-by-term as described above. To do so we shall use an invariant manifold theorem that we shall discuss in the next section. In section 4 we use this theorem to show the local existence of the solution to the HJB equation (2.9).

Suppose one has computed approximate solutions to the FBI PDE up to degree d and the HJB PDE up to degree $d+1$, and one has the desired $\alpha(x, \bar{x})$ up to degree d. Despite the formal nature of these, one can explicitly verify where it gives the desired solution. The function $\pi(x - \theta(\bar{x}, \bar{x}))$ is a potential Lyapunov function for the approximate tracking manifold $x = \theta(\bar{x})$ on which the error $e = O(\bar{x})^{d+1}$. Using this and the true closed loop dynamics, one can estimate the basin of attraction of the approximate tracking manifold.

3. Stable and partial center manifold theorem. The following theorem was proven by Aulbach, Flockerzi, and Knobloch [6] and Aulbach and Flockerzi [7]. We were unaware of their work and suspected that such a theorem must hold because of the formal discussion of the last section. We present our independent proof because [6] and [7] are not widely known nor readily available. Moreover, Theorem 3.2 is new and its proof depends on the proof of Theorem 3.1.

THEOREM 3.1 (see [6], [7]). *Given an ODE of the form*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix}, \tag{3.1}$$

where $x_i \in \mathbf{R}^{\mathbf{n_i}}$, $n = n_1 + n_2 + n_3$, *and* $f_i(x)$ *is* C^k *for* $k \geq 2$. *Suppose that*

(3.2) *the eigenvalues of* A_1 *have negative real part,*

(3.3) *the eigenvalues of* A_2 *have nonnegative real part,*

(3.4) *the eigenvalues of* A_3 *have nonpositive real part,*

$$f_i(0,0,0) = 0, \ i = 1, 2, 3, \tag{3.5}$$

$$\frac{\partial f_i}{\partial x_j}(0,0,0) = 0, \ i, j = 1, 2, 3, \tag{3.6}$$

$$f_i(0, x_2, 0) = 0, \ i = 1, 3. \tag{3.7}$$

Then there exists, around $x = (0,0,0)$, *a local* C^{k-2} *invariant manifold*

$$x_3 = \phi(x_1, x_2), \tag{3.8}$$

where

$$\phi(0, x_2) = 0, \tag{3.9}$$

$$\frac{\partial \phi}{\partial x}(0, 0) = 0 \textit{ if } k > 2. \tag{3.10}$$

Remarks. The condition (3.7) implies that $\{x_1 = 0,\ x_3 = 0\}$ is an invariant manifold. When the spectrum of A_2 lies on the imaginary axis, we call this a partial center manifold as it corresponds to only some of the eigenvalues on the imaginary axis. We call (3.8) a local stable and partial center manifold because it contains the local stable manifold and part of the local center manifold. The partial center manifold provides the needed gap between the eigenvalues that are associated to the invariant manifold and those that are not. The necessity of the existence of the partial center manifold to the existence of the stable and partial center manifold could be argued as follows. If the stable and partial center manifold exists, then its intersection with the center manifold should yield the partial center manifold. The flaw in this argument is that center manifolds are not necessarily unique and the intersection of manifolds is not necessarily a manifold [5], [9]. Still it is plausible. In the above theorem there is a loss of smoothness, from a C^k dynamics to a C^{k-2} local stable and partial center manifold. This is probably an artifact of the proof. In the stable manifold theorem and the theorem of Aulbach and Flockerzi [7] there is no loss of smoothness, and in the center manifold theorem there is a loss of smoothness from C^k to C^{k-1}.

Proof. The first step to make suitable linear changes of coordinates on each of the three subspaces so that there exist $\alpha \geq 16\beta > 0$ such that for all $x_1,\ x_2,\ x_3$

$$\begin{aligned} x_1' A_1 x_1 &\leq -\alpha |x_1|^2, \\ x_2' A_2 x_2 &\leq \beta |x_2|^2, \\ -x_3' A_3 x_3 &\leq \beta |x_3|^2. \end{aligned}$$

This is possible by Lemma 1 of [15].

The next step is to use a cut-off function to redefine f. Let $\nu(x)$ be a scalar valued C^∞ function, $0 \leq \nu(x) \leq 1$, $\nu(x) = 1$ for $0 \leq |x| \leq 1$, and $\nu(x) = 0$ for $|x| \geq 2$. For any $\epsilon > 0$, define

$$f(x; \epsilon) := f(\nu(x/\epsilon)x).$$

Since $f(x; \epsilon)$ agrees with $f(x)$ for $|x| \leq \epsilon$, it suffices to prove the theorem for some $\epsilon > 0$.

Next we show that there exists a continuous function $k(\epsilon)$ with $k(0) = 0$ and a constant $K > 0$ such that for all $x,\ \bar{x} \in \mathbf{R^n}$ and for $i = 1, 2, 3$

$$|f_i(x; \epsilon) - f_i(\bar{x}; \epsilon)|^2 \leq k^2(\epsilon)|x - \bar{x}|^2 \tag{3.11}$$

and for $i = 1, 3$

$$|f_i(x; \epsilon) - f_i(\bar{x}; \epsilon)|^2 \leq k^2(\epsilon) \left| \begin{array}{c} x_1 - \bar{x}_1 \\ x_3 - \bar{x}_3 \end{array} \right|^2 + K \left| \begin{array}{c} \bar{x}_1 \\ \bar{x}_3 \end{array} \right|^2 |x_2 - \bar{x}_2|^2. \tag{3.12}$$

Note that

$$|f_i(x; \epsilon) - f_i(\bar{x}; \epsilon)|^2 \leq \left| \frac{\partial f_i}{\partial x}(\xi; \epsilon) \right|^2 |x - \bar{x}|^2,$$

where ξ is some point on the line between x and $\bar{x}$. Also for $i = 1, 3$

$$\begin{aligned}
|f_i(x;\epsilon) - f_i(\bar{x};\epsilon)|^2 &\le |f_i(x;\epsilon) - f_i(\bar{x}_1, x_2, \bar{x}_3;\epsilon) + f_i(\bar{x}_1, x_2, \bar{x}_3;\epsilon) - f_i(\bar{x};\epsilon)|^2 \\
&\le 2|f_i(x;\epsilon) - f_i(\bar{x}_1, x_2, \bar{x}_3;\epsilon)|^2 + 2|f_i(\bar{x}_1, x_2, \bar{x}_3;\epsilon) - f_i(\bar{x};\epsilon)|^2 \\
&\le 2\left|\frac{\partial f_i}{\partial x}(\xi_1, x_2, \xi_3;\epsilon)\right|^2 \left|\begin{array}{c} x_1 - \bar{x}_1 \\ 0 \\ x_3 - \bar{x}_3 \end{array}\right|^2 \\
&\quad +2\left|\frac{\partial f_i}{\partial x_2}(\bar{x}_1, \xi_2, \bar{x}_3;\epsilon)\right|^2 |x_2 - \bar{x}_2|^2,
\end{aligned}$$

where (ξ_1, x_2, ξ_3) is some point on the line between x and $(\bar{x}_1, x_2, \bar{x}_3)$ and $(\bar{x}_1, \xi_2, \bar{x}_3)$ is some point on the line between $(\bar{x}_1, x_2, \bar{x}_3)$ and $\bar{x}$. Furthermore,

$$\left|\frac{\partial f_i}{\partial x_2}(\bar{x}_1, \xi_2, \bar{x}_3;\epsilon)\right|^2 \le \left|\frac{\partial^2 f_i}{\partial x \partial x_2}(\xi;\epsilon)\right|^2 \left|\begin{array}{c} \bar{x}_1 \\ 0 \\ \bar{x}_3 \end{array}\right|^2,$$

where ξ is some point on the line between $(0, \xi_2, 0)$ and $(\bar{x}_1, \xi_2, \bar{x}_3)$.

Now $\nu(x)$ and its partials are continuous functions with compact support so there exists a constant M such that

$$\begin{aligned}
\left|\frac{\partial \nu}{\partial x}(x)\right| &\le M, \\
\left|\frac{\partial^2 \nu}{\partial x^2}(x)\right| &\le M
\end{aligned}$$

for all x. Since f_i satisfies (3.6) we can choose M large enough so that

$$\begin{aligned}
\left|\frac{\partial f_i}{\partial x}(x)\right| &\le M|x|, \\
\left|\frac{\partial^2 f_i}{\partial x^2}(x)\right| &\le M
\end{aligned}$$

for all $|x| \le 1$. Then for $0 < \epsilon < 1/2$

$$\begin{aligned}
\left|\frac{\partial f_i}{\partial x}(x;\epsilon)\right| &\le \left|\frac{\partial f_i}{\partial x}(\nu(x/\epsilon)x)\right| \left|\nu(x/\epsilon) + \frac{\partial \nu}{\partial x}(x/\epsilon)\frac{x}{\epsilon}\right| \\
&\le 2M(1+2M)\epsilon,
\end{aligned}$$

$$\begin{aligned}
\left|\frac{\partial^2 f_i}{\partial x^2}(x;\epsilon)\right| &\le \left|\frac{\partial^2 f_i}{\partial x^2}(\nu(x/\epsilon)x)\right| \left|\nu(x/\epsilon) + \frac{\partial \nu}{\partial x}(x/\epsilon)\frac{x}{\epsilon}\right|^2 \\
&\quad + \left|\frac{\partial f_i}{\partial x}(\nu(x/\epsilon)x)\right| \left|\frac{\partial \nu}{\partial x}(x/\epsilon)\frac{2}{\epsilon} + \frac{\partial^2 \nu}{\partial x^2}(x/\epsilon)\frac{x}{\epsilon^2}\right| \\
&\le M(1+2M)^2 + 8M^2.
\end{aligned}$$

Let

$$\begin{aligned}
k^2(\epsilon) &= 2\left(2M(1+2M)\epsilon\right)^2, \\
K &= 2\left(M(1+2M)^2 + 8M^2\right)^2;
\end{aligned}$$

then $k(\epsilon)$ is continuous and goes to 0 as ϵ goes to 0.

Henceforth we suppress the ϵ and write $f(x)$ for $f(x;\epsilon)$.

Let $k_1,\ k_2$ be any positive constants and X denote the space of all Lipschitz continuous functions $\phi(x_1,x_2)$ defined for $|x_1|<\epsilon$ and any x_2 such that

$$\phi(0,x_2)=0, \tag{3.13}$$

$$|\phi(x_1,x_2)-\phi(\bar{x}_1,\bar{x}_2)|^2 \le k_1|x_1-\bar{x}_1|^2+k_2|\bar{x}_1|\ |x_2-\bar{x}_2|^2. \tag{3.14}$$

Taking $\bar{x}_1=0$, these imply that

$$|\phi(x_1,x_2)|^2 \le k_1|x_1|^2$$

so we can define

$$\|\phi\|^2=\sup\left\{\frac{|\phi(x_1,x_2)|^2}{|x_1|} : |x_1|<\epsilon\right\}. \tag{3.15}$$

With this norm, X is a complete space.

For $|x_1|<\epsilon,\ x_2\in \mathbf{R^{n_2}}$, and $\phi\in X$, define

$$\xi_i(t)=\xi_i(t;x_1,x_2,\phi)$$

for $i=1,2$ to be the solution of

$$\dot{\xi}_i=A_i\xi_i+f_i(\xi_1,\xi_2,\phi(\xi_1,\xi_2)), \tag{3.16}$$

$$\xi_i(0)=x_i. \tag{3.17}$$

Define a mapping T on X as follows:

$$(T\phi)(x_1,x_2)=\int_\infty^0 e^{-A_3 s} f_3(\xi_1(s),\xi_2(s),\phi(\xi_1(s),\xi_2(s)))\ ds. \tag{3.18}$$

We would like to show that for ϵ sufficiently small, T is a contraction on X.

Suppose $x_1=0$; then $\xi_1(t)=0$ because of (3.7) and for the same reason

$$(T\phi)(0,x_2)=0$$

so $T\phi$ satisfies (3.13).

Suppose $\phi,\ \bar{\phi}\in X$; then for any $|x_1|<\epsilon,\ x_2\in\mathbf{R^{n_2}},\ |\bar{x}_1|<\epsilon,\ \bar{x}_2\in\mathbf{R^{n_2}}$, and $x_3=\phi(x_1,x_2),\ \bar{x}_3=\bar{\phi}(\bar{x}_1,\bar{x}_2)$, then by the above for $i=1,2,3$

$$\begin{aligned}|f_i(x)-f_i(\bar{x})|^2 &\le 2|f_i(x)-f_i(\bar{x}_1,\bar{x}_2,\phi(\bar{x}_1,\bar{x}_2))|^2\\ &\quad +2|f_i(\bar{x}_1,\bar{x}_2,\phi(\bar{x}_1,\bar{x}_2),)-f_i(\bar{x})|^2\\ &\le 2k^2(\epsilon)\left((1+k_1)|x_1-\bar{x}_1|^2+(1+k_2\epsilon)\,|x_2-\bar{x}_2|^2+\epsilon\|\phi-\bar{\phi}\|^2\right)\end{aligned} \tag{3.19}$$

and for $i=1,3$

$$\begin{aligned}|f_i(x)-f_i(\bar{x})|^2 &\le 2|f_i(x)-f_i(\bar{x}_1,\bar{x}_2,\phi(\bar{x}_1,\bar{x}_2))|^2\\ &\quad +2|f_i(\bar{x}_1,\bar{x}_2,\phi(\bar{x}_1,\bar{x}_2))-f_i(\bar{x})|^2\\ &\le l_1(\epsilon)|x_1-\bar{x}_1|^2+l_2(\epsilon)|\bar{x}_1|\ |x_2-\bar{x}_2|^2+l_3(\epsilon)|\bar{x}_1|\ \|\phi-\bar{\phi}\|^2,\end{aligned} \tag{3.20}$$

where the functions

$$\begin{aligned} l_1(\epsilon) &= 2k^2(\epsilon)(1+k_1), \\ l_2(\epsilon) &= 2\left(k^2(\epsilon)k_2 + K(1+k_1)\epsilon\right), \\ l_3(\epsilon) &= 2k^2(\epsilon) \end{aligned}$$

go to 0 as $\epsilon \to 0$.

Suppose $\xi_i(t),\ \bar{\xi}_i(t)$ for $i = 1, 2$ are the solutions of

$$\begin{aligned} \dot{\xi}_i &= A_i\xi_i + f_i(\xi_1, \xi_2, \phi(\xi_1, \xi_2)), \\ \xi_i(0) &= x_i, \end{aligned}$$

$$\begin{aligned} \dot{\bar{\xi}}_i &= A_i\bar{\xi}_i + f_i(\bar{\xi}_1, \bar{\xi}_2, \bar{\phi}(\bar{\xi}_1, \bar{\xi}_2)), \\ \bar{\xi}_i(0) &= \bar{x}_i \end{aligned}$$

and $\xi_3(t) = \phi(\xi_1(t), \xi_2(t)),\ \bar{\xi}_3(t) = \bar{\phi}(\bar{\xi}_1(t), \bar{\xi}_2(t))$.

Then since $2ab \leq a^2 + b^2$ and $(a+b)^2 \leq 2a^2 + 2b^2$

$$\begin{aligned} &\frac{d}{dt}\frac{|\xi_1 - \bar{\xi}_1|^2 + |\xi_2 - \bar{\xi}_2|^2}{2} \\ &\leq (\xi_1 - \bar{\xi}_1)'\left(A_1(\xi_1 - \bar{\xi}_1) + f_1(\xi) - f_1(\bar{\xi})\right) \\ &\quad + (\xi_2 - \bar{\xi}_2)'\left(A_2(\xi_2 - \bar{\xi}_2) + f_2(\xi) - f_2(\bar{\xi})\right) \\ &\leq -\alpha|\xi_1 - \bar{\xi}_1|^2 + \beta|\xi_2 - \bar{\xi}_2|^2 + k(\epsilon)\left(|\xi_1 - \bar{\xi}_1| + |\xi_2 - \bar{\xi}_2|\right) \\ &\quad \left[2(1+k_1)|\xi_1 - \bar{\xi}_1|^2 + 2(1+k_2\epsilon)|\xi_2 - \bar{\xi}_2|^2 + 2\epsilon\|\phi - \bar{\phi}\|^2\right]^{\frac{1}{2}} \\ &\leq \left(-\alpha + k(\epsilon)(2+k_1)\right)|\xi_1 - \bar{\xi}_1|^2 + \left(\beta + k(\epsilon)(2+k_2\epsilon)\right)|\xi_2 - \bar{\xi}_2|^2 \\ &\quad + k(\epsilon)\epsilon\|\phi - \bar{\phi}\|^2. \end{aligned}$$

We assume ϵ is small enough so that

$$\begin{aligned} -\alpha + k(\epsilon)(2+k_1) &\leq 2\beta \\ k(\epsilon)(2+k_2\epsilon) &\leq \beta \\ k(\epsilon)\epsilon &\leq 2\beta; \end{aligned}$$

then

$$\frac{d}{dt}\left(|\xi_1 - \bar{\xi}_1|^2 + |\xi_2 - \bar{\xi}_2|^2\right) \leq 4\beta\left(|\xi_1 - \bar{\xi}_1|^2 + |\xi_2 - \bar{\xi}_2|^2 + \|\phi - \bar{\phi}\|^2\right)$$

and by Gronwall's inequality

$$|\xi_1(t) - \xi_1(t)|^2 + |\xi_2(t) - \xi_2(t)|^2 \leq e^{4\beta t}\left(|x_1 - \bar{x}_1|^2 + |x_2 - \bar{x}_2|^2 + \|\phi - \bar{\phi}\|^2\right). \tag{3.21}$$

With this inequality in hand we can obtain a stricter one by using (3.20) instead of (3.19).

$$\frac{d}{dt}\frac{|\xi_1 - \bar{\xi}_1|^2}{2} \leq -\alpha|\xi_1 - \bar{\xi}_1|^2 + |\xi_1 - \bar{\xi}_1| \left[l_1(\epsilon)|\xi_1 - \bar{\xi}_1|^2 + l_2(\epsilon)|\bar{\xi}_1|\ |\xi_2 - \bar{\xi}_2|^2 + l_3(\epsilon)|\bar{\xi}_1|\ \|\phi - \bar{\phi}\|^2\right]^{\frac{1}{2}}. \tag{3.22}$$

Now suppose that $\bar{x}_1 = 0,\ \bar{x}_2 = 0$ so that $\bar{\xi}_1 = 0,\ \bar{\xi}_2 = 0$; then

$$\frac{d}{dt}\frac{|\xi_1|^2}{2} \le \left(-\alpha + l_1^{\frac{1}{2}}(\epsilon)\right)|\xi_1|^2.$$

If ϵ is small enough so that

$$l_1^{\frac{1}{2}}(\epsilon) \le \frac{\alpha}{2},$$

then by Gronwall

$$|\xi_1(t)|^2 \le e^{-\alpha t}|x_1|^2. \tag{3.23}$$

For $a,\ b,\ c \ge 0$ we have $\sqrt{a+b+c} \le \sqrt{a} + \sqrt{b} + \sqrt{c}$ so (3.22) becomes

$$\begin{aligned}\frac{d}{dt}\frac{|\xi_1-\bar{\xi}_1|^2}{2} &\le \left(-\alpha + l_1^{\frac{1}{2}}(\epsilon)\right)|\xi_1-\bar{\xi}_1|^2\\ &\quad + l_2^{\frac{1}{2}}(\epsilon)|\xi_1-\bar{\xi}_1|\ |\bar{\xi}_1|^{\frac{1}{2}}|\xi_2-\bar{\xi}_2|\\ &\quad + l_2^{\frac{1}{2}}(\epsilon)|\xi_1-\bar{\xi}_1|\ |\bar{\xi}_1|^{\frac{1}{2}}\|\phi-\bar{\phi}\|\\ &\le \left(-\alpha + l_1^{\frac{1}{2}}(\epsilon) + l_2^{\frac{1}{2}}(\epsilon) + l_3^{\frac{1}{2}}(\epsilon)\right)|\xi_1-\bar{\xi}_1|^2\\ &\quad + l_2^{\frac{1}{2}}(\epsilon)|\bar{\xi}_1|\ |\xi_2-\bar{\xi}_2|^2\\ &\quad + l_3^{\frac{1}{2}}(\epsilon)|\bar{\xi}_1|\ \|\phi-\bar{\phi}\|^2.\end{aligned}$$

Assume ϵ is small enough so that

$$l_1^{\frac{1}{2}}(\epsilon) + l_2^{\frac{1}{2}}(\epsilon) + l_3^{\frac{1}{2}}(\epsilon) \le \frac{\alpha}{2},$$

then using (3.23), (3.21), this becomes

$$\begin{aligned}\frac{d}{dt}\frac{|\xi_1-\bar{\xi}_1|^2}{2} &\le -\frac{\alpha}{2}|\xi_1-\bar{\xi}_1|^2\\ &\quad + l_2^{\frac{1}{2}}(\epsilon)e^{(4\beta-\frac{\alpha}{2})t}|\bar{x}_1|\left(|x_1-\bar{x}_1|^2 + |x_2-\bar{x}_2|^2 + \|\phi-\bar{\phi}\|^2\right)\\ &\quad + l_3^{\frac{1}{2}}(\epsilon)e^{-\frac{\alpha}{2}t}|\bar{x}_1|\ \|\phi-\bar{\phi}\|^2.\end{aligned}$$

Since $16\beta \le \alpha$,

$$\begin{aligned}\frac{d}{dt}|\xi_1-\bar{\xi}_1|^2 &\le -\alpha|\xi_1-\bar{\xi}_1|^2\\ &\quad + 2l_2^{\frac{1}{2}}(\epsilon)e^{-\frac{\alpha}{4}t}|\bar{x}_1|\left(|x_1-\bar{x}_1|^2 + |x_2-\bar{x}_2|^2\right)\\ &\quad + 2(l_2^{\frac{1}{2}}(\epsilon) + l_3^{\frac{1}{2}}(\epsilon))e^{-\frac{\alpha}{4}t}|\bar{x}_1|\ \|\phi-\bar{\phi}\|^2\end{aligned}$$

so by Gronwall

$$\begin{aligned}|\xi_1(t)-\bar{\xi}_1(t)|^2 &\le e^{-\alpha t}|x_1-\bar{x}_1|^2\\ &\quad + \frac{8}{3}l_2^{\frac{1}{2}}(\epsilon)e^{-\frac{\alpha}{4}t}|\bar{x}_1|\left(|x_1-\bar{x}_1|^2 + |x_2-\bar{x}_2|^2\right)\\ &\quad + \frac{8}{3}(l_2^{\frac{1}{2}}(\epsilon) + l_3^{\frac{1}{2}}(\epsilon))e^{-\frac{\alpha}{4}t}|\bar{x}_1|\ \|\phi-\bar{\phi}\|^2.\end{aligned} \tag{3.24}$$

Next we use (3.20) to estimate

$$
\begin{aligned}
\left|(T\phi)(x_1,x_2)-(T\bar\phi)(\bar x_1,\bar x_2)\right|^2 &= \left|\int_0^\infty e^{-A_3 s}\left(f_3(\xi(s))-f_3(\bar\xi(s))\right)\,ds\right|^2\\
&\le \int_0^\infty e^{2\beta s}\left|f_3(\xi(s))-f_3(\bar\xi(s))\right|^2\,ds\\
&\le \int_0^\infty e^{2\beta s}\left[l_1(\epsilon)|\xi_1(s)-\bar\xi_1(s)|^2\right.\\
&\qquad +l_2(\epsilon)|\bar\xi_1(s)|\;|\xi_2(s)-\bar\xi_2(s)|^2\\
&\qquad \left.+l_3(\epsilon)|\bar\xi_1(s)|\;\|\phi-\bar\phi\|^2\right]\,ds.
\end{aligned}
$$

From (3.21), (3.23), (3.24) and $16\beta\le\alpha$

$$
\begin{aligned}
&\left|(T\phi)(x_1,x_2)-(T\bar\phi)(\bar x_1,\bar x_2)\right|^2 \le \int_0^\infty e^{2\beta s}\Big[l_1(\epsilon)\big(\,e^{-\alpha s}|x_1-\bar x_1|^2\\
&\quad +\frac{8}{3}l_2^{\frac12}(\epsilon)e^{-\frac{\alpha}{4}s}|\bar x_1|\left(|x_1-\bar x_1|^2+|x_2-\bar x_2|^2\right)\\
&\quad +\frac{8}{3}\left(l_2^{\frac12}(\epsilon)+l_3^{\frac12}(\epsilon)\right)e^{-\frac{\alpha}{4}s}|\bar x_1|\;\|\phi-\bar\phi\|^2\,\big)\\
&\quad +l_2(\epsilon)e^{-\frac{\alpha}{2}s}|\bar x_1|\;e^{4\beta s}\left(|x_1-\bar x_1|^2+|x_2-\bar x_2|^2+\|\phi-\bar\phi\|^2\right)\\
&\quad +l_3(\epsilon)e^{-\frac{\alpha}{2}s}|\bar x_1|\;\|\phi-\bar\phi\|^2\Big]\,ds\\
&\quad \le m_1(\epsilon)|x_1-\bar x_1|^2+m_2(\epsilon)|\bar x_1|\;|x_2-\bar x_2|^2\\
&\quad +m_3(\epsilon)|\bar x_1|\;\|\phi-\bar\phi\|^2,
\end{aligned}
\tag{3.25}
$$

where

$$
\begin{aligned}
m_1(\epsilon)&=l_1(\epsilon)\left(\frac{8}{7\alpha}+\frac{64}{3\alpha}l_2^{\frac12}(\epsilon)\epsilon\right)+\frac{8}{\alpha}l_2(\epsilon)\epsilon,\\
m_2(\epsilon)&=\frac{64}{3\alpha}l_1(\epsilon)l_2^{\frac12}(\epsilon)+\frac{8}{\alpha}l_2(\epsilon),\\
m_3(\epsilon)&=\frac{64}{3\alpha}l_1(\epsilon)\left(l_2^{\frac12}(\epsilon)+l_3^{\frac12}(\epsilon)\right)+\frac{8}{\alpha}l_2(\epsilon)+\frac{8}{3\alpha}l_3(\epsilon).
\end{aligned}
$$

Notice that $m_i(\epsilon)\to 0$ as $\epsilon\to 0$.

By letting $\bar\phi=\phi$ we see that

$$
\begin{aligned}
|(T\phi)(x_1,x_2)-(T\phi)(\bar x_1,\bar x_2)|^2 &\le 2m_1(\epsilon)|x_1-\bar x_1|^2\\
&\quad +2m_2(\epsilon)|\bar x_1|\;|x_2-\bar x_2|^2,
\end{aligned}
$$

so $(T\phi)(x_1,x_2)$ satisfies (3.14) for ϵ sufficiently small and T maps X to X.

By letting $\bar x=x$ we see that

$$
\left|(T\phi)(x_1,x_2)-(T\bar\phi)(x_1,x_2)\right|^2\le m_3(\epsilon)|\bar x_1|\;\|\phi-\bar\phi\|^2,
$$

so $T : X \to X$ is a contraction for ϵ sufficiently small. Hence there exists a unique $\phi \in X$ such that

$$\phi = T\phi.$$

Let $\xi_i(t)$ satisfy (3.16), (3.17) for $i = 1, 2$ and $\xi_3(t) = \phi(\xi_1(t), \xi_2(t))$. By the definition of T (3.18),

$$\begin{aligned}\xi_3(t) &= (T\phi)(\xi_1(t), \xi_2(t))\\ &= \int_\infty^0 e^{-A_3 s} f_3(\xi(t+s))\, ds\\ &= \int_\infty^t e^{-A_3(t-s)} f_3(\xi(s))\, ds,\end{aligned}$$

so $\xi(t)$ is a solution of the differential equation (3.1) and (3.8) defines a C^0 invariant manifold.

Now suppose $k > 2$. We wish to show that the invariant manifold (3.8) is C^1. Consider the dynamics tangent to (3.1),

$$\begin{bmatrix} \dot z_1 \\ \dot z_2 \\ \dot z_3 \end{bmatrix} = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} g_1(x,z) \\ g_2(x,z) \\ g_3(x,z) \end{bmatrix}, \tag{3.26}$$

where

$$g_i(x,z) = \frac{\partial f_i}{\partial x}(x) z.$$

The combined system (3.1), (3.26) satisfies the hypothesis of Theorem 3.1, so a C^0 invariant manifold

$$\begin{bmatrix} x_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} \phi(x_1, x_2, z_1, z_2) \\ \psi(x_1, x_2, z_1, z_2) \end{bmatrix} \tag{3.27}$$

can be found by the extension of the above contraction, call it S. Suppose $\phi(x_1, x_2)$ is a C^1 element of X; define

$$\psi(x_1, x_2, z_1, z_2) = \frac{\partial \phi}{\partial(x_1, x_2)}(x_1, x_2) \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}. \tag{3.28}$$

If

$$(\bar\phi(x_1, x_2, z_1, z_2), \bar\psi(x_1, x_2, z_1, z_2)) = S(\phi(x_1, x_2), \psi(x_1, x_2, z_1, z_2)),$$

then it is straightforward to verify that

$$\bar\phi(x_1, x_2, z_1, z_2) = T(\phi(x_1, x_2)),$$

so $\bar\phi(x_1, x_2, z_1, z_2) = \bar\phi(x_1, x_2) \in X$ and

$$\bar\psi(x_1, x_2, z_1, z_2) = \frac{\partial \bar\phi}{\partial(x_1, x_2)}(x_1, x_2) \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Hence if we start the contraction at $\phi,\ \psi$ satisfying (3.28), it will converge to a $\phi,\ \psi$ satisfying (3.28), so the invariant manifold (3.8) is C^1. By repeated application of this technique we can show it is C^{k-2}.

It remains to verify (3.10) if $k > 2$. Clearly (3.9) implies

$$\frac{\partial \phi}{\partial x_2}(0,0) = 0.$$

Since (3.8) defines a C^{k-2} invariant manifold, we can take its time derivative to obtain the PDE

$$\begin{aligned} & A_3\phi(x_1,x_2) + f_3(x_1,x_2,\phi(x_1,x_2)) \\ & = \sum_{j=1}^{2} \frac{\partial \phi_i}{\partial x_j}(x_1,x_2)\left(A_j x_j + f_j(x_1,x_2,\phi(x_1,x_2))\right). \end{aligned} \tag{3.29}$$

Taking the linear terms from both sides, we obtain a homogeneous linear equation

$$A_3 \frac{\partial \phi}{\partial x_1}(0,0) - \frac{\partial \phi}{\partial x_1}(0,0) A_1 = 0. \tag{3.30}$$

The eigenvalues of the linear mapping

$$B \mapsto A_3 B - B A_1$$

have positive real part because they are of the form $\lambda_3 - \lambda_1$, where $\lambda_3,\ \lambda_1$ are eigenvalues of $A_3,\ A_1$, respectively, and so the real part of $\lambda_3 - \lambda_1$ is positive. Hence (3.30) is nonsingular and

$$\frac{\partial \phi}{\partial x_1}(0,0) = 0. \qquad \square$$

The next theorem gives a term-by-term approximation of the stable and partial center manifold.

THEOREM 3.2. *Suppose the hypothesis of Theorem* 3.1 *holds for* $k > 3$ *and let* $\phi(x_1,x_2)$ *define the* C^{k-2} *stable and partial center manifold. Suppose* $\psi(x_1,x_2)$ *is a* C^{k-2} *function satisfying* (3.9) *and the PDE* (3.29) *through terms of degree* $k-3$,

$$\begin{aligned} & A_3\psi(x_1,x_2) + f_3(x_1,x_2,\psi(x_1,x_2)) \\ & = \sum_{j=1}^{2} \frac{\partial \psi}{\partial x_j}(x_1,x_2)\left(A_j x_j + f_j(x_1,x_2,\psi(x_1,x_2))\right) + O(x_1,x_2)^{k-2}. \end{aligned} \tag{3.31}$$

Then ϕ *and* ψ *agree to degree* $k-3$,

$$\psi(x_1,x_2) = \phi(x_1,x_2) + O(x_1,x_2)^{k-2}. \tag{3.32}$$

Proof. The theorem holds because the Taylor series coefficients to degree $k-3$ of any ψ satisfying (3.31) are uniquely determined. To see this we use induction on r. Assume that ϕ and ψ satisfy (3.31) to degree r and their Taylor series coefficients agree up to $r-1$.

Assume that $A_1,\ A_2,\ A_3$ are semisimple so that there exist bases of left and right eigenvectors. If they are not semisimple, the argument is essentially the same involving bases of left and right generalized eigenvectors but the details are messier, and hence are ommitted. Suppose for $j = 1, 2$, $k = 1, \dots, n_i$, and $l = 1, \dots, n_j$

$$w_{j,l}A_j = \lambda_{j,l}w_{j,l}, \tag{3.33}$$

$$A_3v^{3,k} = \lambda_{3,k}v^{3,k}. \tag{3.34}$$

Let $\psi^{[r]}(x_1, x_2)$ be the part of $\psi(x_1, x_2)$ that is a homogeneous polynomial of degree r. A basis for the n_3-vector fields homogeneous of degree r in $x_1,\ x_2$ consists of the vector fields

$$\phi^{3,k}_{j_1,l_1;\dots;j_r,l_r}(x) = v^{3,k}(w_{j_1,l_1}x_{j_1})\dots(w_{j_r,l_r}x_{j_r}), \tag{3.35}$$

where $k = 1, \dots, n_i$, $j_s = 1, 2$, $l_s = 1, \dots, n_{j_s}$ and the pairs $(j_1, l_1) \le \dots \le (j_r, l_r)$ are in lexographic order.

Thus

$$\psi^{[r]}(x_1, x_2) = \sum_{k;j_1,l_1;\dots;j_r,l_r} \gamma^{j_1,l_1;\dots;j_r,l_r}_{3,k}\psi^{3,k}_{j_1,l_1;\dots;j_r,l_r}(x_1, x_2) \tag{3.36}$$

with

$$\gamma^{j_1,l_1;\dots;j_r,l_r}_{3,k} = 0 \tag{3.37}$$

if $j_1 = \dots = j_r = 2$ so that (3.9) is satisfied.

If we extract the degree r terms from (3.31), we obtain

$$A_3\psi^{[r]}3(x_1, x_2) - \sum_{j=1}^{2} \frac{\partial\psi^{[r]}_i}{\partial x_j}(x_1, x_2)A_jx_j = h^{[r]}(x_1, x_2), \tag{3.38}$$

where $h^{[r]}(x_1, x_2)$ depends only on the ODE (3.1) and the lower degree part of $\psi(x_1, x_2)$ which has been determined by (3.31). Now

$$\begin{aligned} A_3\psi^{3,k}_{j_1,l_1;\dots;j_r,l_r}(x_1, x_2) - \sum_{j=1}^{2} \frac{\partial\psi^{3,k}_{j_1,l_1;\dots;j_r,l_r}(x_1, x_2)}{\partial x_j}(x_1, x_2)A_jx_j \\ = (\lambda_{3,k} - \lambda_{j_1,l_1} - \dots - \lambda_{j_r,l_r})\,\psi^{3,k}_{j_1,l_1;\dots;j_r,l_r}(x_1, x_2). \end{aligned}$$

The real part of $\lambda_{3,k}$ is nonnegative and the real parts of λ_{j_s,l_s} are negative if $j_s = 1$ and are nonpositive if $j_s = 2$. Because of (3.37), we can restrict our attention to $\psi^{3,k}_{j_1,l_1;\dots;j_r,l_r}$, where at least one $j_s = 1$ so such $\gamma^{j_1,l_1;\dots;j_r,l_r}_{3,k}$ are uniquely determined by (3.38). ∎

Notice that if (3.7) is not satisfied to degree r, then (3.38) might not be solvable for then $h^{[r]}(x_1, x_2)$ might contain terms of the form $\psi^{3,k}_{j_1,l_1;\dots;j_r,l_r}(x_1, x_2)$, where $j_1 = \dots = j_r = 2$ and $\lambda_{3,k} - \lambda_{2,l_1} - \dots - \lambda_{2,l_r}$ might be zero.

4. Local solvability of the HJB PDE. The principle theorem of this paper is the following.

THEOREM 4.1. *Suppose the plant* (2.1) *and exosystem* (2.2) *are* C^k*, the linear part of the plant is stabilizable and detectable when* $\bar{x} = 0$*, the linear part of the exosystem is stable, the FBI PDE* (2.3) *has a* C^k *solution in some neighborhood of* 0 *in* $\bar{x}$ *space. Then in some neighborhood of* $0, 0$ *in* $x, \bar{x}$ *space there exists a* C^{k-2} *solution to HJB PDE* (2.9) *satisfying* (2.12).

Proof. The proof generalizes the standard approach [22] to showing the existence of local solutions to HJB PDEs. The graph of gradient of the solution π of the HJB PDE (2.9) is an invariant manifold of the associated Hamiltonian system of ODEs. In the standard case, the Hamiltonian ODEs have a hyperbolic fixed point at the origin and the invariant manifold is the stable manifold of this fixed point. But in this case, the Hamiltonian ODEs do not have a hyperbolic fixed point at the origin and the desired invariant manifold is a stable and partial center manifold.

Consider the Hamiltonian associated to the optimal control problem (2.8),

$$\begin{aligned} H(\lambda,\mu,z,\bar{x},v) &= \lambda\tilde{f}(z,v,\bar{x})+\mu\bar{f}(\bar{x})+l(z,v,\bar{x}) \\ &= \lambda\big(Az+Bv+\tilde{f}^{[2]}(z,v,\bar{x})\big) \\ &\quad +\mu\left(\bar{A}\bar{x}+\bar{f}^{[2]}(\bar{x})\right) \\ &\quad +\tfrac{1}{2}\left(z'Qz+2z'Sv+v'Rv\right) \\ &\quad +\tilde{l}^{[3]}(z,v,\bar{x})+O(\lambda,\mu,z,\bar{x},v)^4. \end{aligned} \tag{4.1}$$

The Pontryagin maximum principle asserts that the optimal control is

$$v=\gamma(\lambda,\mu,z,\bar{x})=\arg\min_v H(\lambda,\mu,z,\bar{x},v). \tag{4.2}$$

For small $\lambda,\mu,z,\bar{x}$ this is given by solving

$$\frac{\partial H}{\partial v}(\lambda,\mu,z,\bar{x},v)=0,$$

which yields

$$\begin{aligned}\gamma = -R^{-1}&\left(B'\lambda'+S'z+\left(\frac{\partial\tilde{f}^{[2]}}{\partial v}\right)'\lambda'+\left(\frac{\partial\tilde{l}^{[3]}}{\partial v}\right)'\right)\\ &+O(\lambda,\mu,z,\bar{x})^3.\end{aligned}$$

The HJB PDE (2.9) can be expressed in terms of the Hamiltonian as

$$H\left(\frac{\partial\pi}{\partial z},\frac{\partial\pi}{\partial\bar{x}},z,\bar{x},\gamma\left(\frac{\partial\pi}{\partial z},\frac{\partial\pi}{\partial\bar{x}},z,\bar{x}\right)\right)=0. \tag{4.3}$$

The Hamiltonian ODEs are

$$\begin{aligned} \dot{z}' &= \frac{\partial H}{\partial\lambda}(\lambda,\mu,z,\bar{x},\gamma(\lambda,\mu,z,\bar{x})),\\ \dot{\lambda} &= -\frac{\partial H}{\partial z}(\lambda,\mu,z,\bar{x},\gamma(\lambda,\mu,z,\bar{x})),\\ \dot{\bar{x}}' &= \frac{\partial H}{\partial\mu}(\lambda,\mu,z,\bar{x},\gamma(\lambda,\mu,z,\bar{x})),\\ \dot{\mu} &= -\frac{\partial H}{\partial\bar{x}}(\lambda,\mu,z,\bar{x},\gamma(\lambda,\mu,z,\bar{x})) \end{aligned} \tag{4.4}$$

and these are C^{k-1} since the Hamiltonian is C^k.

The linearization of this system around $0,0,0,0$ is

$$\left[\begin{array}{c}\dot{z}\\ \dot{\lambda}'\\ \hline \dot{\bar{x}}\\ \dot{\mu}'\end{array}\right]=\left[\begin{array}{cc}\mathbf{H_{11}} & \mathbf{H_{12}}\\ \mathbf{H_{21}} & \mathbf{H_{22}}\end{array}\right]\left[\begin{array}{c}z\\ \lambda'\\ \hline \bar{x}\\ \mu'\end{array}\right], \tag{4.5}$$

where

$$
\mathbf{H} = \left[\begin{array}{cc|cc} A - BR^{-1}S' & -BR^{-1}B' & 0 & 0 \\ -Q + SR^{-1}S' & -A' + SR^{-1}B' & 0 & 0 \\ \hline 0 & 0 & \bar{A} & 0 \\ 0 & 0 & 0 & -\bar{A}' \end{array}\right]. \tag{4.6}
$$

The column span of

$$
\begin{bmatrix} I_{n\times n} & 0 \\ P & 0 \\ 0 & I_{\bar{n}\times\bar{n}} \\ 0 & 0 \end{bmatrix} \tag{4.7}
$$

is an $n + \bar{n}$ dimensional stable and partial center subspace of the linear Hamiltonian system (4.5), where P is the unique nonnegative definite solution of the algebraic Riccati equation (2.13). We know that such a solution exists because the linear part of the plant was assumed to be stabilizable and detectable [3]. Half of the eigenvalues of the upper left $2n \times 2n$ block $\mathbf{H_{11}}$ lie in the open left half plane and half lie in the open right half plane. The asymptotically stable subspace is spanned by the first n columns of (4.7). As for the lower right $2\bar{n} \times 2\bar{n}$ block $\mathbf{H_{22}}$, by assumption the eigenvalues of $\bar{A}$ are in the closed left half plane and hence those of $-\bar{A}'$ are in the closed right half plane. A stable subspace is spanned by the last $\bar{n}$ columns of (4.7). Furthermore, the submanifold $z = 0,\ \lambda = 0,\ \mu = 0$ is an invariant submanifold of the nonlinear Hamiltonian system (4.4), so the conditions of the stable and partial center manifold Theorem are satisfied. There exists an $n + \bar{n}$ dimensional stable and partial center manifold in the $2(n + \bar{n})$ dimensional $z, \lambda, \bar{x}, \mu$ space which is tangent to the column span of (4.7) at $0, 0, 0, 0$. Hence this manifold is given by

$$
\begin{aligned} \lambda &= \phi(z, \bar{x}), \\ \mu &= \psi(z, \bar{x}), \end{aligned} \tag{4.8}
$$

where ϕ, ψ are C^{k-3} and

$$
\begin{aligned} \phi(0, \bar{x}) &= 0, \\ \psi(0, \bar{x}) &= 0. \end{aligned} \tag{4.9}
$$

This submanifold is Lagrangian, i.e., a maximal dimension submanifold on which the canonical two form

$$
\omega = d\lambda\ dz + d\mu\ d\bar{x}
$$

vanishes [1], [4]. To see that it vanishes we note that ω is invariant under the Hamiltonian flow (4.4) and this flow is converging to the $\bar{n}$ dimensional submanifold $z = 0,\ \lambda = 0,\ \mu = 0$, where ω clearly vanishes. The submanifold (4.8) is of maximal dimension, $n + \bar{n}$, in $2(n + \bar{n})$ variables.

Hence the one form

$$
\phi(z, \bar{x})\ dz + \psi(z, \bar{x})\ d\bar{x}
$$

is closed locally around $0, 0$ in $z, \bar{x}$ space and so there exists a C^{k-2} function $\pi(z, \bar{x})$ such that

$$
\frac{\partial \pi}{\partial z}(z, \bar{x}) = \phi(z, \bar{x}),
$$

$$\begin{aligned}\frac{\partial \pi}{\partial \bar{x}}(z,\bar{x}) &= \psi(z,\bar{x}),\\ \pi(0,\bar{x}) &= 0,\\ \frac{\partial \pi}{\partial z}(0,\bar{x}) &= 0,\\ \frac{\partial \pi}{\partial \bar{x}}(0,\bar{x}) &= 0.\end{aligned}$$

Note that π satisfies (2.12).

Differentiating (4.8) with respect to t along the Hamiltonian flow (4.4) yields

$$\begin{aligned}\frac{\partial H}{\partial \lambda}\frac{\partial^2 \pi}{\partial z^2} + \frac{\partial H}{\partial \mu}\frac{\partial^2 \pi}{\partial z \partial \bar{x}} + \frac{\partial H}{\partial z} &= 0,\\ \frac{\partial H}{\partial \lambda}\frac{\partial^2 \pi}{\partial z \partial \bar{x}} + \frac{\partial H}{\partial \mu}\frac{\partial^2 \pi}{\partial \bar{x}^2} + \frac{\partial H}{\partial z} &= 0\end{aligned}$$

or equivalently

$$\begin{aligned}\frac{\partial}{\partial z} H\left(\frac{\partial \pi}{\partial z}, \frac{\partial \pi}{\partial \bar{x}}, z, \bar{x}, \gamma\left(\frac{\partial \pi}{\partial z}, \frac{\partial \pi}{\partial \bar{x}}, z, \bar{x}\right)\right) &= 0,\\ \frac{\partial}{\partial \bar{x}} H\left(\frac{\partial \pi}{\partial z}, \frac{\partial \pi}{\partial \bar{x}}, z, \bar{x}, \gamma\left(\frac{\partial \pi}{\partial z}, \frac{\partial \pi}{\partial \bar{x}}, z, \bar{x}\right)\right) &= 0.\end{aligned}$$

Clearly π satisfies the HJB PDE (4.3) at $z = 0,\ \bar{x} = 0$, so it satisfies it in a neighborhood of this point. Moreover π is of the form

$$\pi(z,\bar{x}) = \frac{1}{2} z' P z + O(z,\bar{x})^3 \qquad \Box \tag{4.10}$$

The next theorem shows that the solution to the HJB PDE (2.9) can be computed term-by-term.

THEOREM 4.2. *Suppose the hypotheses of Theorem* 4.1 *hold for* $k > 3$ *and let* $\pi(z,\bar{x})$ *be the* C^{k-2} *solution of the HJB PDE* (2.9) *satisfying* (2.12). *Suppose* $\phi(x_1, x_2)$ *is a* C^{k-2} *function satisfying the HJB PDE through terms of degree* $k-3$ *and satisfying* (2.12). *Then* π *and* ψ *agree to degree* $k-3$,

$$\pi(z,\bar{x})) = \psi(z,\bar{x}) + O(z,\bar{x})^{k-2} \tag{4.11}$$

Proof (sketch). Clearly $\pi(z,\bar{x})$ satisfies the term-by-term equations, so the result follows if we can show that these equations have unique solutions satisfying (2.12). We showed above that the quadratic terms agree, and as for the cubic terms, consider (2.14). The first equation is a linear equation for $\pi^{[3]}$. For simplicity assume that $A + BK$ and $\bar{A}$ have bases of left eigenvectors

$$\begin{array}{llll}\xi_i(A+BK) & = & \lambda_i \xi_i, & i = 1,\ldots,n,\\ \zeta_j \bar{A} & = & \mu_j \xi_i, & j = 1,\ldots,\bar{n},\end{array} \tag{4.12}$$

otherwise we use bases of generalized eigenvectors. Since the linear part of the plant is stabilizable and detectable, the linear part of the closed loop system is asymptotically stable, Re $\lambda_i < 0$, and by assumption the linear part of the exosystem is stable, Re $\mu_j \le 0$.

Now any cubic polynomial $\pi^{[3]}(z,\bar{x})$ satisfying (2.12) can be expressed as

$$\begin{aligned}\pi^{[3]}(z,\bar{x}) = \sum c_{i_1,i_2,i_3}\ \xi_{i_1}z\ \xi_{i_2}z\ \xi_{i_3}z \\ + \sum d_{i_1,i_2,j_3}\ \xi_{i_1}z\ \xi_{i_2}z\ \zeta_{j_3}\bar{x}\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial\pi^{[3]}}{\partial z}(z,\bar{x})(A+BK)z + \frac{\partial\pi^{[3]}}{\partial\bar{x}}(z,\bar{x})(\bar{A}\bar{x})\\ &= \sum c_{i_1,i_2,i_3}\ (\lambda_{i_1}+\lambda_{i_2}+\lambda_{i_3})\,\xi_{i_1}z\ \xi_{i_2}z\ \xi_{i_3}z\\ &\quad + \sum d_{i_1,i_2,j_3}\ (\lambda_{i_1}+\lambda_{i_2}+\mu_{j_3})\,\xi_{i_1}z\ \xi_{i_2}z\ \zeta_{j_3}\bar{x}.\end{aligned}$$

It follows from (2.11) that

$$z'P\tilde{f}^{[2]}(z,Kz,\bar{x}) + l^{[3]}(z,Kz,\bar{x}) = O(z,\bar{x})^2,$$

so

$$\begin{aligned}&z'P\tilde{f}^{[2]}(z,Kz,\bar{x}) + l^{[3]}(z,Kz,\bar{x})\\ &\quad = \sum k_{i_1,i_2,i_3}\ \xi_{i_1}z\ \xi_{i_2}z\ \xi_{i_3}z\\ &\qquad + \sum l_{i_1,i_2,j_3}\ \xi_{i_1}z\ \xi_{i_2}z\ \zeta_{j_3}\bar{x}\end{aligned}$$

for some k, l's. Hence there is a unique $\pi^{[3]}$ satisfying (2.14) and (2.12) given by

$$\begin{aligned}c_{i_1,i_2,i_3} &= -\frac{k_{i_1,i_2,i_3}}{\lambda_{i_1}+\lambda_{i_2}+\lambda_{i_3}},\\ d_{i_1,i_2,j_3} &= -\frac{l_{i_1,i_2,j_3}}{\lambda_{i_1}+\lambda_{i_2}+\mu_{j_3}}\end{aligned}$$

because the denominators are not zero, Re $\lambda_i < 0$, and Re $\mu_j \le 0$. The higher degree terms are handled in a similar fashion. □

5. H_∞ regulation. One can also use nonlinear H_∞ control techniques to stabilize the transverse dynamics in a robust fashion. Consider a smooth plant

$$\begin{aligned}\dot{x} &= f(x,u,\bar{x}) + g(x,\bar{x},w)\\ &= Ax + Bu + F\bar{x} + Gw\\ &\quad + f^{[2]}(x,u,\bar{x}) + g^{[2]}(x,\bar{x},w) + O(x,u,\bar{x},w)^3\\ e &= h(x,u,\bar{x})\\ &= Cx + Du + H\bar{x}\\ &\quad + h^{[2]}(x,u,\bar{x}) + O(x,u,\bar{x})^3\end{aligned} \tag{5.1}$$

which is perturbed by an unknown noise $w(t)$ and by a smooth nonlinear exosystem

$$\begin{aligned}\dot{\bar{x}} &= \bar{f}(\bar{x},w)\\ &= \bar{A}\bar{x} + \bar{B}w + \bar{f}^{[2]}(\bar{x},w) + O(\bar{x},w)^3.\end{aligned} \tag{5.2}$$

Notice that there is no direct interaction between the control and the noise in the dynamics and the noise w does not directly affect the error e.

The goal is as before, to find a feedforward and feedback control $u = \alpha(x, \bar{x})$ to drive $e(t)$ as close to zero as possible for any $x(0),\ \bar{x}(0)$ despite the unknown noise. More precisely, for any choice of α the closed loop system defines a map from the initial conditions $x(0),\ \bar{x}(0)$ and the noise $w(t)$ to the variables that we want to keep small, $u(t),\ e(t)$. We would like the gain of this mapping to be as small as possible. This is a very difficult problem to solve directly so we settle for a suboptimal solution. Given an attenuation level $\delta > 0$, we seek an $u = \alpha(x, \bar{x})$ so that the map from $x(0),\ \bar{x}(0),\ w(t)$ to $u(t),\ e(t)$ has gain less than δ. This goal needs to be modified because as before we should not penalize those parts of $x(0),\ u(t)$ that are necessary for exact tracking.

As before we start by solving the FBI equations (2.3) for exact tracking and transform the combined system into transverse coordinates (2.4) to obtain

$$
\begin{aligned}
\dot{z} &= \tilde{f}(z, v, \bar{x}) + \tilde{g}(z, \bar{x}, w) \\
&= Ax + Bv + \tilde{G}w + \tilde{f}^{[2]}(z, v, \bar{x}, w) + \tilde{g}^{[2]}(z, \bar{x}, w) + O(z, v, \bar{x}, w)^3 \\
\dot{\bar{x}} &= \bar{f}(\bar{x}) = \bar{A}\bar{x} + \bar{B}w + \bar{f}^{[2]}(\bar{x}, w) + O(\bar{x}, w)^3 \\
e &= \tilde{h}(z, v, \bar{x}) = Cz + Dv + \tilde{h}^{[2]}(z, v, \bar{x}) + O(z, v, \bar{x})^3,
\end{aligned}
\tag{5.3}
$$

where

$$
\begin{aligned}
\tilde{f}(z, v, \bar{x}) &= f(z + \theta(\bar{x}), v + \beta(\bar{x}), \bar{x}) - \frac{\partial \theta}{\partial \bar{x}}(\bar{x})\bar{f}(\theta(\bar{x}), \bar{x}) \\
\tilde{g}(z, \bar{x}, w) &= g(z + \theta(\bar{x}), \bar{x}, w) \\
\tilde{h}(z, v, \bar{x}) &= h(z + \theta(\bar{x}), v + \beta(\bar{x}), \bar{x}) \\
\tilde{G} &= G - TB.
\end{aligned}
\tag{5.4}
$$

We wish to find the control $v = \gamma(z, \bar{x})$ that maximizes

$$
\pi(z, \bar{x}) = \inf_w \frac{1}{2} \int_0^t \delta^2 |w(s)|^2 - |e(s)|^2 - |v(s)|^2 \, ds,
\tag{5.5}
$$

where the infimum is over all $t \geq 0$, with $w(s)$ generating a trajectory satisfying $z(0) = 0,\ \bar{x}(0) = 0,\ z(t) = z,\ \bar{x}(t) = \bar{x}$. For any control $v = \gamma(z, \bar{x})$, the function $\pi(z, \bar{x})$ is the minimum required net energy that must be supplied to the combined system to go from the origin $0,\ 0$ to $z,\ \bar{x}$. Energy is supplied to the system at the rate $\frac{\delta^2}{2}|w(s)|^2$ and extracted from the system at the rate $\frac{1}{2}(|e(s)|^2 + |v(s)|^2)$. The goal is to supremize the energy necessary to reach any $z,\ \bar{x}$. See [23] and [18, 19] for more on nonlinear H_∞ control.

An immediate consequence of the definition of $\pi(z, \bar{x})$ is that along any trajectory of the system

$$
\pi(z(s), \bar{x}(s))]_{t_1}^{t_2} \leq \frac{1}{2} \int_{t_1}^{t_2} \delta^2 |w(s)|^2 - |e(s)|^2 - |v(s)|^2 \, ds.
\tag{5.6}
$$

This is called a dissipation inequality; if we view $\pi(z, \bar{x})$ as the energy stored in the combined system when it is in state $z,\ \bar{x}$ then the change in stored energy over any time interval is less than or equal to the net energy supplied to the system over that time interval.

If there exists a control $v = \gamma(z, w)$ so that $\pi(z, \bar{x}) \geq 0$, then

$$
\frac{1}{2} \int_{t_1}^{t_2} |e(s)|^2 + |v(s)|^2 \, ds \leq \pi(z(t_1), \bar{x}(t_1)) + \frac{\delta^2}{2} \int_{t_1}^{t_2} |w(s)|^2 \, ds.
\tag{5.7}
$$

If this holds, then the energy of the tracking error plus the energy of the control used to reduce it is less than the energy of the initial mismatch between the plant and exosystem, $\pi(z(t_1), \bar{x}(t_1))$, plus the energy of the disturbance.

We can view (5.5) as the cost criterion of a differential game pitting the control v against the noise w. The optimal $\pi,\ v^*,\ w^*$ satisfy the HJI PDE

$$
\begin{aligned}
0 &= \frac{\partial \pi}{\partial z}(z,\bar{x})\left(\tilde{f}(z,v,\bar{x}) + \tilde{g}(z,\bar{x},w)\right) \\
&\quad + \frac{\partial \pi}{\partial \bar{x}}(z,\bar{x})\bar{f}(\bar{x},w) + l(z,v,\bar{x},w) \\
v^*, w^* &= \arg\ \min_v \max_w \left\{ \frac{\partial \pi}{\partial z}(z,\bar{x})\left(\tilde{f}(z,v,\bar{x}) + \tilde{g}(z,\bar{x},w)\right)\right. \\
&\quad \left. + \frac{\partial \pi}{\partial \bar{x}}(z,\bar{x})\bar{f}(\bar{x},w) + l(z,v,\bar{x},w) \right\},
\end{aligned}
\tag{5.8}
$$

where

$$
\begin{aligned}
l(z,v,\bar{x},w)) &= \frac{1}{2}\left(|e|^2 + |v|^2\right) - \frac{\gamma^2}{2}|w|^2 \\
&= \frac{1}{2}\left(z'Qz + 2z'Sv + v'Rv\right) - \frac{\gamma^2}{2}|w|^2 \\
&\quad + l^{[3]}(z,v,\bar{x}) + O(z,v,\bar{x})^4.
\end{aligned}
$$

Van der Schaft [23] considered the local solvability of the HJI PDE when the plant is stabilizable and detectable and there is no exosystem (so $x = z,\ u = v,\ G = \tilde{G}$). He showed that a local solution exists if the linear quadratic part of the problem admits a stable solution. That is, there exists a $P \geq 0$ satisfying the Riccati equation

$$
\begin{aligned}
0 &= A'P + PA + Q + \frac{1}{\gamma^2}P\tilde{G}\tilde{G}'P \\
&\quad -(PB+S)R^{-1}(PB+S)'
\end{aligned}
\tag{5.9}
$$

and such that the closed loop spectrum is in the open left half plane,

$$
\sigma\left(A - BR^{-1}(B'P + S') + \frac{1}{\gamma^2}\tilde{G}\tilde{G}'P\right) < 0. \tag{5.10}
$$

The optimal linear feedback and worst case noise are

$$
\begin{aligned}
v^* &= -R^{-1}(B'P + S')z, \\
w^* &= \frac{1}{\gamma^2}\tilde{G}'Pz.
\end{aligned}
$$

Following the approach described in section 4 using the stable and partial center manifold theorem, one can prove the following theorems.

THEOREM 5.1. *Suppose the plant* (5.1) *and exosystem* (5.2) *are C^k, the linear part of the plant is stabilizable and detectable when $\bar{x} = 0$, the linear part of the exosystem is stable, and the FBI PDE* (2.3) *has a C^k solution in some neighborhood of* 0 *in $\bar{x}$ space. If there exists a $P \geq 0$ satisfying the Riccati equation* (5.9) *and such that the closed loop spectrum is in the open left half plane* (5.10), *then in some neighborhood of* $0, 0$ *in $x, \bar{x}$ space there exists a C^{k-2} solution to HJB PDE* (5.8) *satisfying* (2.12).

THEOREM 5.2. *Suppose the hypotheses of Theorem* 5.1 *hold for* $k > 3$ *and let* $\pi(z, \bar{x})$ *be the* C^{k-2} *solution of the HJI PDE* (5.8) *satisfying* (2.12). *Suppose* $\phi(x_1, x_2)$ *is a* C^{k-2} *function satisfying the HJI PDE through terms of degree* $k-3$ *and satisfying* (2.12). *Then* π *and* ψ *agree to degree* $k-3$,

$$\pi(z, \bar{x})) = \psi(z, \bar{x}) + O(z, \bar{x})^{k-2}. \tag{5.11}$$

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