NORMAL FORMS OF LINEARLY UNCONTROLLABLE NONLINEAR CONTROL SYSTEMS WITH A SINGLE INPUT

A. J. Krener * ,1 W. Kang ** Dong Eui Chang *** ,2

* Department of Mathematics, University of California, Davis, CA 95616-8633
ajkrener@ucdavis.edu
** Department of Mathematics, Naval Postgraduate School, Monterey, CA 93943
wkang@math.nps.navy.mil
*** Control and Dynamical Systems, California Institute of Technology, Pasadena, CA 91125
dchang@cds.caltech.edu

Abstract: We derive the controller normal form and the dual normal form for linearly uncontrollable nonlinear control systems with a single input. The invariants under the state and feedback transformation of degree d are found for d ≥ 2. Copyright © IFAC 2001

Keywords: nonlinear control systems, normal forms, invariants

1. INTRODUCTION

The Poincaré normal form is known to be useful in the bifurcation analysis of dynamical systems. Kang and Krener (1992) initiated the extension of Poincaré normal form theory to nonlinear control systems, where they considered quadratic controller normal forms of linearly controllable nonlinear control systems with a single input. The controller normal form has been proved useful in the analysis of control bifurcations (see Chang et al. (2000), Kang (1998), and Krener et al. (2001)). Kang (1996) extended Kang and Krener (1992) by deriving the controller normal form of arbitrary degree of linearly controllable nonlinear control systems with a single input, which was further extended in Tall and Respondek (2000), refining the normal form in Kang (1996) by using a larger transformation group. On the other hand, Chang et al. (2000) and Krener et al. (2001) extended Kang and Krener (1992) by deriving the controller normal form up to degree three of linearly uncontrollable nonlinear control systems with a single input. In this paper, we extend Chang et al. (2000), Kang (1996) and Tall and Respondek (2000) by deriving the controller normal form and the dual normal form of arbitrary degree of linearly uncontrollable nonlinear control systems with a single input and finding a set of invariants which help compute the normal form.

2. MAIN RESULTS

Consider a smooth nonlinear control system

\[ \dot{x} = f(x, u) \]  

(2.1)

where \( x \) is \( n \) dimensional, \( u \) is one dimensional and \( f(0,0) = 0 \). It is well known that by a linear change of state coordinates, linear state feedback

---

1 Research supported in part by NSF 9970998 and by AFOSR F49620-01-1-0202
2 Research supported in part by AFOSR F49620-99-1-0190
and Taylor series expansion, the system can be brought to the form

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 
\end{bmatrix} = \begin{bmatrix}
A_1 & 0 \\
0 & A_2
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
0 \\
B_2
\end{bmatrix} u \\
+ \sum_{k=2}^{d} \begin{bmatrix}
f_1^{[k]}(x_1, x_2, u) \\
f_2^{[k]}(x_1, x_2, u)
\end{bmatrix} \\
+ O(x_1, x_2, u)^{d+1}
\tag{2.2}
\]

where \(x_1, x_2\) are \(n_1, n_2\) dimensional, \(n_1 + n_2 = n\), \(A_1\) is in Jordan form, \(A_2, B_2\) are in controller (Brunovsky) form and \(f_i^{[k]}(x_1, x_2, u)\) is an \(n_i\) dimensional vector-valued homogeneous polynomial of degree \(k\) in its arguments. The linear change of coordinates that brings \(A_1\) to Jordan form may be complex, in which case some of the coordinates \(x_{1,j}\) are complex. The complex coordinates come in conjugate pairs. The corresponding \(f_{1,i}^{[k]}\)'s are complex-valued and come in conjugate pairs.

A pair \(A_2, B_2\) is in controller form if

\[
A_2 = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & & & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\tag{2.3}
\]

**Theorem 2.1.** Consider a smooth \((C^\infty)\) control system (2.2) where \(A_1\) is in Jordan form and the input is scalar. There exist a change of coordinates and a feedback of the form

\[
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} = \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} - \sum_{k=2}^{d} \begin{bmatrix}
\phi_1^{[k]}(x_1, x_2) \\
\phi_2^{[k]}(x_1, x_2)
\end{bmatrix}
\tag{2.4}
\]

\[\nu = u - \sum_{k=2}^{d} \alpha^{[k]}(x_1, x_2, u)
\tag{2.5}
\]

which transform the system (2.2) into the normal form

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix} = \begin{bmatrix}
A_1 & 0 \\
0 & A_2
\end{bmatrix} \begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} + \begin{bmatrix}
0 \\
B_2
\end{bmatrix} \nu \\
+ \sum_{k=2}^{d} \begin{bmatrix}
f_1^{[k],\nu}(z_1, z_2, \nu) \\
f_2^{[k],\nu}(z_1, z_2, \nu)
\end{bmatrix} \\
+ O(z_1, z_2, \nu)^{d+1}
\tag{2.6}
\]

where \(f_i^{[k],\nu}\) is in the Poincaré normal form of degree \(k\) (see Arnold (1983)) and the other vector fields are as follows:

\[
\dot{f}_1^{[k], c} = \sum_{i=1}^{n_1} e_{1,i} f_i^{[k-1]}(z_2, z_2, \nu) \\
+ \sum_{j=1}^{n_2+1} Q_{ij}^{[k-2]}(z_1, z_2, \nu) z_2^j
\tag{2.7}
\]

\[
\dot{f}_2^{[k]} = \sum_{i=1}^{n_1} \sum_{j=i+2}^{n_2+1} e_{2,i} R_{ij}^{[k-2]}(z_1, z_2, \nu) z_2^j
\tag{2.8}
\]

where \(e_{r,v}\) is the \(r\)'th unit vector in \(z_i\) space, \(P_i^{[m]}(\cdot), Q_i^{[m]}(\cdot)\) and \(R_i^{[m]}(\cdot)\) are arbitrary homogeneous polynomials of degree \(m\), and for notational convenience we have defined \(z_2, z_2, \nu = z\).

**Proof.** First notice that the following \(d\)'th order transformation with \(d > 1\)

\[
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} = \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} - \begin{bmatrix}
\phi_1^{[d]}(x_1, x_2) \\
\phi_2^{[d]}(x_1, x_2)
\end{bmatrix}
\tag{2.10}
\]

\[v = u - \alpha^{[d]}(x_1, x_2, u)
\tag{2.11}
\]

does not change the terms of degree less than \(d\). Without loss of generality, we may assume that the given system is in the normal form up to degree \((d - 1)\). Consider two systems in the same normal forms up to degree \((d - 1)\) as follows:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
A_1 & 0 \\
0 & A_2
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
0 \\
B_2
\end{bmatrix} u \\
+ \sum_{k=2}^{d-1} \begin{bmatrix}
f_1^{[k]}(x_1, x_2, u) \\
f_2^{[k]}(x_1, x_2, u)
\end{bmatrix} \\
+ O(x_1, x_2, u)^{d+1}
\tag{2.12}
\]

and

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix} = \begin{bmatrix}
A_1 & 0 \\
0 & A_2
\end{bmatrix} \begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} + \begin{bmatrix}
0 \\
B_2
\end{bmatrix} v \\
+ \sum_{k=2}^{d-1} \begin{bmatrix}
f_1^{[k]}(z_1, z_2, v) \\
f_2^{[k]}(z_1, z_2, v)
\end{bmatrix} \\
+ O(z_1, z_2, v)^{d+1}
\tag{2.13}
\]

where \(\dim x_i = \dim z_i, i = 1, 2\).

Define the two vector spaces \(U\) and \(V\) as follows.
\[ U = \left\{ (\phi_i^d(z), \alpha_i^d(z, v)) \mid \phi_i^d = \begin{bmatrix} \phi_i^d_1 \\ \phi_i^d_2 \end{bmatrix} \right\}, \]
\[ V = \left\{ y_i^d(z, v) \mid y_i^d = \begin{bmatrix} y_i^d_1 \\ y_i^d_2 \end{bmatrix} \right\} \]

where \( \phi_i^d \) and \( y_i^d \) are \( n \)-dimensional vector-valued homogeneous polynomials of degree \( d \) in their arguments with \( i = 1, 2 \) and \( \alpha_i^d \) is a homogeneous polynomial of degree \( d \). Define a linear map \( L : U \to V \)

\[
L(\phi_i^d, \alpha_i^d)_i = A_1 \phi_i^d(z) - \frac{\partial \phi_i^d}{\partial z_1}(z) A_1 y_1 - \frac{\partial \phi_i^d}{\partial z_2}(z) A_2 y_2 + B_1 \alpha_i^d(z, v)
\]

for \( i = 1, 2 \). One can see that the transformation in (2.10) and (2.11) transforms the system (2.12) to the system (2.13) if

\[
\tilde{f}_i^d(z, v) = f_i^d(z, v) + L(\phi_i^d, \alpha_i^d)(z, v)
\]

where

\[
f_i^d = \begin{bmatrix} f_i^d_1 \\ f_i^d_2 \end{bmatrix}, \quad \tilde{f}_i^d = \begin{bmatrix} \tilde{f}_i^d_1 \\ \tilde{f}_i^d_2 \end{bmatrix}
\]

Let \( p : V \to \text{Im} L \) be the quotient map. To find a normal form is to find a subspace \( W \) of \( V \) such that \( W \) is isomorphic to \( \text{Im} L \) via \( p \). Namely, it boils down to finding a subspace \( W \) of \( V \) such that \( V = W \oplus \text{Im} L \). In this proof, we decompose the spaces \( U, V \) and the map \( L \) as \( U = \oplus_{i=1}^{n_1} U_i, V = \oplus_{i=1}^{n_2} V_i \) and \( L = \oplus_{i,j=1}^{n_1,n_2} L_{ij} \) with \( L_i := L_{i|U_i} : U_i \to V_i \) in order to decompose the problem of deriving the normal forms into three subproblems.

1. \( \tilde{f}_i^d(z, v) \): Define the two vector spaces \( U_1 \) and \( V_1 \) by \( U_1 = \left\{ \sum_{i=1}^{n_1} e_{1,i} \phi_i^d(z_1) \right\} \) and \( V_1 = \left\{ \sum_{i=1}^{n_1} e_{1,i} \tilde{f}_i^d(z_1) \right\} \). Then \( L_1 := L_{i|U_1} : U_1 \to V_1 \) becomes

\[
L_1(\phi_1) = A_1 \phi_1(z_1) - \frac{\partial \phi_1}{\partial z_1}(z_1) A_1 y_1
\]

which is the same as the Lie derivative \((-1)L_{A_1}\) in Arnold (1983, p. 179). Hence, one can transform any element \( \phi_1 \) of \( U_1 \) into a Poincaré normal form of degree \( d \). In particular, when \( A_1 \) is diagonal, the Poincaré normal form of arbitrary degree \( k \geq 2 \) is given by (2.9) (see Arnold (1983) for more detail on the Poincaré normal forms).

2. \( \tilde{f}_j^d(z, v) \): Define the two vector spaces \( U_2 \) and \( V_2 \) by

\[
U_2 = \left\{ \sum_{i=1}^{n_1} e_{1,i} \phi_i^d(z_1, z_2) \mid \text{deg}_{2} \phi_i^d \geq 1 \right\},
\]
\[
V_2 = \left\{ \sum_{i=1}^{n_1} e_{1,i} \tilde{f}_i^d(z_1, z_2, v) \mid \text{deg}_{2} \tilde{f}_i^d \geq 1 \right\}
\]

where \( \text{deg}_2 \) is the map which assigns to a polynomial in \((z_1, z_2, v)\) the lowest degree of \((z_2, v)\) in the polynomial. Then \( L_2 := L_{i|U_2} : U_2 \to V_2 \) becomes

\[
L_2(\phi_1) = A_1 \phi_1(z) - \frac{\partial \phi_1}{\partial z_1}(z_1) A_1 z_1 - \frac{\partial \phi_1}{\partial z_2}(z) A_2 z_2 - \frac{\partial \phi_1}{\partial v}(z_2) B_2 v
\]

(2.15)

for \( \phi_1 \in U_2 \). We claim \( \ker L_2 = \{0\} \). Suppose \( \phi_1 \in \ker L_2 \). It satisfies

\[
A_1 \phi_1(z) - \frac{\partial \phi_1}{\partial z_1}(z_1) A_1 z_1 - \frac{\partial \phi_1}{\partial z_2}(z) A_2 z_2 - \frac{\partial \phi_1}{\partial v}(z_2) B_2 v = 0.
\]

(2.16)

Recall the special forms of \( A_2 \) and \( B_2 \) in (2.3). Since only the term \( \frac{\partial \phi_1}{\partial v}(z_2) B_2 v \) in (2.16) depends on \( v \), it follows that \( \frac{\partial \phi_1}{\partial v}(z_2) B_2 v = 0 \). Hence \( \phi_1 \) does not depend on \( z_2, v \). With this result in mind, notice that the term \( \frac{\partial \phi_1}{\partial z_2}(z) A_2 z_2 \) is the only term in (2.16) which depends on \( z_2, v \). It follows that \( \frac{\partial \phi_1}{\partial z_2} = 0 \). Inductively, one can show \( \frac{\partial \phi_1}{\partial z_i} = 0 \) for \( i = 1, \ldots, n_2 \), which implies \( \phi_1 = 0 \) since \( \text{deg}_{2} \phi_1 \geq 1 \). Hence \( \ker L_2 = \{0\} \).

Define a subspace \( W_2 \) of \( V_2 \) by

\[
W_2 = \left\{ \sum_{i=1}^{n_1} e_{1,i} [P_i^{d-1}]_{j} z_{2,j} \right\}
\]

which is exactly the normal form in (2.7). We claim \( V_2 = W_2 \oplus \text{Im} L_2 \). First, we check the dimension condition. One can compute

\[
\dim U_2 = n_1 \left[ \begin{array}{c} n + d - 1 \\ d \end{array} \right] - \left[ \begin{array}{c} n_1 + d - 1 \\ d \end{array} \right],
\]
\[
\dim V_2 = n_1 \left[ \begin{array}{c} n + d - 1 \\ d \end{array} \right] - \left[ \begin{array}{c} n + d - 1 - 1 \\ d \end{array} \right],
\]
\[
\dim W_2 = n_1 \left[ \begin{array}{c} n + d - 1 \\ d - 1 \end{array} \right]
\]

where \( n = n_1 + n_2 \). Since \( \ker L_2 = \{0\} \), \( \dim \text{Im} L_2 = \dim U_2 \). Hence \( \dim V_2 = \dim W_2 + \dim \text{Im} L_2 \) since

\[
\left[ \begin{array}{c} n + d - 1 \\ d \end{array} \right] = \left[ \begin{array}{c} n + d - 1 \\ d - 1 \end{array} \right] + \left[ \begin{array}{c} n + d - 1 \\ d - 1 \end{array} \right].
\]

Second, we show \( W_2 \cap \text{Im} L_2 = \{0\} \). Suppose \( f \in W_2 \cap \text{Im} L_2 \). Then there exists \( \phi_1 \in U_2 \) such
that \( f = L_2(\phi_1) \). We will use the similar argument to the one used in showing \( \ker L_2 = \{0\} \). Namely, we inductively compare the terms dependent on \( v, z_1, z_2, \ldots, z_{n_2} \) in \( L_2(\phi_1) \) and \( f \in W_2 \). The monomials dependent on \( v \) in \((2.15)\) are in \( \frac{\partial}{\partial z_2} B_2 v \) and the degree of \( v \) in the monomials is 1 whereas the degree of \( v \) in the monomials dependent on \( v \) in \( f \in W_2 \) is 2 or 4. It follows that \( \frac{\partial \phi_1}{\partial z_2} = 0 \). Inductively, one can get \( \frac{\partial \phi_1}{\partial z_j} = 0 \) for \( 1 \leq j \leq n_2 \).

Since \( \deg \phi_1 \geq 1 \), it follows that \( \phi_1 = 0 \) and thus \( f = 0 \). This proves \( W_2 \cap \ker L_2 = \{0\} \).

3. \( \bar{\mathcal{J}}_2^{[d]} \): Define the two vector spaces \( U_3 \) and \( V_3 \) by

\[
U_3 = \left\{ \left( \sum_{j=1}^{n_2} e_j \phi_j^{[d]}(z_1, z_2), \alpha^{[d]}(z_1, z_2, v) \right) \right\},
\]

\[
V_3 = \left\{ \sum_{j=1}^{n_2} e_j \phi_j^{[d]}(z_1, z_2, v) \right\}.
\]

where \( \alpha^{[d]}(z_1, z_2, v) \) is a homogeneous polynomial of degree \( d \) in \((z_1, z_2, v)\) in the definition of \( U_3 \).

The dimensions of \( U_3 \) and \( V_3 \) are

\[
\dim U_3 = n_2 \left( \binom{n + d - 1}{d} + \binom{n + d}{d} \right) \tag{2.17}
\]

\[
\dim V_3 = n_2 \left( \binom{n + d}{d} \right) \tag{2.18}
\]

where \( n = n_1 + n_2 \). Then the map \( L_3 := L|_{U_3} : U_3 \to V_3 \) becomes

\[L_3(\phi_2, \alpha) = A_2 \phi_2(z) + B_2 \alpha(z, v) - \frac{\partial \phi_1}{\partial z_1}(z) A_1 z_1 - \frac{\partial \phi_2}{\partial z_2}(z) A_2 z_2 - \frac{\partial \phi_2}{\partial z_2}(z) B_2 v \tag{2.19}\]

for \( \phi_2 \in U_3 \). We now compute the dimension of \( \ker L_3 \). Suppose \((\phi_2, \alpha) \in \ker L_3 \) with \( \phi_2(z_1, z_2) = \sum_{j=1}^{n_2} e_j \phi_j, \phi_j(z_1, z_2) \). It satisfies

\[A_2 \phi_2(z) + B_2 \alpha(z, v) - \frac{\partial \phi_1}{\partial z_1}(z) A_1 z_1 - \frac{\partial \phi_2}{\partial z_2}(z) A_2 z_2 - \frac{\partial \phi_2}{\partial z_2}(z) B_2 v = 0 \tag{2.20}\]

Recall the structure of \( A_2 \) and \( B_2 \) in \((2.3)\). The monomials dependent on \( v \) in the first \((n_2 - 1)\) rows of the LHS of \((2.20)\) are in the first \((n_2 - 1)\) rows of \( \frac{\partial \phi_2}{\partial z_2}(z) B_2 v \). It follows \( \frac{\partial \phi_2}{\partial z_2} = 0 \) for \( 1 \leq i \leq n_2 - 1 \). Namely, \( \phi_j \)'s are independent of \( z_2, n_2 \) for \( 1 \leq i \leq n_2 - 1 \). Now, the monomials dependent on \( z_2, n_2 \) in the first \((n_2 - 2)\) rows of the LHS of \((2.20)\) are in \( \frac{\partial \phi_2}{\partial z_2}(z) A_2 z_2 \). Hence, \( \frac{\partial \phi_2}{\partial z_2} = 0 \) for \( 1 \leq i \leq n_2 - 2 \). Inductively, one gets

\[\frac{\partial \phi_2}{\partial z_2} = 0, \quad \forall 1 \leq i < j \leq n_2. \tag{2.21}\]

In particular,

\[\frac{\partial \phi_2}{\partial z_2} = 0 \quad \forall 2 \leq j \leq n_2. \]

Then \( \phi_2 \) belongs to the space \( D \) of homogeneous polynomials of degree \( d \) in \((z_1, z_2, v)\). Also, notice that once we choose \( \phi_2 \) from \( D \), the remaining terms \((\phi_2, \ldots, \phi_2, \alpha)^t = A_2 \phi_2(z) + B_2 \alpha(z, v) \) are uniquely determined so that \((2.20)\) is satisfied. Hence \( \ker L_3 \) is isomorphic to the space \( D \), which implies

\[\dim \ker L_3 = \binom{n_1 + d}{d} \tag{2.22}\]

Define a subspace \( W_3 \) of \( V_3 \) by

\[W_3 = \left\{ \sum_{i=1}^{n_2 - 1} \sum_{j=1}^{n_2 + 1} e_{ij} R_{ij}^{[d-2]}(z_1, z_2, \ldots, z_{n_2}, v) \right\} \]

which is exactly the normal form in \((2.8)\). The dimension of \( W_3 \) is given by

\[\dim W_3 = \binom{n_1 + d}{d} - \binom{n + d - 1}{d} + \binom{n + d - 1}{d - 1} \tag{2.23}\]

where \( n = n_1 + n_2 \). We claim \( V_3 = W_3 \oplus \ker L_3 \). The dimension condition, \( \dim V_3 = \dim W_3 + \dim \ker L_3 \), can be easily checked by \((2.17), (2.18), (2.22), (2.23)\) and the formula, \( \dim U_3 = \dim \ker L_3 + \dim \ker L_3 \). We now show \( W_3 \cap \ker L_3 = \{0\} \).

Suppose there exists \((\phi_2, \alpha) \in U_3 \) such that \( L_3(\phi_2, \alpha) \in W_3 \cap \ker L_3 \). This can be written as

\[A_2 \phi_2(z) + B_2 \alpha(z, v) - \frac{\partial \phi_1}{\partial z_1}(z) A_1 z_1 - \frac{\partial \phi_2}{\partial z_2}(z) A_2 z_2 - \frac{\partial \phi_2}{\partial z_2}(z) B_2 v = 0 \tag{2.24}\]

for some \( R_{ij}^{[d-2]} \)'s. Compare the terms dependent on \( v \) in the first \((n_2 - 1)\) rows on both sides of \((2.24)\). Those terms on the RHS of \((2.24)\) are in the first \((n_2 - 1)\) rows of \( \frac{\partial \phi_2}{\partial z_2}(z) B_2 v \) and they depend on \( v \) linearly, whereas those terms on the RHS of \((2.24)\) depend on \( v^2 \) or \( v^4 \). Hence, \( \frac{\partial \phi_2}{\partial z_2} = 0 \) for \( 1 \leq i \leq n_2 - 1 \). As in the computation of the dimension of \( \ker L_3 \), one can inductively get \((2.21)\), which with \((2.19)\) implies that \( L_3(\phi_2, \alpha) \) is of the form

\[L_3(\phi_2, \alpha) = \sum_{i=1}^{n_2} \sum_{j=1}^{n_2 + 1} e_{ij} R_{ij}^{[d]}(z_1, z_2, \ldots, z_{n_2}). \]
which belongs to \( W_3 \) iff \( L_3(\phi_2, \alpha) = 0 \). Thus \( W_3 \cap \text{Im } L_3 = \{ \{0\} \} \). This completes the proof of the theorem. \( \square \)

**Remarks.** 1. The proof of Theorem 2.1 is not constructive. There are two basic operations on polynomials for systems with linear normal form: pull-up and push-down. The linear map \( L \) in the proof can be expressed as the composition of these two maps. For more details, refer to Krener et al. (2001) and Krener and Li (2001).

2. The normal form is not unique. A given system might be transformed into two different normal forms. The reason is as follows. We brought a given system into a normal form by the successive annihilation of the terms of degree 2, 3, etc. Notice that the lower degree transformations affect the higher degree terms, which we didn’t exploit in the derivation of the normal form. Tall and Respondek (2000) used this enlarged transformation group for the linearly controllable systems to refine the normal form, which is a canonical form. The problem of the derivation of the canonical form of linearly uncontrollable nonlinear systems is still open.

Next, we study the relationship between the normal form of degree \( d \) and invariants under the transformations in (2.10) and (2.11). This will give us a method of deriving the normal form of a given system. First we denote by \( O(k) \) all the polynomials in \( (x_1, x_2, u) \) of degree \( k \). Consider the transform in (2.11). Let \( G = \frac{\partial \pi}{\partial x} \) and \( \bar{G} = \frac{\partial \pi}{\partial y} \).

By the chain rule,

\[
G = \frac{\partial \pi}{\partial u} = (1 + O(d - 1)) \bar{G}.
\]

Let \( F(x_1, x_2, u) \) be the RHS of (2.2). It is straightforward to show by induction that for \( 0 \leq k \leq n_2 \)

\[
(-1)^k \text{ad}_F^k(G) = \frac{\partial}{\partial x_{2,n_2+1-k}} + O(1, \frac{\partial}{\partial x})
\]

and

\[
(-1)^{n_2-k} \text{ad}_F^{n_2+1}(G) = O(1, \frac{\partial}{\partial x})
\]

(2.25)

From (2.25), (2.26), and (2.27), we have the following lemma.

**Lemma 2.2.** The following polynomials are invariant under the transformations in (2.10) and (2.11):

\[
\pi_{1,i}^{(d-1)} \left( \text{ad}_F^{n_2+1}(G) \right),
\]

\[
\pi_{1,i}^{(d-2)} \left( \left[ \text{ad}_F^{n_2+1}(G), \text{ad}_F^k(G) \right] \right),
\]

\[
\pi_{2,j}^{(d-1)} \left( \left[ \text{ad}_F^{n_2+1}(G), \text{ad}_F^k(G) \right] \right)
\]

for \( 0 \leq k \leq n_2, 1 \leq i \leq n_1, 1 \leq j \leq n_2 - k \). \( \pi_{a,b} \) takes the terms of degree \( m \) from the \( \frac{\partial}{\partial x_{a,b}} \) -th component of a given vector-valued polynomial in \( (x_1, x_2, u) \).

We now show that the terms \( f_1^{[d],e} \) and \( f_2^{[d]} \) in the normal form of degree \( d \) are directly derived from the invariants in Lemma 2.2. Let \( F \) be the RHS of (2.6). One can show by induction that for \( 0 \leq p \leq n_2, l = n_2 + 2 - p \),

\[
(-1)^p \text{ad}_F^p(G) = \frac{\partial}{\partial x_{2,n_2+1-p}} + O(1, \frac{\partial}{\partial x})
\]

and

\[
\text{ad}_F^{n_2+1}(G) = \text{ad}_F^{n_2+1}(G) + \sum_{r=0}^{n_2} O(d - 1) \frac{\partial}{\partial x_{2,n_2+1-r}}
\]

(2.26)

where the \( h_{i,j}(\cdot) \)'s are some vector-valued polynomials of degree \( (d - 2) \) in their arguments. Then, for \( 0 \leq p \leq n_2 \) and \( q = n_2 + 1 - p \),
\[
\left[ \text{ad}_F^{n+1}(G), \text{ad}_F^n(G) \right] = \sum_{i=1}^{n_1} \frac{\partial^2 Q_{ij}^{d-3}}{\partial z_{2,i}^2} (z_1, z_{2,i}, \ldots, z_{2,q}) \frac{\partial}{\partial z_{2,i}} + \sum_{j=1}^{n_2+1} \frac{\partial^2 R_{ij}^{d-2}}{\partial z_{2,j}^2} (z_1, z_{2,i}, \ldots, z_{2,j}) \frac{\partial}{\partial z_{2,j}} + \Delta(d-3) \frac{\partial}{\partial z} + O(d-1) \frac{\partial}{\partial z},
\]

where the \( h_i^{d-3}() \)'s are some vector-valued homogeneous polynomials of degree (\( d - 3 \)) in their arguments and \( \Delta(d-3) \) denotes all the polynomials in \((x_1, x_2, u)\) of degree \((d - 3)\).

Let \( M_i = \{z_{2,i+1} = z_{2,i+2} = \ldots = z_{2,n_2+1} = 0\}\).

From (2.31) we have for \( 1 \leq i \leq n_1 \)
\[
\left. \pi[i]^{d-1} \left( (-1)^{n_2+1} \text{ad}_F^{n_2+1}(G) \right) \right|_{M_i} = p[i]^{d-1}(z_1) + \frac{\partial Q_{ij}^{d-2}}{\partial z_{2,i}} (z_1, z_{2,i}, z_{2,j}^2),
\]

One can also show that for \( 1 \leq i \leq n_1, 1 \leq j \leq n_2 + 1 \)
\[
\left. \pi[i]^{d-2} \left( \left[ \text{ad}_F^{n_2+2-j}(G), \text{ad}_F^{n_2+1-j}(G) \right] \right) \right|_{M_i} = \frac{\partial^2 Q_{ij}^{d-2}}{\partial z_{2,j}^2} (z_1, z_{2,i}, \ldots, z_{2,j}) z_{2,j},
\]

and that for \( 1 \leq i \leq n_2 - 1 \) and \( i + 2 \leq j \leq n_2 + 1 \)
\[
\left. \pi[i]^{d-2} \left( \left[ \text{ad}_F^{n_2+2-j}(G), \text{ad}_F^{n_2+1-j}(G) \right] \right) \right|_{M_i} = \frac{\partial^2 R_{ij}^{d-2}}{\partial z_{2,j}^2} (z_1, z_{2,i}, \ldots, z_{2,j}) z_{2,j}. \]

From (2.32), (2.33), (2.34) and Lemma 2.2, it follows that the terms \( f[i]^{d-1} \) and \( f[i]^{d} \) in the normal form of degree \( d \geq 2 \) are invariant under the transformations in (2.10) and (2.11) and readily computed from the invariants in Lemma 2.2.

We now show that when \( A_i \) is diagonal, the coefficients \( \beta[i]^{d-j} \) in (2.9) are invariants under the transformations in (2.10) and (2.11). Clearly, they are potentially changed only by \( \phi[i]^{d-j}(x_1) \). Therefore we need only consider coordinates changes of the form
\[
\tilde{x}_{1,i} = x_{1,i} + c x_{1,j_1} \cdots x_{1,j_a},
\]

where \( 1 \leq i \leq n_1, 1 \leq j_1 \leq \ldots \leq j_a \leq n_1 \) because more general ones are just compositions of these. This coordinate change only affects a piece of the dynamics (2.2),
\[
\tilde{x}_{1,i} = \lambda_i x_{1,i} + \sum_{k=2}^{d} f[i]_j^k(x_1, x_2, u) + O(d+1).
\]

is transformed to
\[
\tilde{x}_{1,i} = \lambda_i x_{1,i} + \sum_{k=2}^{d} f[i]_j^k(x_1, x_2, u) - c \left( \sum_{i=1}^{d} \lambda_i \right) x_{1,j_1} \cdots x_{1,j_a} + O(d+1).
\]

Clearly, if \( \lambda_i = \sum_{i=1}^{d} \lambda_i \), then the coefficient of \( x_{1,j_1} \cdots x_{1,j_a} \) is unchanged. It follows that the \( \beta[i]^{d-j} \)'s are invariant under the transformations in (2.10) and (2.11).

We now show the dual norm form without proof. The replacement of \( \tilde{f}[i]^{d-1} \) in (2.8) by the following gives the dual norm form:
\[
\sum_{i=1}^{n_1} \sum_{j=n_2+2}^{n_2+1} \phi[i]^{d-j}(z_1, z_{2,i}, \ldots, z_{2,j}) z_{2,j},
\]

for some homogeneous polynomials \( \tilde{f}[i]^{d-j} \) of degree \( (d - 2) \). It may not be possible to transform \( \tilde{f}[i]^{d-j} \) in (2.7) because of resonances.

3. REFERENCES


