NORMAL FORMS OF LINEARLY UNCONTROLLABLE NONLINEAR CONTROL SYSTEMS WITH A SINGLE INPUT

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Abstract: We derive the controller normal form and the dual normal form for linearly uncontrollable nonlinear control systems with a single input. The invariants under the state and feedback transformation of degree d are found for $d \geq 2$.

Keywords: nonlinear control systems, normal forms, invariants

1. INTRODUCTION

The Poincaré normal form is known to be useful in the bifurcation analysis of dynamical systems. Kang and Krener (1992) initiated the extension of Poincaré normal form theory to nonlinear control systems, where they considered quadratic controller normal forms of linearly controllable nonlinear control systems with a single input. The controller normal form has been proved useful in the analysis of control bifurcations (see Chang et al (2000),Kang (1998), and Krener et al. (2001)). Kang (1996) extended Kang and Krener (1992) by deriving the controller normal form of arbitrary degree of linearly controllable nonlinear control systems with a single input, which was further extended in Tall and Respondek (2000), refining the normal form in Kang (1996) by using a larger transformation group. On the other hand, Chang et al. (2000) and Krener et al. (2001) extended Kang and Krener (1992) by deriving the controller normal form up to degree three of linearly *uncontrollable* nonlinear control systems with a single input. In this paper, we extend Chang et al. (2000), Kang (1996) and Tall and Respondek (2000) by deriving the controller normal form and the dual normal form of arbitrary degree of linearly *uncontrollable* nonlinear control systems with a single input and finding a set of invariants which help compute the normal form.

2. MAIN RESULTS

Consider a smooth nonlinear control system

$$\dot{x} = f(x, u) \tag{2.1}$$

where x is n dimensional, u is one dimensional and $f(0,0) = 0$. It is well known that by a linear change of state coordinates, linear state feedback

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and Taylor series expansion, the system can be brought to the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u + \sum_{k=2}^{d} \begin{bmatrix} f_1^{[k]}(x_1, x_2, u) \\ f_2^{[k]}(x_1, x_2, u) \end{bmatrix} + O(x_1, x_2, u)^{d+1} \tag{2.2}$$

where x_1, x_2 are n_1, n_2 dimensional, $n_1 + n_2 = n$, A_1 is in Jordan form, A_2, B_2 are in controller (Brunovsky) form and $f_i^{[k]}(x_1, x_2, u)$ is an n_i dimensional vector-valued homogeneous polynomial of degree k in its arguments. The linear change of coordinates that brings A_1 to Jordan form may be complex, in which case some of the coordinates $x_{1,i}$ are complex. The complex coordinates come in conjugate pairs. The corresponding $f_{1,i}^{[k]}$'s are complex-valued and come in conjugate pairs.

A pair A_2, B_2 is in controller form if

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \ddots & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{2.3}$$

Theorem 2.1. Consider a smooth (C^∞) control system (2.2) where A_1 is in Jordan form and the input is scalar. There exist a change of coordinates and a feedback of the form

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \sum_{k=2}^{d} \begin{bmatrix} \phi_1^{[k]}(x_1, x_2) \\ \phi_2^{[k]}(x_1, x_2) \end{bmatrix} \tag{2.4}$$

$$v = u - \sum_{k=2}^{d} \alpha^{[k]}(x_1, x_2, u) \tag{2.5}$$

which transform the system (2.2) into the normal form

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} v + \sum_{k=2}^{d} \begin{bmatrix} \tilde{f}_1^{[k],\mathrm{P}}(z_1) + \tilde{f}_1^{[k],c}(z_1, z_2, v) \\ \tilde{f}_2^{[k]}(z_1, z_2, v) \end{bmatrix} + O(z_1, z_2, v)^{d+1} \tag{2.6}$$

where $\tilde{f}^{[k],\mathrm{P}}$ is in the Poincaré normal form of degree k (see Arnold (1983)) and the other vector fields are as follows:

$$\tilde{f}_1^{[k],c} = \sum_{i=1}^{n_1} \mathbf{e}_{1,i} [P_i^{[k-1]}(z_1) z_{2,1} + \sum_{j=1}^{n_2+1} Q_{ij}^{[k-2]}(z_1, z_{2,1}, \dots, z_{2,j}) z_{2,j}^2] \tag{2.7}$$

$$\tilde{f}_2^{[k]} = \sum_{i=1}^{n_2-1} \sum_{j=i+2}^{n_2+1} \mathbf{e}_{2,i} R_{ij}^{[k-2]}(z_1, z_{2,1}, \dots, z_{2,j}) z_{2,j}^2. \tag{2.8}$$

where $\mathbf{e}_{i,r}$ is the r^{th} unit vector in z_i space, $P_i^{[m]}(\cdot)$, $Q_{ij}^{[m]}(\cdot)$ and $R_{ij}^{[m]}(\cdot)$ are arbitrary homogeneous polynomials of degree m, and for notational convenience we have defined $z_{2,n_2+1} = v$.

In particular, when A_1 is diagonal, the Poincaré normal form $\tilde{f}^{[k],\mathrm{P}}(z_1)$ is given by

$$\tilde{f}_1^{[k],\mathrm{P}} = \sum_{i=1}^{n_1} \mathbf{e}_{1,i} \sum_{\substack{1 \le j_1 \le \dots \le j_k \le n_1 \\ \lambda_{j_1} + \dots + \lambda_{j_k} = \lambda_i}} \beta_i^{j_1, \dots, j_k} z_{1,j_1} \cdots z_{1,j_k}. \tag{2.9}$$

Proof. First notice that the following d^{th} order transformation with $d > 1$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} \phi_1^{[d]}(x_1, x_2) \\ \phi_2^{[d]}(x_1, x_2) \end{bmatrix} \tag{2.10}$$

$$v = u - \alpha^{[d]}(x_1, x_2, u) \tag{2.11}$$

does not change the terms of degree less than d. Without loss of generality, we may assume that the given system is in the normal form up to degree $(d-1)$. Consider two systems in the same normal forms up to degree $(d-1)$ as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u + \sum_{k=2}^{d-1} \begin{bmatrix} \tilde{f}_1^{[k]}(x_1, x_2, u) \\ \tilde{f}_2^{[k]}(x_1, x_2, u) \end{bmatrix} + \begin{bmatrix} f_1^{[d]}(x_1, x_2, u) \\ f_2^{[d]}(x_1, x_2, u) \end{bmatrix} + O(x_1, x_2, u)^{d+1} \tag{2.12}$$

and

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} v + \sum_{k=2}^{d-1} \begin{bmatrix} \tilde{f}_1^{[k]}(z_1, z_2, v) \\ \tilde{f}_2^{[k]}(z_1, z_2, v) \end{bmatrix} + \begin{bmatrix} \tilde{f}_1^{[d]}(z_1, z_2, v) \\ \tilde{f}_2^{[d]}(z_1, z_2, v) \end{bmatrix} + O(z_1, z_2, v)^{d+1} \tag{2.13}$$

where $\dim x_i = \dim z_i$, $i = 1, 2$.

Define the two vector spaces U and V as follows.

$$U = \left\{ (\phi^{[d]}(z), \alpha^{[d]}(z,v)) \mid \phi^{[d]} = \begin{bmatrix} \phi_1^{[d]} \\ \phi_2^{[d]} \end{bmatrix} \right\},$$
$$V = \left\{ g^{[d]}(z,v) \mid g^{[d]} = \begin{bmatrix} g_1^{[d]} \\ g_2^{[d]} \end{bmatrix} \right\}$$

where $\phi_i^{[d]}$ and $g_i^{[d]}$ are n_i-dimensional vector valued homogeneous polynomials of degree d in their arguments with $i = 1, 2$ and $\alpha^{[d]}$ is a homogeneous polynomial of degree d. Define a linear map $L : U \to V$ by

$$\begin{aligned} L(\phi^{[d]}, \alpha^{[d]})_i = A_i\phi_i^{[d]}(z) - \frac{\partial \phi_i^{[d]}}{\partial z_1}(z)A_1 z_1 \\ -\frac{\partial \phi_i^{[d]}}{\partial z_2}(z)A_2 z_2 - \frac{\partial \phi_i^{[d]}}{\partial z_2}(z)B_2 v \\ +B_i\alpha^{[d]}(z,v) \end{aligned} \quad (2.14)$$

for $i = 1, 2$. One can see that the transformation in (2.10) and (2.11) transforms the system (2.12) to the system (2.13) iff

$$\bar{f}^{[d]}(z,v) = f^{[d]}(z,v) + L(\phi^{[d]}, \alpha^{[d]})(z,v)$$

where

$$f^{[d]} = \begin{bmatrix} f_1^{[d]} \\ f_2^{[d]} \end{bmatrix}, \quad \bar{f}^{[d]} = \begin{bmatrix} \bar{f}_1^{[d]} \\ \bar{f}_2^{[d]} \end{bmatrix}.$$

Let $p : V \to V/\mathrm{Im}\, L$ be the quotient map. To find a normal form is to find a subspace W of V such that W is isomorphic to $V/\mathrm{Im}\, L$ via p. Namely, it boils down to finding a subspace W of V such that $V = W \oplus \mathrm{Im}\, L$. In this proof, we decompose the spaces U, V and the map L as $U = \oplus_{i=1}^3 U_i$, $V = \oplus_{i=1}^3 V_i$, and $L = \oplus_{i=1}^3 L_i$ with $L_i := L|_{U^i} : U_i \to V_i$, in order to decompose the problem of deriving the normal forms into three subproblems.

1. $\bar{f}_1^{[d],P}$: Define the two vector spaces U_1 and V_1 by $U_1 = \left\{\sum_{i=1}^{n_1} \mathbf{e}_{1,i}\phi_{1,i}^{[d]}(z_1)\right\}$ and $V_1 = \left\{\sum_{i=1}^{n_1} \mathbf{e}_{1,i}f_{1,i}^{[d]}(z_1)\right\}$. Then $L_1 := L|_{U_1} : U_1 \to V_1$ becomes

$$L_1(\phi_1) = A_1\phi_1(z_1) - \frac{\partial \phi_1}{\partial z_1}(z_1)A_1 z_1$$

which is the same as the Lie derivative $(-1)L_{A_1}$ in Arnold (1983, p. 179). Hence, one can transform any element ϕ_1 of U_1 into a Poincaré normal form of degree d. In particular, when A_1 is diagonal, the Poincaré normal form of arbitrary degree $k \geq 2$ is given by (2.9) (see Arnold (1983) for more detail on the Poincaré normal forms).

2. $\bar{f}_1^{[d],c}$: Define the two vector spaces U_2 and V_2 by

$$U_2 = \left\{ \sum_{i=1}^{n_1} \mathbf{e}_{1,i}\phi_{1,i}^{[d]}(z_1,z_2) \mid \deg_2\phi_{1,i}^{[d]} \geq 1 \right\},$$
$$V_2 = \left\{ \sum_{i=1}^{n_1} \mathbf{e}_{1,i}f_{1,i}^{[d]}(z_1,z_2,v) \mid \deg_2 f_{1,i}^{[d]} \geq 1 \right\}$$

where $\deg_2$ is the map which assigns to a polynomial in (z_1, z_2, v) the lowest power of (z_2, v) in the polynomial. Then $L_2 := L|_{U_2} : U_2 \to V_2$ becomes

$$\begin{aligned} L_2(\phi_1) = A_1\phi_1(z) - \frac{\partial \phi_1}{\partial z_1}(z)A_1 z_1 - \frac{\partial \phi_1}{\partial z_2}(z)A_2 z_2 \\ -\frac{\partial \phi_1}{\partial z_2}(z)B_2 v \end{aligned} \quad (2.15)$$

for $\phi_1 \in U_2$. We claim $\ker L_2 = \{0\}$. Suppose $\phi_1 \in \ker L_2$. It satisfies

$$\begin{aligned} A_1\phi_1(z) - \frac{\partial \phi_1}{\partial z_1}(z)A_1 z_1 - \frac{\partial \phi_1}{\partial z_2}(z)A_2 z_2 \\ - \frac{\partial \phi_1}{\partial z_2}(z)B_2 v = 0. \end{aligned} \quad (2.16)$$

Recall the special forms of A_2 and B_2 in (2.3). Since only the term $\frac{\partial \phi_1}{\partial z_2}(z)B_2 v$ in (2.16) depends on v, it follows that $\frac{\partial \phi_1}{\partial z_{2,n_2}} = 0$. Hence ϕ_1 does not depend on z_{2,n_2}. With this result in mind, notice that the term $\frac{\partial \phi_1}{\partial z_2}(z)A_2 z_2$ is the only term in (2.16) which depends on z_{2,n_2}. It follows $\frac{\partial \phi_1}{\partial z_{2,n_2-1}} = 0$. Inductively, one can show $\frac{\partial \phi_1}{\partial z_{2,i}} = 0$ for $i = 1, \ldots, n_2$, which implies $\phi_1 = 0$ since $\deg_2\phi_1 \geq 1$. Hence $\ker L_2 = \{0\}$.

Define a subspace W_2 of V_2 by

$$W_2 = \left\{ \sum_{i=1}^{n_1} \mathbf{e}_{1,i}[P_i^{[d-1]}(z_1)z_{2,1} + \sum_{j=1}^{n_2+1} Q_{ij}^{[d-2]}(z_1, z_{2,1}, \ldots, z_{2,j})z_{2,j}^2] \right\}$$

which is exactly the normal form in (2.7). We claim $V_2 = W_2 \oplus \mathrm{Im}\, L_2$. First, we check the dimension condition. One can compute

$$\mathrm{dim} U_2 = n_1 \left[\binom{n+d-1}{d} - \binom{n_1+d-1}{d} \right],$$
$$\mathrm{dim} V_2 = n_1 \left[\binom{n+d}{d} - \binom{n_1+d-1}{d} \right],$$
$$\mathrm{dim} W_2 = n_1 \binom{n+d-1}{d-1}$$

where $n = n_1 + n_2$. Since $\ker L_2 = \{0\}$, $\dim \mathrm{Im}\, L_2 = \dim U_2$. Hence $\mathrm{dim} V_2 = \mathrm{dim} W_2 + \dim \mathrm{Im}\, L_2$ since

$$\binom{n+d}{d} = \binom{n+d-1}{d} + \binom{n+d-1}{d-1}.$$

Second, we show $W_2 \cap \mathrm{Im}\, L_2 = \{0\}$. Suppose $f \in W_2 \cap \mathrm{Im} L_2$. Then there exists $\phi_1 \in U_2$ such that $f = L_2(\phi_1)$. We will use the similar argument to the one used in showing $\ker L_2 = \{0\}$. Namely, we inductively compare the terms dependent on $v, z_{2,n_2}, \dots, z_{2,2}$ in $L_2(\phi_1)$ and $f \in W_2$. The monomials dependent on v in (2.15) are in $\frac{\partial \phi_1}{\partial z_2} B_2 v$ and the degree of v in the monomials is 1 whereas the degree of v in the monomials dependent on v in $f \in W_2$ is 2 or 4. It follows that $\frac{\partial \phi_1}{\partial z_{2,n_2}} = 0$. Inductively, one can get $\frac{\partial \phi_1}{\partial z_{2,j}} = 0$ for $1 \le j \le n_2$. Since $\deg_2 \phi_1 \ge 1$, it follows that $\phi_1 = 0$ and thus $f = 0$. This proves $W_2 \cap \mathrm{Im}\, L_2 = \{0\}$.

3. $\tilde{f}_2^{[d]}$: Define the two vector spaces U_3 and V_3 by

$$U_3 = \left\{ \left(\sum_{i=1}^{n_2} \mathbf{e}_{2,i} \phi_{2,i}^{[d]}(z_1, z_2),\ \alpha^{[d]}(z_1, z_2, v) \right) \right\},$$
$$V_3 = \left\{ \sum_{i=1}^{n_1} \mathbf{e}_{2,i} f_{2,i}^{[d]}(z_1, z_2, v) \right\}.$$

where $\alpha^{[d]}(z_1, z_2, v)$ is a homogeneous polynomial of degree d in (z_1, z_2, v) in the definition of U_3. The dimensions of U_3 and V_3 are

$$\dim U_3 = n_2 \binom{n+d-1}{d} + \binom{n+d}{d} \quad (2.17)$$
$$\dim V_3 = n_2 \binom{n+d}{d} \quad (2.18)$$

where $n = n_1 + n_2$. Then the map $L_3 := L|_{U_3} : U_3 \to V_3$ becomes

$$L_3(\phi_2, \alpha) = A_2 \phi_2(z) + B_2 \alpha(z, v) - \frac{\partial \phi_1}{\partial z_1}(z) A_1 z_1 - \frac{\partial \phi_2}{\partial z_2}(z) A_2 z_2 - \frac{\partial \phi_2}{\partial z_2}(z) B_2 v \quad (2.19)$$

for $\phi_2 \in U_3$. We now compute the dimension of $\ker L_3$. Suppose $(\phi_2, \alpha) \in \ker L_3$ with $\phi_2(z_1, z_2) = \sum_{i=1}^{n_2} \mathbf{e}_{2,i} \phi_{2,i}(z_1, z_2)$. It satisfies

$$A_2 \phi_2(z) + B_2 \alpha(z, v) - \frac{\partial \phi_2}{\partial z_1}(z) A_1 z_1 - \frac{\partial \phi_2}{\partial z_2}(z) A_2 z_2 - \frac{\partial \phi_2}{\partial z_2}(z) B_2 v = 0. \quad (2.20)$$

Recall the structure of A_2 and B_2 in (2.3). The monomials dependent on v in the first (n_2-1) rows of the LHS of (2.20) are in the first (n_2-1) rows of $\frac{\partial \phi_2}{\partial z_2}(z) B_2 v$. It follows $\frac{\partial \phi_{2,i}}{\partial z_{2,n_2}} = 0$ for $1 \le i \le n_2 - 1$. Namely, $\phi_{2,i}$'s are independent of z_{2,n_2} for $1 \le i \le n_2 - 1$. Now, the monomials dependent on z_{2,n_2} in the first $(n_2 - 2)$ rows of the LHS of (2.20) are in $\frac{\partial \phi_2}{\partial z_2}(z) A_2 z_2$. Hence, $\frac{\partial \phi_{2,i}}{\partial z_{2,n_2-1}} = 0$ for $1 \le i \le n_2 - 2$. Inductively, one gets

$$\frac{\partial \phi_{2,i}}{\partial z_{2,j}} = 0, \quad \forall\, 1 \le i < j \le n_2. \quad (2.21)$$

In particular,

$$\frac{\partial \phi_{2,1}}{\partial z_{2,j}} = 0 \quad \forall\, 2 \le j \le n_2.$$

Then $\phi_{2,1}$ belongs to the space D of homogeneous polynomials of degree d in $(z_1, z_{2,1})$. Also notice that once we choose $\phi_{2,1}$ from D, the remaining terms $(\phi_{2,2}, \dots, \phi_{2,n_2}, \alpha)^t = A_2 \phi_2(z) + B_2 \alpha(z, v)$ are uniquely determined so that (2.20) is satisfied. Hence $\ker L_3$ is isomorphic to the space D, which implies

$$\dim \ker L_3 = \binom{n_1 + d}{d}. \quad (2.22)$$

Define a subspace W_3 of V_3 by

$$W_3 = \left\{ \sum_{i=1}^{n_2-1} \sum_{j=i+2}^{n_2+1} \mathbf{e}_{2i} R_{ij}^{[d-2]}(z_1, z_{2,1}, \dots, z_{2,j}) z_{2,j}^2 \right\}$$

which is exactly the normal form in (2.8). The dimension of W_3 is given by

$$\dim W_3 = \binom{n_1 + d}{d} - \binom{n+d-1}{d} + (n_2 - 1) \binom{n+d-1}{d-1} \quad (2.23)$$

where $n = n_1 + n_2$. We claim $V_3 = W_3 \oplus \mathrm{Im} L_3$. The dimension condition, $\dim V_3 = \dim W_3 + \dim \mathrm{Im} L_3$, can be easily checked by (2.17), (2.18), (2.22), (2.23) and the formula, $\dim U_3 = \dim \ker L_3 + \dim \mathrm{Im} L_3$. We now show $W_3 \cap \mathrm{Im} L_3 = \{0\}$. Suppose there exists $(\phi_2, \alpha) \in U_3$ such that $L_3(\phi_2, \alpha) \in W_3 \cap \mathrm{Im} L$. This can be written as

$$A_2\, \phi_2(z) + B_2 \alpha(z, v) - \frac{\partial \phi_1}{\partial z_1}(z) A_1 z_1 - \frac{\partial \phi_2}{\partial z_2}(z) A_2 z_2 - \frac{\partial \phi_2}{\partial z_2}(z) B_2 v = \sum_{i=1}^{n_2-1} \sum_{j=i+2}^{n_2+1} \mathbf{e}_{2,i} R_{ij}^{[d-2]}(z_1, z_{2,1}, \dots, z_{2,j}) z_{2,j}^2 \quad (2.24)$$

for some $R_{ij}^{[d-2]}$'s. Compare the terms dependent on v in the first $(n_2 - 1)$ rows on both sides of (2.24). Those terms on the LHS of (2.24) are in the first $(n_2 - 1)$ rows of $\frac{\partial \phi_2}{\partial z_2}(z) B_2 v$ and they depend on v linearly, whereas those terms on the RHS of (2.24) depend on v^2 or v^4. Hence, $\frac{\partial \phi_{2,i}}{\partial z_{2,n_2}} = 0$ for $1 \le i \le n_2 - 1$. As in the computation of the dimension of $\ker L_3$, one can inductively get (2.21), which with (2.19) implies that $L_3(\phi_2, \alpha)$ is of the form

$$L_3(\phi_2, \alpha) = \sum_{i=1}^{n_2} \sum_{j=1}^{i+1} \mathbf{e}_{2,i} \tilde{R}_{ij}^{[d]}(z_1, z_{2,1}, \dots, z_{2,j}),$$

which belongs to W_3 iff $L_3(\phi_2, \alpha)=0$. Thus $W_3 \cap \operatorname{Im} L_3 = \{0\}$. This completes the proof of the theorem. □

Remarks. 1. The proof of Theorem 2.1 is not constructive. There are two basic operations on polynomials for systems with linear normal form; pull-up and push-down. The linear map L in the proof can be expressed as the composition of these two maps. For more details, refer to Krener et al. (2001) and Krener and Li (2001).

2. The normal form is not unique. A given system might be transformed into two different normal forms. The reason is as follows. We brought a given system into a normal form by the successive annihilation of the terms of degree 2, 3, etc. Notice that the lower degree transformations affect the higher degree terms, which we didn't exploit in the derivation of the normal form. Tall and Respondek (2000) used this enlarged transformation group for the linearly controllable systems to refine the normal form, which is a canonical form. The problem of the derivation of the canonical form of linearly *uncontrollable* nonlinear systems is still open.

Next, we study the relationship between the normal form of degree d and invariants under the transformations in (2.10) and (2.11). This will give us a method of deriving the normal form of a given system. First we denote by $O(k)$ all the polynomials in (x_1, x_2, u) of degree $\geq k$. Consider the transform in (2.11). Let $G = \frac{\partial}{\partial u}$ and $\bar{G} = \frac{\partial}{\partial v}$. By the chain rule,

$$G = \frac{\partial v}{\partial u}\bar{G} = (1 + O(d-1))\bar{G}.$$

Let $F(x_1, x_2, u)$ be the RHS of (2.2). It is straightforward to show by induction that for $0 \leq k \leq n_2$

$$(-1)^k \mathbf{ad}_F^k(G) = \frac{\partial}{\partial x_{2,n_2+1-k}} + O(1)\frac{\partial}{\partial x},$$
$$(-1)^{n_2+1}\mathbf{ad}_F^{n_2+1}(G) = O(1)\frac{\partial}{\partial x} \qquad (2.25)$$

and

$$\mathbf{ad}_F^k(G) = \mathbf{ad}_F^k(\bar{G}) + \sum_{r=0}^{k} O(d-1)\frac{\partial}{\partial x_{2,n_2+1-r}} + O(d)\frac{\partial}{\partial x}$$

$$\mathbf{ad}_F^{n_2+1}(G) = \mathbf{ad}_F^{n_2+1}(\bar{G}) + \sum_{r=0}^{n_2} O(d-1)\frac{\partial}{\partial x_{2,n_2+1-r}} + O(d)\frac{\partial}{\partial x} \qquad (2.26)$$

so

$$\left[\mathbf{ad}_F^{k+1}(G), \mathbf{ad}_F^k(G)\right] = \left[\mathbf{ad}_F^{k+1}(\bar{G}), \mathbf{ad}_F^k(\bar{G})\right] + \sum_{r=0}^{k+1} O(d-2)\frac{\partial}{\partial x_{2,n_2+1-r}} + O(d-1)\frac{\partial}{\partial x}. \qquad (2.27)$$

From (2.25), (2.26), and (2.27), we have the following lemma.

Lemma 2.2. The following polynomials are invariant under the transformations in (2.10) and (2.11):

$$\pi_{1,i}^{[d-1]}\left(\mathbf{ad}_F^{n_2+1}(G)\right), \qquad (2.28)$$
$$\pi_{1,i}^{[d-2]}\left(\left[\mathbf{ad}_F^{k+1}(G), \mathbf{ad}_F^k(G)\right]\right), \qquad (2.29)$$
$$\pi_{2,j}^{[d-2]}\left(\left[\mathbf{ad}_F^{k+1}(G), \mathbf{ad}_F^k(G)\right]\right) \qquad (2.30)$$

for $0 \leq k \leq n_2$, $1 \leq i \leq n_1$, $1 \leq j \leq n_2 - k - 1$ where $\pi_{a,b}^{[m]}$ takes the terms of degree m from the $\frac{\partial}{\partial x_{a,b}}$-th component of a given vector-valued polynomial in (x_1, x_2, u).

We now show that the terms $f_1^{[d],c}$ and $f_2^{[d]}$ in the normal form of degree d are directly derived from the invariants in Lemma 2.2. Let F be the RHS of (2.6). One can show by induction that for $0 \leq p \leq n_2$, $l = n_2 + 2 - p$,

$$\begin{aligned}(-1)^p \mathbf{ad}_F^p(G) &= \frac{\partial}{\partial z_{2,n_2+1-p}} + \sum_{k=2}^{d}\left[\sum_{i=1}^{n_1}\frac{\partial[Q_{il}^{[k-2]}(z_1, z_{2,1}, \ldots, z_{2,l})z_{2,l}^2]}{\partial z_{2,l}}\frac{\partial}{\partial z_{1,i}} + \sum_{i=1}^{l-2}\frac{\partial[R_{il}^{[k-2]}(z_1, z_{2,1}, \ldots, z_{2,l})z_{2,l}^2]}{\partial z_{2,l}}\frac{\partial}{\partial z_{2,i}}\right] \\ &\quad + \sum_{j=l+1}^{n_2+1} h_{p,j}(z_1, z_{2,1}, \ldots, z_{2,j})z_{2,j} + O(d)\frac{\partial}{\partial z}\end{aligned}$$

and

$$\begin{aligned}(-1)^{n_2+1}\mathbf{ad}_F^{n_2+1}(G) &= \sum_{k=2}^{d}\left[\sum_{i=1}^{n_1} P_i^{[k-1]}(z_1)\frac{\partial}{\partial z_{1,i}} + \sum_{i=1}^{n_1}\frac{\partial[Q_{i1}^{[k-2]}(z_1, z_{2,1})z_{2,1}^2]}{\partial z_{2,1}}\frac{\partial}{\partial z_{1,i}}\right] \\ &\quad + \sum_{j=2}^{n_2+1} h_{n_2+1,j}(z_1, z_{2,1}, \ldots, z_{2,j})z_{2,j} + O(d)\frac{\partial}{\partial z}\end{aligned} \qquad (2.31)$$

where the $h_{i,j}(\cdot)$'s are some *vector-valued* polynomials of degree $\leq (d-2)$ in their arguments. Then, for $0 \leq p \leq n_2$ and $q = n_2 + 1 - p$,

$$
\begin{aligned}
&\left[\mathbf{ad}_F^{p+1}(G), \mathbf{ad}_F^{p}(G)\right] \\
&= \sum_{i=1}^{n_1} \frac{\partial^2 [Q_{iq}^{[d-2]}(z_1, z_{2,1}, \dots, z_{2,q}) z_{2,q}^2]}{\partial z_{2,q}^2} \frac{\partial}{\partial z_{1,i}} \\
&+ \sum_{i=1}^{q-2} \frac{\partial^2 [R_{iq}^{[d-2]}(z_1, z_{2,1}, \dots, z_{2,q}) z_{2,q}^2]}{\partial z_{2,q}^2} \frac{\partial}{\partial z_{2,i}} \\
&+ \sum_{j=q+1}^{n_2+1} \bar{h}_j^{[d-3]}(z_1, z_{2,1}, \dots, z_{2,j}) z_{2,j} \\
&+ \triangle(d-3)\frac{\partial}{\partial z} + O(d-1)\frac{\partial}{\partial z}
\end{aligned}
$$

where the $\bar{h}_j^{[d-3]}(\cdot)$'s are some *vector-valued* homogeneous polynomials of degree $(d-3)$ in their arguments and $\triangle(d-3)$ denotes all the polynomials in (x_1, x_2, u) of degree $\leq (d-3)$.

Let $M_i = \{z_{2,i+1} = z_{2,i+2} = \dots = z_{2,n_2+1} = 0\}$. From (2.31) we have for $1 \leq i \leq n_1$

$$
\begin{aligned}
&\pi_{1,i}^{[d-1]}((-1)^{n_2+1}\mathbf{ad}_F^{n_2+1}(G))\Big|_{M_1} \qquad (2.32) \\
&= P_i^{[d-1]}(z_1) + \frac{\partial [Q_{i1}^{[d-2]}(z_1, z_{2,1}) z_{2,1}^2]}{\partial z_{2,1}}.
\end{aligned}
$$

One can also show that for $1 \leq i \leq n_1$, $1 \leq j \leq n_2+1$

$$
\begin{aligned}
&\pi_{1,i}^{[d-2]}\left(\left[\mathbf{ad}_F^{n_2+2-j}(G), \mathbf{ad}_F^{n_2+1-j}(G)\right]\right)\Big|_{M_j} \\
&= \frac{\partial^2 [Q_{ij}^{[d-2]}(z_1, z_{2,1}, \dots, z_{2,j}) z_{2,j}^2]}{\partial z_{2,j}^2} \qquad (2.33)
\end{aligned}
$$

and that for $1 \leq i \leq n_2 - 1$ and $i+2 \leq j \leq n_2+1$

$$
\begin{aligned}
&\pi_{2,i}^{[d-2]}\left(\left[\mathbf{ad}_F^{n_2+2-j}(G), \mathbf{ad}_F^{n_2+1-j}(G)\right]\right)\Big|_{M_j} \\
&= \frac{\partial^2 [R_{ij}^{[d-2]}(z_1, z_{2,1}, \dots, z_{2,j}) z_{2,j}^2]}{\partial z_{2,j}^2}. \qquad (2.34)
\end{aligned}
$$

From (2.32), (2.33), (2.34) and Lemma 2.2, it follows that the terms $\tilde{f}_1^{[d],c}$ and $\tilde{f}_2^{[d]}$ in the normal form of degree $d \geq 2$ are invariant under the transformations in (2.10) and (2.11) and readily computed from the invariants in Lemma 2.2.

We now show that when A_1 is diagonal, the coefficients $\beta_i^{j_1,\cdots,j_d}$ in (2.9) are invariants under the transformations in (2.10) and (2.11). Clearly, they are potentially changed only by $\phi_1^{[d]}(x_1)$. Therefore we need only consider coordinates changes of the form

$$\bar{x}_{1,i} = x_{1,i} + c x_{1,j_1} \cdots x_{1,j_d}$$

where $1 \leq i \leq n_1$, $1 \leq j_1 \leq \dots \leq j_d \leq n_1$ because more general ones are just compositions of these. This coordinate change only affects a piece of the dynamics (2.2).

$$\dot{x}_{1,i} = \lambda_i x_{1,i} + \sum_{k=2}^{d} f_{1,i}^{[k]}(x_1, x_2, u) + O(d+1)$$

is transformed to

$$
\begin{aligned}
\dot{\bar{x}}_{1,i} &= \lambda_i \bar{x}_{1,i} + \sum_{k=2}^{d} f_{1,i}^{[k]}(x_1, x_2, u) \\
&\quad - c\left(\lambda_i - \sum_{l=1}^{d} \lambda_{j_l}\right) x_{1,j_1} \cdots x_{1,j_d} \\
&\quad + O(d+1).
\end{aligned}
$$

Clearly, if $\lambda_i = \sum_{l=1}^{d} \lambda_{j_l}$, then the coefficient of $x_{1,j_1} \cdots x_{1,j_d}$ is unchanged. It follows that the $\beta_i^{j_1,\cdots,j_d}$'s are invariant under the transformations in (2.10) and (2.11).

We now show the dual normal form without proof. The replacement of $\tilde{f}_2^{[k]}$ in (2.8) by the following gives the dual normal form:

$$\sum_{i=1}^{n_2-1} \sum_{j=n_2-i+2}^{n_2+1} \mathbf{e}_{2,i} \tilde{R}_{ij}^{[k-2]}(z_1, z_{2,1}, \dots, z_{2,j}) z_{2,j} v$$

for some homogeneous polynomials $\tilde{R}_{ij}^{[k-2]}$ of degree $(k-2)$. It may not be possible to transform $\tilde{f}_1^{[k],c}$ in (2.7) because of resonances.

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