BIFURCATIONS OF DISCRETE TIME NONLINEAR CONTROL SYSTEMS

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Abstract: This paper presents a normal form for the quadratic and cubic terms of a nonlinear discrete time control system around an equilibrium under the group of smooth changes of state coordinates and smooth invertible state feedback. The linear part of the control system need not be controllable. A ***control bifurcation*** happens at an equilibrium where there is a loss of linear stabilizability. The paper examines the Neimark-Sacker control bifurcation, and show its relationship to the classical bifurcation of the same name.*Copyright© 2001 IFAC*

Keywords: Nonlinear control systems, discrete-time systems, state feedback

1. INTRODUCTION

The theory of normal forms and bifurcations of nonlinear difference equations is well known (Arnold, 1983), (Guckenheimer and Holmes, 1983), (Kuznetsov, 1998), (Wiggins, 1990). Briefly it is as follows. Consider two smooth (C^4) n dimensional difference equations with equilibrium points,

$$\begin{aligned} x^+ &= f(x) \\ 0 &= f(0) \end{aligned} \tag{1}$$

and

$$\begin{aligned} z^+ &= g(z) \\ 0 &= g(0) \end{aligned} \tag{2}$$

where $x^+(t) = x(t+1)$. They are *locally diffeomorphic* if there exists a local diffeomorphism

$$\begin{aligned} z &= \phi(x) \\ 0 &= \phi(0) \end{aligned} \tag{3}$$

which carries (1) to (2),

$$g(\phi(x)) = \frac{\partial \phi}{\partial x}(x) f(x).$$

Such a local diffeomorphism carries trajectories $x(t)$ in its domain onto trajectories $z(t)$ in its range,

$$z(t) = \phi(x(t))$$

hence the two dynamics are locally smoothly equivalent.

The *linear approximation* of (1) around the fixed point $x = 0$ is

$$\delta x^+ = \frac{\partial f}{\partial x}(0)\ \delta x \tag{4}$$

and this is a *hyperbolic fixed point* if $\frac{\partial f}{\partial x}(0)$ has no eigenvalues on the unit circle. The discrete time *Grobman-Hartman Theorem* states that if the equilibrium $x = 0$ of (1) is hyperbolic then it is locally topologically conjugate to its linear approximation (4). A related theorem is that two hyperbolic equilibria are locally topologically conjugate if their linear approximations have the same number of eigenvalues strictly inside the unit circle and the signs of their products are the same

[1] Research supported in part by NSF 9970998.

and also the same number of eigenvalues strictly outside and the signs of their products are the same (Kuznetsov, 1998).

A parametrized system

$$x^+ = f(x, \mu) \tag{5}$$

can have a locus of equilibria

$$x_e = f(x_e, \mu_e).$$

It undergoes a ***local bifurcation*** at an equilibrium x_e, μ_e that is not locally topologically conjugate to nearby equilibria. In light of the above, such a bifurcation can only happen if one or more eigenvalues of the linearized system cross the unit circle or the sign of the product of the strictly stable eigenvalues changes or the sign of the product of the strictly unstable eigenvalues changes.

A standard approach to analyze the behavior of the parametrized system (5) around a bifurcation point is to add the parameter as an additional state with trivial dynamics

$$\mu^+ = \mu \tag{6}$$

then compute the center manifold through the bifurcation point and the dynamics restricted to this manifold (Kuznetsov, 1998). The center manifold is an invariant manifold of the extended difference equation (5, 6) which is tangent at the bifurcation point to the eigenspace of the eigenvalues on the unit circle. In practice, one does not compute the center manifold and its dynamics exactly, in most cases of interest, an approximation of degree two or three suffices. If the other eigenvalues are off the unit circle then this part of the dynamics cannot affect the local topological conjugacy around the bifurcation point. If at the bifurcation point all the eigenvalues of the linear approximation are inside or on the unit circle then the bifurcation point will be locally asymptotically stable for the complete dynamics if and only if the dynamics on the center manifold is locally asymptotically stable. Of course, at some nearby equilibria the dynamics may be unstable.

The next step is to compute the Poincarê normal form of the center manifold dynamics. This is a normal form under smooth changes of coordinates

$$z = \phi(x) = Tx - \phi^{[2]}(x) - \phi^{[3]}(x) \tag{7}$$

where $\phi^{[d]}(x)$ denotes a vector field that is a homogeneous polynomial of degree d in x. The linear part of the change of coordinates T puts the linear part of the center manifold dynamics in Jordan form, the quadratic part of the change of coordinates $\phi^{[2]}$ cancels as much of the quadratic part of the center manifold dynamics as possible, as well as the cubic part of the cordinates do. From its normal form the bifurcation is recognized and understood.

Kang and Krener (Kang and Krener, 1992) developed a quadratic normal form for continuous time nonlinear systems whose linear part is controllable. This was extended to discrete time systems by Barbot, Monaco and Normand-Cyrot (J.-P. Barbot and Normand-Cyrot, 1997). These authors considered a larger group of transformations to bring the system to normal form, including invertible state feedback as well as change of state coordinates. Kang (Kang, 1998*a*), (Kang, 1998*b*) also developed a quadratic normal form for continuous time nonlinear systems whose linear part may have uncontrollable modes. Krener, Kang and Chang, (***Normal Forms and Bifurcations of Control Systems***, n.d.), (A. J. Krener and Chang, n.d.) described the quadratic and cubic normal forms of continuous time nonlinear control systems and also their bifurcations.

This paper develops quadratic and cubic normal forms for discrete time nonlinear control systems of the form

$$\begin{aligned} x^+ = f(x,u) &= Ax + Bu + f^{[2]}(x,u) \\ &+ f^{[3]}(x,u) + O(x,u)^4 \end{aligned} \tag{8}$$

where x, u are of dimensions n, 1. The linear part of the system does not need to be controllable. Moreover, our quadratic normal form differs from that of (J.-P. Barbot and Normand-Cyrot, 1997) for linearly controllable systems.

A ***control bifurcation*** of (8) happens at an equilibrium where the linear approximation loses stabilizability. This is different from the bifurcation of a parametrized system (5) which take place at an equilibrium where there is a loss of structural stability with respect to parameter variations. To emphasize this distinction the paper shall refer to the latter as a ***classical bifurcation***.

The other difference between control and classical bifurcations is that when bringing the control system into normal form, change of state coordinates and state dependent change of control coordinates (invertible state feedback) is used to simplify the dynamics.

2. QUADRATIC NORMAL FORMS

Consider a smooth (C^3) system of the form (8) under the action of linear and quadratic change of state coordinates and state feedback

$$z = \phi(x) = Tx - \phi^{[2]}(x) \tag{1}$$

$$v = \alpha(x,u) = Kx + Lu - \alpha^{[2]}(x,u) \quad (2)$$

where T, L are invertible.

It is well known that there exists a linear change of coordinates T and a linear feedback K, L that transforms the system into the linear normal form

$$\begin{bmatrix} x_1^+ \\ x_2^+ \end{bmatrix} = \begin{bmatrix} f_1(x_1,x_2,u) \\ f_2(x_1,x_2,u) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u + \begin{bmatrix} f_1^{[2]}(x_1,x_2,u) \\ f_2^{[2]}(x_1,x_2,u) \end{bmatrix} + O(x_1,x_2,u)^3 \quad (3)$$

where x_1, x_2 are n_1, n_2 dimensional, $n_1 + n_2 = n$, A_1 is in Jordan form, A_2, B_2 are in controller (Brunovsky) form,

$$A_2 = \begin{bmatrix} 0 & 1 & \dots & 0 \\ & \ddots & \ddots & \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The result generalizing (Li, 1999) is given without detailed proof.
Consider the system (3) where A_1 is diagonal and A_2, B_2 are in Brunovsky form. There exist a quadratic change of coordinates and a quadratic feedback

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} \phi_1^{[2]}(x_1,x_2) \\ \phi_2^{[2]}(x_1,x_2) \end{bmatrix}$$

$$v = u - \alpha^{[2]}(x_1,x_2,u)$$

which transforms the system (3) into the quadratic normal form

$$\begin{bmatrix} z_1^+ \\ z_2^+ \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} v + \begin{bmatrix} \bar{f}_1^{[2;0]}(z_1;z_2,v) + \bar{f}_1^{[1;1]}(z_1;z_2,v) \\ 0 + 0 \end{bmatrix} + \begin{bmatrix} \bar{f}_1^{[0;2]}(z_1;z_2,v) \\ \bar{f}_2^{[0;2]}(z_1;z_2,v) \end{bmatrix} + O(z_1,z_2,v)^3 \quad (4)$$

where $\bar{f}_i^{[d_1;d_2]}(z_1;z_2,v)$ is a polynomial vector field homogeneous of degree d_1 in z_1 and homogeneous of degree d_2 in z_2, v. For notational convenience, define $z_{2,n_2+1} = v$.

The vector field $\bar{f}_1^{[2;0]}$ is in the quadratic normal form of Poincarê,

$$\bar{f}_1^{[2;0]} = \sum_{\lambda_i = \lambda_j \lambda_k} \beta_i^{jk} \, \mathbf{e}_1^i \, z_{1,j} \, z_{1,k} \quad (5)$$

and the other vector fields are as follows,

$$\bar{f}_1^{[1;1]} = \sum_{\lambda_i=0} \sum_{\lambda_j=0} \sum_{k=1}^{n_2+1} \gamma_i^{jk} \, \mathbf{e}_1^i \, z_{1,j} \, z_{2,k} + \sum_{\lambda_i \neq 0} \sum_{\lambda_j \neq 0} \gamma_i^{j1} \, \mathbf{e}_1^i \, z_{1,j} \, z_{2,1}, \quad (6)$$

$$\bar{f}_1^{[0;2]} = \sum_{\lambda_i \neq 0} \sum_{k=1}^{n_2+1} \delta_i^{1k} \, \mathbf{e}_1^i \, z_{2,1} \, z_{2,k}, \quad (7)$$

$$\bar{f}_2^{[0;2]} = \sum_{i=1}^{n_2-1} \sum_{k=i+2}^{n_2+1} \epsilon_i^{1k} \, \mathbf{e}_2^i \, z_{2,1} \, z_{2,k}. \quad (8)$$

The normal form is unique, that is, each system (3) can be transformed into only one such normal form (4-8) . This follows from the fact that the numbers in the above, $\beta_i^{jk}, \gamma_i^{jk}, \delta_i^{1k}, \epsilon_i^{1k}$ for the indicated indices, are moduli, i.e., continuous invariants of the system (3) under quadratic change of coordinates and quadratic feedback. These numbers are defined as follows

$$\beta_i^{jk} = \frac{\partial^2 f_{1,i}}{\partial x_{1,j} \partial x_{1,k}}(0,0,0) \quad (9)$$
for $1 \le i,j,k \le n_1$ and $\lambda_i = \lambda_j \lambda_k$,

$$\gamma_i^{jk} = \frac{\partial^2 f_{1,i}}{\partial x_{1,j} \partial x_{2,k}}(0,0,0) \quad (10)$$
for $1 \le i,j \le n_1, 1 \le k \le n_2 + 1$ and $\lambda_i = \lambda_j = 0$,

$$\gamma_i^{j1} = \sum_{l=0}^{n_2} \left(\frac{\lambda_i}{\lambda_j}\right)^l \frac{\partial^2 f_{1,i}}{\partial x_{1,j} \partial x_{2,l+1}}(0,0,0) \quad (11)$$
for $1 \le i,j \le n_1$ and $\lambda_i \lambda_j \neq 0$,

$$\delta_i^{1k} = \sum_{l=0}^{n_2-k+1} \lambda_i^l \frac{\partial^2 f_{1,i}}{\partial x_{2,l+1} \partial x_{2,k+l}}(0,0,0) \quad (12)$$
for $1 \le i \le n_1, 1 \le k \le n_2 + 1$ and $\lambda_i \neq 0$,

$$\epsilon_i^{1k} = \sum_{l=0}^{n_2-k+1} \frac{\partial^2 f_{2,i+l}}{\partial x_{2,l+1} \partial x_{2,k+l}}(0,0,0) \quad (13)$$
for $1 \le i \le n_2 - 1$ and $i + 2 \le k \le n_2 + 1$.

Remark. If some of the eigenvalues of A_1 are complex then a linear complex change of coordinates is required to bring it to Jordan form. In this case some of the coordinates of z_1 are complex conjugate pairs and some of the coefficients in the normal form are complex. These complex coefficients occur in conjugate pairs so that the real dimension of the coefficient space of the normal form is unchanged. This is also true for the cubic normal form discussed in the next section.

3. CUBIC NORMAL FORMS

Now the cubic normal forms is given without detailed proof.

Consider a smooth (C^4) system

$$\begin{bmatrix} x_1^+ \\ x_2^+ \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2, u) \\ f_2(x_1, x_2, u) \end{bmatrix} \tag{1}$$

$$= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u + \begin{bmatrix} f_1^{[2;0]}(x_1; x_2, u) \\ 0 \end{bmatrix} + \begin{bmatrix} f_1^{[1;1]}(x_1; x_2, u) \\ 0 \end{bmatrix} + \begin{bmatrix} f_1^{[0;2]}(x_1; x_2, u) \\ f_2^{[0;2]}(x_1; x_2, u) \end{bmatrix} + \begin{bmatrix} f_1^{[3]}(x_1; x_2, u) \\ f_2^{[3]}(x_1; x_2, u) \end{bmatrix} + O(x_1, x_2, u)^4$$

where A_1 is diagonal, A_2, B_2 is in Brunovsky form and the quadratic terms are in the normal form of Theorem 2. There exist a cubic change of coordinates and a cubic feedback

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} \phi_1^{[3]}(x_1, x_2) \\ \phi_2^{[3]}(x_1, x_2) \end{bmatrix}$$

$$v = u - \alpha^{[3]}(x_1, x_2, u)$$

which transforms the system (1) into the cubic normal form

$$\begin{bmatrix} z_1^+ \\ z_2^+ \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} v \tag{2}$$

$$+ \begin{bmatrix} f_1^{[2;0]}(z_1; z_2, v) \\ 0 \end{bmatrix} + \begin{bmatrix} f_1^{[1;1]}(z_1; z_2, v) \\ 0 \end{bmatrix} + \begin{bmatrix} f_1^{[0;2]}(z_1; z_2, v) \\ f_2^{[0;2]}(z_1; z_2, v) \end{bmatrix} + \begin{bmatrix} \tilde{f}_1^{[3;0]}(z_1; z_2, v) \\ 0 \end{bmatrix} + \begin{bmatrix} \tilde{f}_1^{[2;1]}(z_1; z_2, v) \\ 0 \end{bmatrix} + \begin{bmatrix} \tilde{f}_1^{[1;2]}(z_1; z_2, v) \\ \tilde{f}_2^{[1;2]}(z_1; z_2, v) \end{bmatrix} + \begin{bmatrix} \tilde{f}_1^{[0;3]}(z_1; z_2, v) \\ \tilde{f}_2^{[0;3]}(z_1; z_2, v) \end{bmatrix} + O(z_1, z_2, v)^4.$$

[illegible] $\tilde{f}^{[3;0]}$ [illegible]

$$\tilde{f}_1^{[2;1]} = \sum_{\lambda_i = 0} \sum_{\lambda_j \lambda_k = 0} \sum_{l=1}^{n_2+1} \gamma_i^{jkl} \; \mathbf{e}_1^i \; z_{1,j} \; z_{1,k} \; z_{2,l} + \sum_{\lambda_i \neq 0} \sum_{\lambda_j \lambda_k \neq 0} \gamma_i^{jk1} \; \mathbf{e}_1^i \; z_{1,j} \; z_{1,k} \; z_{2,1} \tag{4}$$

$$\tilde{f}_1^{[1;2]} = \sum_{\lambda_i = 0} \sum_{\lambda_j = 0} \sum_{k=1}^{n_2+1} \sum_{l=k}^{n_2+1} \delta_i^{jkl} \; \mathbf{e}_1^i \; z_{1,j} \; z_{2,k} \; z_{2,l} + \sum_{\lambda_i \neq 0} \sum_{\lambda_j \neq 0} \sum_{l=1}^{n_2+1} \delta_i^{j1l} \; \mathbf{e}_1^i \; z_{1,j} \; z_{2,1} \; z_{2,l} \tag{5}$$

$$\tilde{f}_1^{[0;3]} = \sum_{\lambda_i \neq 0} \sum_{k=1}^{n_2+1} \sum_{l=k}^{n_2+1} \epsilon_i^{1kl} \; \mathbf{e}_1^i \; z_{2,1} \; z_{2,k} \; z_{2,l} \tag{6}$$

$$\tilde{f}_2^{[1;2]} = \sum_{i=1}^{n_2-1} \sum_{\lambda_j \neq 0} \sum_{l=i+2}^{n_2+1} \zeta_i^{j1l} \; \mathbf{e}_2^i \; z_{1,j} \; z_{2,1} \; z_{2,l} \tag{7}$$

$$\tilde{f}_2^{[0;3]} = \sum_{i=1}^{n_2-1} \sum_{l=i+2}^{n_2+1} \sum_{k=1}^{l} \eta_i^{1kl} \; \mathbf{e}_2^i \; z_{2,1} \; z_{2,k} \; z_{2,l} \tag{8}$$

The normal form is unique, that is, each system (1) can be transformed into only one such normal form (2-8) . This follows from the fact that the numbers in the above, $\beta_i^{jkl}, \gamma_i^{jkl}, \delta_i^{1kl}, \epsilon_i^{1kl}, \zeta_i^{j1l}, \eta_i^{1kl}$ for the indicated indices, are moduli of the system (3) under cubic change of coordinates and cubic feedback. These moduli are defined as follows

$$\beta_i^{jkl} = \frac{\partial^3 f_{1,i}}{\partial x_{1,j} \partial x_{1,k} \partial x_{1,l}}(0,0,0) \tag{9}$$

for $1 \leq i, j, k, l \leq n_1$ and $\lambda_i = \lambda_j \lambda_k \lambda_l$,

$$\gamma_i^{jkl} = \frac{\partial^3 f_{1,i}}{\partial x_{1,j} \partial x_{1,k} \partial x_{2,l}}(0,0,0) \tag{10}$$

for $1 \leq i \leq n_1, 1 \leq j \leq k \leq n_1$,

$1 \leq l \leq n_2 + 1$and $\lambda_i = \lambda_j \lambda_k = 0$,

$$\gamma_i^{jk1} = \sum_{r=0}^{n_2-k+1} \left(\frac{\lambda_i}{\lambda_j \lambda_k} \right)^r \frac{\partial^3 f_{1,i}}{\partial x_{1,j} \partial x_{1,k} \partial x_{2,r+1}}(0,0,0) \tag{11}$$

for $1 \leq i, j \leq n_1, 1 \leq k \leq n_2 + 1$

and $\lambda_i \lambda_j \lambda_k \neq 0$,

$$\delta_i^{jkl} = \frac{\partial^3 f_{1,i}}{\partial x_{1,j} \partial x_{2,k} \partial x_{2,l}}(0,0,0) \tag{12}$$

for $1 \leq i, j \leq n_1, 1 \leq k \leq l \leq n_2 + 1$

and $\lambda_i = \lambda_j = 0$,

δ_i^{j1l} $\sum^{n_2-l+1} \left(\lambda_i \right)^l$ [illegible] $\partial^3 f_{1,i}$ [illegible] $(0,0,0)$

$$(14)$$

for $1 \le i \le n_1, 1 \le k \le l,$
$i + 2 \le l \le n_2 + 1$ and $\lambda_i \ne 0,$

$$\zeta_i^{j1l} = \sum_{r=0}^{n_2-l+1} \lambda_j^{-r} \frac{\partial^3 f_{2,i+r}}{\partial x_{1,j} \partial x_{2,1+r} \partial x_{2,r+l}} (0,0,0) \tag{15}$$

for $1 \le i \le n_2 - 1, i + 2 \le l \le n_2 + 1$
and $\lambda_j \ne 0$

$$\eta_i^{1kl} = \sum_{r=0}^{n_2-l+1} \frac{\partial^3 f_{2,i+r}}{\partial x_{2,1+r} \partial x_{2,k+r} \partial x_{2,l+r}} (0,0,0) \tag{16}$$

for $1 \le i \le n_2 - 1, 1 \le k \le l$
and $i + 2 \le l \le n_2 + 1.$

4. CONTROL BIFURCATIONS

In the above theorems there is a lot more details than are necessary to understand the type of bifurcations that are possible. Recall that in the bifurcation theory of a parametrized system of difference equations, the interesting part of the dynamics is that restricted to the center manifold. This leads to a great reduction in the dimension of space that must be explored. A similar fact holds true when studying control bifurcations. In most applications one will ultimately use state feedback in an attempt to stabilize the system so the coordinates that are linearly stabilizable can be ignored to a large extent. If there are modes which are neutrally stable and are not linearly stabilizable, then the particular choice of feedback will influence the shape of center manifold of the closed loop system and the dynamics thereon. It might be possible to achieve asymptotically stable center manifold dynamics by the proper choice of feedback although it will not be exponentially stable. Let us now discuss an important bifurcation of control systems.

4.1 *Neimark-Sacker Control Bifurcation*

The discrete time analogue of a classical Hopf bifurcation is called a Neimark-Sacker bifurcation. The authors present the control analogue of this bifurcation. The uncontrollable modes are a nonzero complex conjugate pair,

$$A_1 = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}$$

where $\lambda = \rho e^{i\theta}$, $\bar{\lambda} = \rho e^{-i\theta}$, $\theta \ne 0, \pi/2, \pi, 3\pi/2$. The equilibria $z_e(\mu), v_e(\mu)$ are given by

$$\begin{bmatrix} z_{e1,1} \\ z_{e1,2} \end{bmatrix} = \mu^2 (I - A_1)^{-1} \begin{bmatrix} \tilde{\delta}_1 \\ \tilde{\delta}_2 \end{bmatrix} + O(\mu)^3 \tag{1}$$

$$z_{e2,i} = \mu + O(\mu)^2 \qquad i = 1, \ldots, n_2 \tag{2}$$

$$v_e = \mu \tag{3}$$

where

$$\tilde{\delta}_i = \sum_{k=1}^{n_2+1} \delta_i^{1k}.$$

The local linearization around z_e, v_e is

$$\begin{bmatrix} \tilde{z}_1^+ \\ \tilde{z}_2^+ \end{bmatrix} = \left(\begin{bmatrix} A_1 + \mu\Gamma & \mu\Delta \\ 0 & A_2 \end{bmatrix} + O(\mu)^2 \right) \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix} + \left(\begin{bmatrix} \mu B_1 \\ B_2 \end{bmatrix} + O(\mu)^2 \right) \tilde{v} \tag{4}$$

where $\tilde{z} = z - z_e(\mu)$, $\tilde{v} = v - v_e(\mu)$ and

$$\Gamma = \begin{bmatrix} \gamma_1^{11} & \gamma_1^{21} \\ \gamma_2^{11} & \gamma_2^{21} \end{bmatrix}$$

$$\Delta = \begin{bmatrix} \tilde{\delta}_1 + \delta_1^{11} & \delta_1^{12} & \ldots & \delta_1^{1n_2} \\ \tilde{\delta}_2 + \delta_2^{11} & \delta_2^{12} & \ldots & \delta_2^{1n_2} \end{bmatrix}$$

$$B_1 = \begin{bmatrix} \delta_1^{1,n_2+1} \\ \delta_2^{1,n_2+1} \end{bmatrix}.$$

If the transversality condition

$$\begin{bmatrix} \tilde{\delta}_1 + \delta_1^{11} \\ \tilde{\delta}_2 + \delta_2^{11} \end{bmatrix} + A_1 \begin{bmatrix} \delta_1^{12} \\ \delta_2^{12} \end{bmatrix} + \ldots + A_1^{n_2} \begin{bmatrix} \delta_1^{1,n_2+1} \\ \delta_2^{1,n_2+1} \end{bmatrix} \ne 0 \tag{5}$$

is satisfied then the system is linearly controllable hence stabilizable about any equilibrium except $\mu = 0$. Consider a parametrized family of feedbacks

$$v = \kappa(z, \mu)$$
$$\tilde{v} = K_1(\mu)\tilde{z}_1 + K_2(\mu)\tilde{z}_2. \tag{6}$$

If $\rho < 1$ then the system is stabilizable about any equilibrium but if $\rho \ge 1$ then the system is not stabilizable when $\mu = 0$. The case $\rho \ge 1$ is called a Neimark-Sacker control bifurcation. Let us distinguish two subcases, $\rho > 1$ and $\rho = 1$.

If $\rho > 1$ then it requires larger and larger gain to stabilize the system closer and closer to $\mu = 0$. But if the feedback (6) is continuous it will stabilize only for some small $\mu > 0$ or for some small $\mu < 0$ but not both. At $\mu = 0$ the poles of the closed loop system are $\lambda, \bar{\lambda}$ and the poles of $A_2 + B_2 K_2(0)$. The latter can be made stable but the former are unstable. If the feedback is bounded then as $\mu \to 0$ the poles converge to these. The system is controllable for $\mu \ne 0$ so the poles can be placed arbitrarily by feedback. The poles associated primarily with the z_2 subsystem can be kept stable but the two poles associated primarily with the z_1 subsystem will leave the unit disk at some small value(s) of μ. Depending on the choice of feedback, they will leave one at a time as real poles, leave together through ± 1 or leave together as a nonzero complex conjugate pair. If they leave together as a complex conjugate pair that is neither real nor imaginary then generically the system undergoes a Neimark-Sacker bifurcation. If they leave together through ± 1 the situation can be quite complicated and will not be discussed here.

If $\rho = 1$ and the feedback (6) is continuous then generically the system undergoes a Neimark-Sacker bifurcation at $\mu = 0$ provided that $e^{ik\theta} \neq 1$ for $k = 1, 2, 3, 4$. The authors illustrate this with an example.

$$\begin{aligned} z_{1,1}^{+} &= e^{i\pi/4} z_{1,1} + z_2^2 \\ z_{1,2}^{+} &= e^{-i\pi/4} z_{1,2} + z_2^2 \\ x_2 &= u. \end{aligned}$$

The equilibria are

$$\begin{aligned} z_{e1,1} &= c\mu^2 \\ z_{e1,2} &= \bar{c}\mu^2 \\ x_{e2} &= \mu \\ u_e &= \mu \end{aligned}$$

where $c = \left(1 - e^{i\pi/4}\right)^{-1}$ The linear approximations are

$$\begin{aligned} \tilde{z}_{1,1}^{+} &= e^{i\pi/4} \tilde{z}_{1,1} + 2\mu\tilde{z}_2 \\ \tilde{z}_{1,2}^{+} &= e^{-i\pi/4} \tilde{z}_{1,2} + 2\mu\tilde{z}_2 \\ \tilde{x}_2^{+} &= \tilde{u} \end{aligned}$$

where $\tilde{z}_{1,1} = z_{1,1} - c\mu^2$, $\tilde{z}_{1,2} = z_{1,2} - \bar{c}\mu^2$, $\tilde{x}_z = z_2 - \mu$, $\tilde{u} = u - \mu$. The linear approximations are controllable except at $\mu = 0$.

The feedback

$u = \mu + 0.5(z_{1,1} - c\mu^2) + 0.5(z_{1,2} - \bar{c}\mu^2) + 0.5(x_2 - \mu)$

places the poles of the closed loop system inside the open unit disk at $0.7953 \pm 0.5743i$, 0.3957 at $\mu = 0.1$. A pair of poles leaves the unit disk at $e^{\pm\pi/4}$ when $\mu = 0$.

The closed loop dynamics undergoes a Neimark-Sacker classical bifurcation at $\mu = 0$. The discrete time analogue of the first Lyapunov coefficient is found in Kuznetsov, (Kuznetsov, 1998), p. 186, formula (5.74). For this example $a(0) = 46.8$ which indicates that the system undergoes a subcritical Neimark-Sacker bifurcation at $\mu = 0$. For small $\mu > 0$ the equilibrium is exponentially stable but there is an unstable invariant closed curve nearby. For small $\mu < 0$ the equilibrium is unstable as is the bifurcation equilibrium, $\mu = 0$.

5. CONCLUSION

The authors have developed a theory of quadratic and cubic normal forms for discrete time control systems. and have also shown the uniqueness of the normal forms. To avoid notational difficulties, the authors have restricted attention to scalar input systems whose uncontrollable part is diagonalizable. But the result can be easily extended to more general systems. The authors have introduced the concept of control bifurcation and exhibited a simple example of the Neimark-Sacker control bifurcation.

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