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Observers for linearly unobservable nonlinear systems ☆

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Abstract

We provide a method for constructing local observers for some nonlinear systems around a critical point where the linearization is not observable or not detectable. Two examples are provided to illustrate the results of the paper. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

We consider the problem of estimating the current state $x(t)$ of a smooth, nonlinear dynamical system,

$$\dot{x} = f(x),\quad y = h(x),\quad x(0) = x^0 \tag{1}$$

when it is known that the state is close to $x = 0$, a critical point of the dynamics, $f(0) = 0$. By replacing $h(x)$ by $h(x) - h(0)$ we can assume without loss of generality that $h(0) = 0$ also. By smooth we mean C^r for r sufficiently large. Occasionally, we shall assume real analyticity of f and h. The dimension of x is n and the dimension of y is p which is typically less than n.

We wish to construct an observer

$$\dot{\hat{x}} = \hat{f}(\hat{x}, y),\quad \hat{x}(0) = \hat{x}^0 \tag{2}$$

so that the error $\tilde{x}(t) = x(t) - \hat{x}(t)$ goes to zero when x^0 and $\hat{x}^0$ are close to $x = 0$.

Most approaches to this problem start with the assumption that the linearized system

$$\dot{x} = Fx,\quad y = Hx \tag{3}$$

is observable or detectable where

$$F = \frac{\partial f}{\partial x}(0),\quad H = \frac{\partial h}{\partial x}(0).$$

A linear system (3) is observable if the largest F invariant subspace in the kernel of H is just 0. The system is observable iff the spectrum of the matrix $F+BH$ can be arbitrarily assigned up to a complex conjugation by the choice of an $n \times p$ matrix B.

A linear system (3) is detectable if the spectrum of F restricted to the largest F invariant subspace in the

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kernel of H is in the open left half-plane. The system is detectable iff the spectrum of the matrix $F+BH$ can be put in the open left half-plane by choice of B. Observability implies detectability but the converse does not hold.

When the linear system (3) is detectable, a local observer can be easily constructed for the nonlinear system (1). Choose some B so that the spectrum of $F+BH$ is in the open left half-plane and then an observer is given by

$$\begin{aligned}\dot{\hat{x}} &= f(\hat{x}) - B(y-\hat{y}),\\ \hat{y} &= h(\hat{x}),\\ \hat{x}(0) &= \hat{x}^0.\end{aligned} \tag{4}$$

The error dynamics of this observer is

$$\dot{\tilde{x}} = (F+BH)\tilde{x} + O(\tilde{x})O(x,\tilde{x}), \tag{5}$$

which converges to 0 if $x(t)$ stays small and $\tilde{x}(0)$ is small.

To see this, since the spectrum of $F+BH$ lies in the open left half-plane, there exists a positive definite solution P to the Lyapunov equation

$$(F+BH)'P + P(F+BH) = -I.$$

So if $x(t)$ satisfies (1) and $\hat{x}(t)$ satisfies (4) then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{x}'P\tilde{x}) = -|\tilde{x}|^2 + (\tilde{f}+B\tilde{h})'P\tilde{x} + \tilde{x}'P(\tilde{f}+B\tilde{h}), \tag{6}$$

where $\tilde{f}=\tilde{f}(x,\tilde{x})=f(x)-f(x-\tilde{x})-F\tilde{x}$, $\tilde{h}=\tilde{h}(x,\tilde{x})=h(x)-h(x-\tilde{x})-H\tilde{x}$. Assuming that the system is sufficiently smooth, the last two terms on the right side are $O(x)O(\tilde{x})^2$ and so are dominated by $|\tilde{x}|^2$ for small $x,\tilde{x}$. Hence the right side is negative and the error converges to zero.

Recently Kazantzis and Kravaris [3] have proposed an improved method of observer design for systems (1) whose linear part is observable. They seek a local change of coordinates $z=\theta(x)$ which transforms (1) into a linear system up to an output injection

$$\begin{aligned}\dot{z} &= Az - \beta(y),\\ y &= \bar{h}(z) = h(\theta^{-1}(z)),\end{aligned} \tag{7}$$

where the matrix A is Hurwitz. It is easy to construct an observer for such a system in the new coordinate z:

$$\begin{aligned}\dot{\hat{z}} &= A\hat{z} - \beta(y),\\ \hat{x} &= \theta^{-1}(\hat{z}),\end{aligned} \tag{8}$$

where the error dynamics is linear and *exponentially stable*

$$\dot{\tilde{z}} = A\tilde{z}. \tag{9}$$

Alternatively, one can pull the observer back into the original coordinates to obtain

$$\dot{\hat{x}} = f(\hat{x}) - \left(\frac{\partial\theta}{\partial x}(\hat{x})\right)^{-1}(\beta(y)-\beta(h(\hat{x}))). \tag{10}$$

The approach of Kazantzis and Kravaris differs from that of Krener and Respondek [6] and similar approaches [2,5]. Krener and Respondek seek local changes of coordinates $z=\theta(x)$ and $w=\gamma(y)$ which transforms (1) into a linear system up to an output injection with a linear output map

$$\begin{aligned}\dot{z} &= Az - \beta(y),\\ w &= Cz.\end{aligned} \tag{11}$$

Because of the extra requirement that the transformed output be a linear function of the state, it is harder to find $\theta(x)$ and $\gamma(y)$ to do this. They must satisfy a system of first-order PDEs, which is solvable iff rather restrictive integrability conditions are met. Most systems do not satisfy the integrability conditions.

It is easy to see that $z=\theta(x)$ transforms (1) into (7) iff θ satisfies the first-order PDE

$$\frac{\partial\theta}{\partial x}(x)f(x) = A\theta(x) - \beta(y). \tag{12}$$

Assuming $f(x), h(x), \beta(y)$ are smooth enough this equation has a formal power series solution to degree d provided that none of the eigenvalues $\mu=(\mu_1,\ldots,\mu_n)$ of A are resonant of degree $\leqslant d$ with the eigenvalues $\lambda=(\lambda_1,\ldots,\lambda_n)$ of F. A eigenvalue μ_j is resonant of degree $d>0$ with λ if there is a vector $m=(m_1,\ldots,m_n)$ of nonnegative integers such that $|m|=m_1+\cdots+m_n=d$ and $m\cdot\lambda=m_1\lambda_1+\cdots+m_n\lambda_n=\mu_j$.

If there are no resonances of any degree $d>0$ and the system is real analytic, Kazantzis and Kravaris proved that this formal power series solution converges if $\lambda=(\lambda_1,\ldots,\lambda_n)$ is in the Poincaré domain, that is, 0 is not in the convex hull of $\{\lambda_1,\dots,\lambda_n\}$ [1]. If 0 is in the convex hull of $\{\lambda_1,\dots,\lambda_n\}$ then λ is said to be in the Siegel domain [1]. Krener and Xiao [8] proved that even if λ is in the Siegel domain then a Hurwitz A can be chosen so that there are no resonances and the formal series for $\theta(x)$ converges.

One does not need an exact solution of PDE (12) to construct a local observer, a finite series suffices. Suppose $\theta(x)=\theta^{[1]}(x)+\theta^{[2]}(x)+\cdots+\theta^{[d]}(x)$ satisfies

$$\frac{\partial\theta}{\partial x}(x)f(x)=A\theta(x)-\beta(y)+O(x)^{d+1}, \tag{13}$$

where $\theta^{[k]}(x)$ denotes a polynomial vector field that is homogeneous of degree k. Without loss of generality we can let $\theta^{[1]}(x)=x$. Then observer (10) has nearly linear error dynamics in transformed coordinates $\tilde z=\theta(x)-\theta(\hat x)$,

$$\dot{\tilde z}=A\tilde z+O(z,\tilde z)^d O(\tilde z). \tag{14}$$

Since the spectrum of A lies in the open left half-plane then there exists a positive definite solution P to the Lyapunov equation

$$A'P+PA=-I. \tag{15}$$

So if $x(t)$ satisfies (1) and $\hat x(t)$ satisfies (10) then

$$\frac{d}{dt}(\tilde x'P\tilde x)=-|\tilde x|^2+O(x,\tilde x)^d O(\tilde x)^2. \tag{16}$$

From a comparison of (6) and (16) one expects (10) to outperform (4).

2. Solution of the first-order PDE

We focus on the quadratic terms of (12) assuming formal expansions of all the functions involved.

$$f(x)=Fx+f^{[2]}(x)+f^{[3]}(x)+\cdots,$$

$$h(x)=Hx+h^{[2]}(x)+h^{[3]}(x)+\cdots,$$

$$\theta(x)=x+\theta^{[2]}(x)+\theta^{[3]}(x)+\cdots,$$

$$\beta(h(x))=BHx+\beta^{[2]}(x)+\beta^{[3]}(x)+\cdots,$$

$$A=F+BH.$$

They must satisfy the following first-order PDE:

$$\frac{\partial\theta^{[2]}}{\partial x}(x)Fx-A\theta^{[2]}(x)=-f^{[2]}(x)-\beta^{[2]}(x). \tag{17}$$

This equation admits a unique solution $\theta^{[2]}(x)$ for any right-hand side iff none of the eigenvalues of A are resonant of degree 2 with the spectrum of F [7].

Similarly, the degree k part of (12) is given by the solution of

$$\frac{\partial\theta^{[k]}}{\partial x}(x)Fx-A\theta^{[k]}(x)$$
$$=-\sum_{i=1}^{k-1}\frac{\partial\theta^{[i]}}{\partial x}(x)f^{[k-i]}(x)-\beta^{[k]}(x). \tag{18}$$

This equation admits a unique solution $\theta^{[k]}(x)$ for any right side iff none of the eigenvalues of A are resonant of degree k with the spectrum of F [7]. The unknowns $\beta^{[2]}(x),\dots,\beta^{[d]}(x)$ can be chosen to keep $\theta^{[2]}(x),\dots,\theta^{[d]}(x)$ close to 0 so $\theta(x)$ remains a diffeomorphism over a wide region.

If system (1) is real analytic and a slightly stronger condition than no resonances of any degree is satisfied then PDE (12) has a real analytic solution, i.e. the formal power serious solution converges [8]. The stronger condition is that all of the eigenvalues of A must be of type (C,ν) with respect to the spectrum of F for some $C>0$, $\nu>0$. A complex number μ_j is of type (C,ν) for $|m|>d$ with respect to $\lambda=(\lambda_1,\dots,\lambda_n)$ if for any nonzero vector of nonzero integers $m=(m_1,\dots,m_n)$, $|m|>d$ it is true that

$$|\mu_j-m\cdot\lambda|\geqslant\frac{C}{|m|^\nu}. \tag{19}$$

Loosely speaking, a complex number μ_j is of type (C,ν) with respect to λ ($\in\sigma(F)$) if $|\mu_j-m\cdot\lambda|$ is never zero and does not approach zero too fast as $|m|\to\infty$. If $\nu>n/2$ then the set of μ_j's which are not of type (C,ν) for any $C>0$ is a set of measure zero in the complex plane [8].

If the system is linearly observable then the spectrum of A can be set arbitrarily by the linear part of the output injection. Hence, we can choose it to be in the open left half plane and to be of type (C,ν) with respect to the spectrum of F. This will insure that the formal power series for $\theta(x)$ converges.

3. Linearly unobservable systems

We consider systems (1) whose linear part is unobservable. For simplicity of exposition we focus our attention on scalar output systems with only a few unobservable modes. But our techniques can be applied to more general systems.

Assume that the linear part of (1) has n_o observable modes x_o and n_u unobservable modes x_u whose linear dynamics is diagonalizable. Then after a linear change of coordinate we can assume the following form:

$$\begin{bmatrix} \dot{x}_o \\ \dot{x}_u \end{bmatrix} = \begin{bmatrix} F_o & 0 \\ 0 & F_u \end{bmatrix} \begin{bmatrix} x_o \\ x_u \end{bmatrix} + O(x_o, x_u)^2,$$

$$y = [\, H_o \ 0 \,] \begin{bmatrix} x_o \\ x_u \end{bmatrix} + O(x_o, x_u)^2,$$

where F_o, H_o are in the observable form

$$F_o = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ f_1 & f_2 & f_3 & \dots & f_{n_o} \end{bmatrix}, \quad H_o = [\, 1 \ 0 \ 0 \ \dots \ 0 \,]$$

and F_u is diagonal

$$F_u = \begin{bmatrix} \lambda_{u,1} & \dots & 0 \\ & \ddots & \\ 0 & \dots & \lambda_{u,n_u} \end{bmatrix}.$$

If any of the eigenvalues of F_u lies in the open right half-plane then it is not hard to show that it is impossible to construct a smooth observer (2) with asymptotically stable error dynamics [4]. Henceforth, we assume that the spectrum of F_u lies in the closed left half-plane.

Next, we seek a change of coordinates and an output injection which carries the system to (7) or something similar. Let

$$A = \begin{bmatrix} A_o & 0 \\ 0 & A_u \end{bmatrix}$$

with $A_u = F_u$.

Theorem 1. *Suppose that the system is linearly detectable, that is, the spectrum $\sigma(F_u)$ of F_u lies in the open left half-plane. Then:*

1. *for any formal power series $\beta(x)$, there is a formal power series for $\theta(x)$ satisfies* (12);
2. *if in addition the system is real analytic and the spectrum of F_u is of type (C, ν) for $|m| > 1$ with respect to the spectrum of F then the formal power series converges.*

Proof. (a) Without loss of generality we can assume that the linear part of the change of coordinates is the identity. Thus let

$$\theta(x) := x + \theta_2(x),$$

where $\theta_2(x)$ is to be determined. Let

$$\theta_2(x) = \theta^{[2]}(x) + \theta^{[3]}(x) + \cdots + \theta^{[k]}(x) + \cdots.$$

Then $\theta^{[k]}(x)$, $k \geqslant 2$ can be obtained from Eq. (18) since the eigenvalues of A_u are resonant of degree 1 with respect to the spectrum of F_u due to the linear detectability assumption.

(b) The detail proof can be found in [8]. □

Now we consider the coordinate transformation

$$\begin{bmatrix} z_o \\ z_u \end{bmatrix} = \begin{bmatrix} \theta_o(x_o, x_u) \\ \theta_u(x_o, x_u) \end{bmatrix} = \begin{bmatrix} x_o \\ x_u \end{bmatrix} + O(x_o, x_u)^2$$

and note that the linear part of the output injection only affects $\dot{z}_o$

$$\begin{bmatrix} \beta_o(y) \\ \beta_u(y) \end{bmatrix} = \begin{bmatrix} B_o \\ 0 \end{bmatrix} y + O(y)^2,$$

where

$$B_o = \begin{bmatrix} b_1 \\ \vdots \\ b_{n_o} \end{bmatrix}.$$

Then

$$A_o = F_o + B_oH_o = \begin{bmatrix} b_1 & 1 & 0 & \dots & 0 \\ b_2 & 0 & 1 & \dots & 0 \\ & & & \ddots & \\ b_{n_o-1} & 0 & 0 & \dots & 1 \\ b_{n_o}+f_1 & f_2 & f_3 & \dots & f_{n_o} \end{bmatrix},$$

$$A_u = F_u = \begin{bmatrix} \lambda_{u,1} & \dots & 0 \\ & \ddots & \\ 0 & \dots & \lambda_{u,n_u} \end{bmatrix}.$$

The spectrum of A_o is arbitrarily assignable up to the complex conjugation by choice of B_o. In fact the characteristic polynomial of A_o is

$$p_o(s) = s^{n_o} + \sum_{k=0}^{n_o-1}\left(-b_{n_o-k} - f_{k+1} + \sum_{j=1}^{n_o-k-1} b_j f_{k+j+1}\right)s^k$$

and clearly the coefficients of this can be arbitrarily assigned by choice of $b_1,\dots,b_{n_o}$. First b_1 is used to set the coefficient of s^{k-1} then b_2 is used to set the coefficient of s^{k-2}, etc.

On the other hand the matrix A_u and its spectrum cannot be changed by the output injection. Clearly, the eigenvalues of A_u are resonant of degree 1 with the spectrum of F_u but this causes no difficulty as we have chosen the linear part of θ to be the identity. When the system is linearly detectable, that is the spectrum of F_u is in the open left half-plane, and if the spectrum of F_u is not resonant with the spectrum of F for degrees $k = 2,\dots,d$ then for any formal power series for $\beta(x)$ there is a formal power series for $\theta(x)$ that satisfies (12). From this formal series we can construct an observer (10) with nearly linear error dynamics (14). This leads to the following corollary. In the next section we give an example of this technique applied to the Lorentz equations.

Corollary 1. *If assumption* (2) *of Theorem* 1 *holds, then* $z = \theta(x)$ *defines a local change of coordinate transformation which transforms* (1) *into* (7). *An observer for* (1) *can be constructed in the form of* (8) *and the error dynamics is linear which is given by* (9). *If* A *is chosen to be Hurwitz, then the error dynamic is exponentially stable.*

Next suppose that there is only one unobservable mode and the corresponding eigenvalue $\lambda_{u,1} = 0$. Then the spectrum of F_u is resonant with the spectrum of F for all degrees

$$k\lambda_{u,1} = \lambda_{u,1} = 0 \tag{20}$$

and Eq. (12) may not be solvable.

Theorem 2. *Suppose that* (20) *are the only resonances. Then for any* $\beta^{[k]}$ *we can always find* $\theta^{[k]}$ *such that*

$$\frac{\partial\theta^{[k]}}{\partial x}(x)Fx - A\theta^{[k]}(x) + \sum_{i=1}^{k-1}\frac{\partial\theta^{[i]}}{\partial x}(x)f^{[k-i]}(x) + \beta^{[k]}(x) = c_k \mathbf{e}_u z_u^k, \tag{21}$$

where $\mathbf{e}_u$ *is the unit vector in the unobservable direction*

$$\mathbf{e}_u = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Moreover, if for some odd d, $c_2 = \dots = c_{d-1} = 0$ *and* $c_d < 0$ *then observer* (10) *is locally convergent.*

Proof. Eq. (12) is unsolvable when the homogeneous vector-valued polynomial $\theta^{[k]}$ of degree k equal to the degree of resonance. However, we can construct the $\theta^{[k]}$ as follows: let

$$\beta^{[k]}(x) = \sum_{j=1}^{n}\sum_{|m|=k}\beta_{j,m}\mathbf{e}^j x^m$$

and set

$$\theta^{[k]}(x) = \sum_{j=1}^{n}\sum_{|m|=k}\theta_{j,m}\mathbf{e}^j x^m$$

with

$$\theta_{j,m} = \frac{\beta_{j,m}}{m\cdot\lambda - \mu_j}$$

for those m and j for which the denominator is different from zero. After substituting this $\theta^{[k]}$ into (12), we obtain (21).

Next, let us consider the error dynamics in $\tilde{z}=z-\hat{z}$ coordinates. It takes the form

$$\dot{\tilde{z}} = A\tilde{z} + c_d e_u(z_u^d - \hat{z}_u^d) + O(z,\tilde{z})^d O(\tilde{z}). \tag{22}$$

Since the spectrum of A_o can be placed in the open left half-plane then there exists a positive definite solution P_o to the Lyapunov equation

$$A_o'P_o + P_oA_o = -I^{n_o\times n_o}. \tag{23}$$

Define

$$P = \begin{bmatrix} P_o & 0 \\ 0 & 1 \end{bmatrix}$$

then

$$\frac{d}{dt}(\tilde{z}'P\tilde{z}) = -|\tilde{z}_o|^2 + c_d\tilde{z}_u(z_u^d - \hat{z}_u^d) + O(z,\tilde{z})^d O(\tilde{z})^2,$$

which is negative definite for small $z,\tilde{z}$. □

4. Examples

We consider an observer for the Lorenz equations for the standard parameter values. We assume that the first state is directly measurable but the others are not.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{bmatrix},$$

$$y = x_1.$$

The first two states are linearly observable but the third state is not. The spectrum of F is $\{-22.83,\ 11.83,\ -2.67\}$ and, in particular, the eigenvalue of the unobservable mode is -2.67 so the system is linearly detectable.

We let

$$\beta(y) = \begin{bmatrix} -10 \\ -50 \\ 0 \end{bmatrix} y$$

then $A = F + BH$ has the spectrum $\{-10.5 \pm j11.39,\ -2.67\}$.

There are no resonances at least through degree 3 so we are able to find $\theta(x)=\theta^{[1]}(x)+\theta^{[2]}(x)+\theta^{[3]}(x)$ which transforms the Lorentz system to

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -20 & 10 & 0 \\ -22 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} - \begin{bmatrix} -10 \\ -50 \\ 0 \end{bmatrix} y$$
$$+O(z,y)^4.$$

The transformation which accomplishes this is

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$+ \begin{bmatrix} 0.0249x_1x_3 - 0.0025x_2x_3 \\ 0.0111x_1x_3 + 0.0208x_2x_3 \\ 0.0020x_1^2 + 0.0012x_1x_2 - 0.0184x_2^2 \end{bmatrix}$$
$$+10^{-3} \begin{bmatrix} -0.1647x_1^3 - 0.0290x_1^2x_2 \\ 0.0834x_1^3 + 0.2158x_1^2x_2 \\ -0.2288x_1^2x_3 - 0.1197x_1x_2x_3 \end{bmatrix}$$
$$+10^{-3} \begin{bmatrix} -0.3190x_1x_2^2 + 0.5617x_1x_3^2 \\ -0.1552x_1x_2^2 + 0.3745x_1x_3^2 \\ -0.5983x_2^2x_3 \end{bmatrix}$$
$$+10^{-3} \begin{bmatrix} 0.0492x_2^3 - 0.0509x_2^2x_3 \\ -0.2354x_2^3 + 0.4921x_2^2x_3 \\ 0 \end{bmatrix}$$
$$+O(x)^4.$$

In Fig. 1 we show the dynamics of the Lorenz system for 20 s starting from $x_1=10$, $x_2=20$, $x_3=30$. Notice $x_1(t) \approx x_2(t)$ and these two variables oscillate with $x_3(t)$. The oscillation shift erratically from positive values of $x_1(t)$, $x_2(t)$ to negative ones.

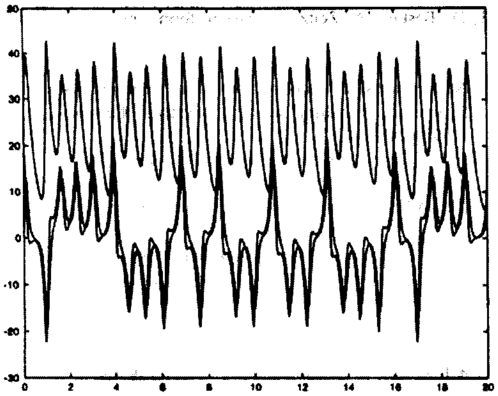

Fig. 1. Dynamics of Lorenz system for 20 s.

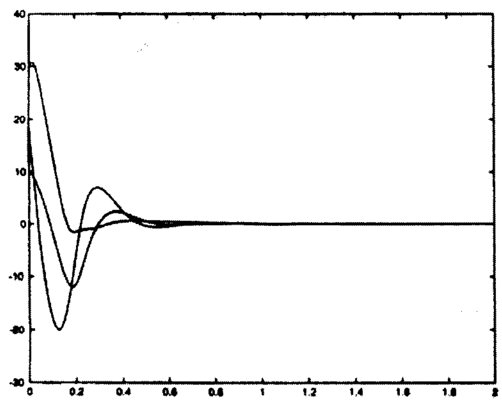

Fig. 2. Error of Lorenz observer for 2 s.

In Fig. 2 we show the error of the observer for the Lorenz system for two seconds starting from $\hat{x}_1 = 0$, $\hat{x}_2 = 0$, $\hat{x}_3 = 30$. Notice that the observer converges in < 1 s which is roughly equal to the period of the oscillations of $x_1(t) \approx x_2(t)$ with respect to $x_3(t)$.

Next, we consider an example where there is one unobservable mode whose eigenvalue is 0.

$$\dot{x}_1 = -x_1 + x_2^2,$$
$$\dot{x}_2 = -x_1 x_2, \tag{24}$$
$$y = x_1.$$

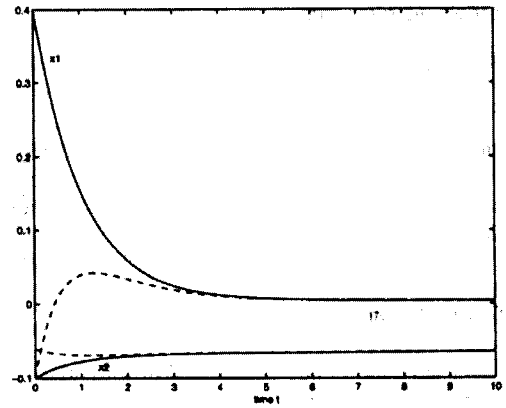

Fig. 3. Dynamics of the second example and its observer for 10 s..

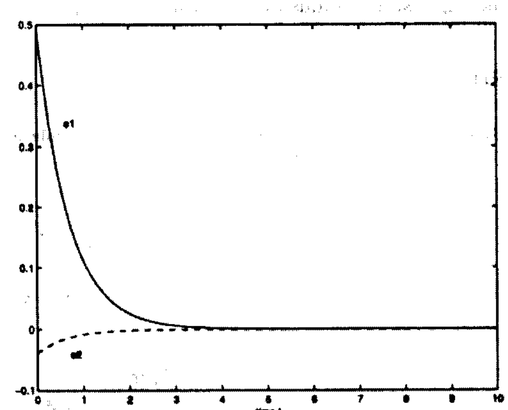

Fig. 4. Error of the observer for 10 s.

Let

$$B = \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}$$

then

$$A = F + BH = \begin{bmatrix} -\frac{3}{2} & 0 \\ 0 & 0 \end{bmatrix}.$$

If $\beta^{[2]} = \beta^{[3]} = 0$ then

$$z_1 = x_1 - \tfrac{2}{3} x_2^2 - \tfrac{8}{3} x_1 x_2^2,$$
$$z_2 = x_2 - x_1 x_2 + \tfrac{1}{7} x_1^2 x_2$$

carries the system to

$$\dot{z}_1 = -\tfrac{3}{2} z_1 + \tfrac{1}{2} y + O(z)^4,$$

$$\dot{z}_2 = -z_2^3 + O(z)^4$$

and an observer can be constructed by the above method. Fig. 3 shows the simulations of the dynamics of the system and the observer, and Fig. 4 provides the error dynamics.

5. Conclusion

We have shown how the method of Kazantzis and Kravaris can be extended to some systems that are not linearly observable and even to some systems which are not linearly detectable. Two examples included in the paper serve to illustrate the results.

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