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Design of Reduced-Order Observers of Nonlinear Systems through Change of Coordinates

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Abstract

We extend our recent results [9], [10] to the design of reduced-order observers for nonlinear systems. The approach method is to use change of coordinates, which is based on the solution of a system of first-order nonlinear PDEs. The sufficient condition for the solution of the PDEs is provided under very general conditions. The approach is also applicable when the system is only detectable. The method proposed in this paper is constructive and can be applied approximately to any sufficiently smooth, linearly observable system yielding a local observer with approximately linear error dynamics.

Keywords: Nonlinear systems; Nonlinear observers; Output injection; Linearizable error dynamics.

1 Introduction

We consider the problem of estimating the current state $x(t)$ of a smooth, nonlinear dynamical system,

$$\begin{array}{rcl} \dot{x} & = & f(x) \\ y & = & h(x) \\ x(0) & = & x_0 \end{array} \quad (1.1)$$

from the past observation $y(s), s \leq t$, where $f(0) = 0$ and $h(0) = 0$. By smooth we mean that f and h are C^r for r being sufficiently large. Occasionally we shall assume real analyticity of f and h. The dimension of x is n and the dimension of y is p which is typically less than n. For the system (1.1), an observer is a dynamical system driven by the observation y

$$\dot{\hat{x}} = \hat{f}(\hat{x}, y) \quad (1.2)$$

such that the estimation error $x - \hat{x}$ goes to zero as $t \to \infty$. A local observer is one whose estimation error converges to zero when $x(0)$ and $\hat{x}(0)$ in a neighborhood of the origin.

Based on the idea of [6] and [7], one technique of constructing an observer proposed by Kazantzis and Kravaris [3],[4], [5] is to seek a local change of coordinates $z = \theta(x)$ which transforms (1.1) into a linear system up to an output injection

$$\begin{array}{rcl} \dot{z} & = & Az + \beta(y) \\ y & = & h(\theta^{-1}(z)) \end{array} \quad (1.3)$$

where the matrix A is Hurwitz. It is easy to construct an observer for such a system in the new coordinate z:

$$\begin{array}{rcl} \dot{\hat{z}} & = & A\hat{z} + \beta(y) \\ \hat{x} & = & \theta^{-1}(\hat{z}) \end{array} \quad (1.4)$$

where the error dynamics $\tilde{z} := z - \hat{z}$ is linear

$$\dot{\tilde{z}} = A\tilde{z} \quad (1.5)$$

and is *exponentially stable.* Alternatively one can pull the observer back into the original coordinates to obtain

$$\dot{\hat{x}} = f(\hat{x}) + \left(\frac{\partial\theta}{\partial x}(\hat{x})\right)^{-1} (\beta(y) - \beta(h(\hat{x})). \quad (1.6)$$

It is easy to see that $z = \theta(x)$ transforms (1.1) into (1.3) if and only if θ satisfies a system of the

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first-order PDEs

$$\frac{\partial \theta}{\partial x}(x)f(x) \quad = \quad A\theta(x) - \beta(y). \qquad (1.7)$$

Let $\frac{\partial f}{\partial x}(0) := F$ and its eigenvalues be $(\lambda_1, \ldots, \lambda_n) := \lambda$. If there are no resonances of any degree $d > 0$ and the system is real analytic, applying Liapunov Theorem Kazantzis and Kravaris proved that this formal power series solution converges if $\lambda = (\lambda_1, \ldots, \lambda_n)$ is in the Poincaré domain, that is, 0 is not in the convex hull of $\{\lambda_1, \ldots, \lambda_n\}$ (in other words, eigenvalues in the Poincaré domain implies that either all of them in the left plane or all of them in the right plane [1]). If 0 is in the convex hull of $\{\lambda_1, \ldots, \lambda_n\}$ then λ is said to be in the Siegel domain [1]. Krener and Xiao [9] proved that even if λ is in the Siegel domain then a Hurwitz A can be chosen so that there are no resonances and the formal series for $\theta(x)$ converges.

The above approach through change of coordinates requires the observer to have the same size as the original systems, called a full-order observer, which may result in a high computational cost that sometimes may be intolerable when the system has a large size. Hence a reduced-order observer design is an attractive choice from a practical viewpoint.

In this paper we provide a method for designing the reduced-order observer through change of coordinates, whose idea is presented in [9] and [10]. Similar to our prior approach as described in the introduction, we introduce change of coordinates $z = \theta(x)$ in which the image of θ has a lower dimension. This leads to a reduction of the order of the system of the PDEs (1.7) and reduces the computation cost, while maintaining the error dynamics to be linear and thus exponentially stable.

The paper is organized as follows. Section 2 briefly discusses the existence and uniqueness of the solution of a system of first-order nonlinear PDEs. A detail discussion of design of reduced-order observer is given in section 3. Section 4 studies the case when the system is only detectable. Two examples are provided in section 5 to illustrate the results of the paper.

2 Solution of a System of First-Order PDEs

We first introduce the following definition:

Definition 1 *Given an $n \times n$ matrix F with spectrum $\sigma(F) = \lambda = (\lambda_1, \ldots, \lambda_n)$ and constants $C > 0$, $\nu > 0$, we say a complex number μ is of type (C, ν) with respect to $\sigma(F)$ if for any vector $m = (m_1, m_2, \ldots, m_n)$ of nonnegative integers, $|m| = \sum m_i > 0$, we have*

$$|\mu - m \cdot \lambda| \geq \frac{C}{|m|^{\nu}}. \qquad (2.8)$$

Loosely speaking, a complex number μ is of type (C, ν) with respect to $\sigma(F) = \lambda$ if $|\mu - m \cdot \lambda|$ is never zero and does not approach zero too fast as $|m| \to \infty$. If ν is large enough then the set of μ's which are of type (C, ν) for some $C > 0$ is dense in the complex plane [9].

A minor modification of the proof given in [9] leads to the following theorem.

Theorem 1 *Assume that $f : R^n \to R^n, h : R^n \to R^p$ and $\beta : R^p \to R^{n-p}$ are analytic vector fields with $f(0) = 0$, $h(0) = 0$, $\beta(0) = 0$ and $F = \frac{\partial f}{\partial x}(0)$, $H = \frac{\partial h}{\partial x}(0)$, $B = \frac{\partial \beta}{\partial x}(0)$. Let the eigenvalues of F be $\lambda = (\lambda_1, \ldots, \lambda_n)$. Let A be an $(n-p) \times (n-p)$ matrix and its eigenvalues be $(\mu_1, \ldots, \mu_{n-p})$. Suppose that*

1. *if λ is in the Pointcaré domain, then there does not exist non-native integers m_1, m_2, $\ldots$, m_{n-p} not all zero such that*

$$\sum_{i=1}^{n-p} m_i \lambda_i = \mu_j;$$

2. *if λ is in the Siegel domain, then there exists a $C > 0, \nu > 0$ such that all the eigenvalues of A are of type (C, ν) w.r.t. $\sigma(F)$.*

Then there exists a unique analytic solution $z = \theta(x)$ to the system of PDEs

$$\frac{\partial \theta}{\partial x}(x)f(x) \quad = \quad A\theta(x) + \beta(h(x)). \qquad (2.9)$$

locally around the origin, where $\theta : R^n \to R^{n-p}$.

One can show that the assumption 1 of the theorem actually implies that all the eigenvalues of A are of type (C,ν) w.r.t. $\sigma(F)$ [9]. Hence the assumption of type (C,ν) w.r.t. $\sigma(F)$ is essential for the formal series of $\theta(x)$ to be convergent.

Note that A has only $n-p$ eigenvalues, thus the requirement of A being of type (C,ν) w.r.t. $\sigma(F)$ is easier to be verified than one in [9].

3 Reduced-Order Observers

We assume that the following $p\times n$ matrix has rank p:

$$H := \begin{bmatrix} \frac{\partial h_1}{\partial x}(0) \\ \vdots \\ \frac{\partial h_p}{\partial x}(0) \end{bmatrix}. \tag{3.10}$$

We wish to construct an observer with order $(n-p)\times(n-p)$:

$$\begin{aligned} \dot{\hat{z}} &= Az+\beta(y) \\ \hat{x} &= \Psi(\hat{z},y) \end{aligned} \tag{3.11}$$

such that the error $\tilde{x}(t) = x(t) - \hat{x}(t)$ goes to zero when x_0 and $\hat{x}_0$ are close to $x=0$, where $\Psi : R^n \to R^n$ will be discussed next. Thus we seek a coordinate transform $z=\theta(x)$ which can translate (1.1) into the first part of (3.11). Clearly, there exists such a coordinate transform if and only if θ satisfies (2.9).

Theorem 2 *Suppose that the assumptions of Theorem 1 are satisfied and (1.1) is linearly observable. Let*

$$\Phi(x) := \begin{bmatrix} h_1(x) \\ \vdots \\ h_p(x) \\ \theta(x) \end{bmatrix}. \tag{3.12}$$

Then Φ is an analytic function. Moreover, there always exists an output injection function β such that Φ is an invertible map in a neighborhood of the origin.

The proof of Theorem 2 will be given in the end of this section.

Let $z_0 = \theta(x_0)$ and $\hat{z}(0) = \hat{z}_0$. Define the error to be $\tilde{z}(t) := z(t) - \hat{z}(t)$. Then if we choose A to be Hurwitz, the error dynamics is given by

$$\dot{\tilde{z}}(t) = A\tilde{z}(t) \tag{3.13}$$

and is exponentially stable. Under the assumption of Theorem 2, we can denote $x = \Phi^{-1}(y,z) := \Psi(y,z)$. Let us define $\hat{x} = \Phi^{-1}(y,\hat{z}) = \Psi(y,\hat{z})$ and

$$\tilde{x} := x - \hat{x}. \tag{3.14}$$

Since $z(t) \to \hat{z}(t)$ as $t\to\infty$, one can see that

$$\begin{aligned} \tilde{x}(t) &= x(t)-\hat{x}(t) \\ &= \Phi^{-1}(y,z(t)) - \Phi^{-1}(y,\hat{z}(t)) \\ &= \Psi(y,z(t)) - \Psi(y,\hat{z}(t)) \to 0, \end{aligned} \tag{3.15}$$

in a neighborhood of $x=0$. This leads to the following theorem:

Theorem 3 *Under the assumptions of Theorem 1 and Theorem 2, if A is chosen to be Hurwitz, then*

$$\tilde{z}(t) = z(t) - \hat{z}(t) \to 0 \qquad \text{as } t\to\infty. \tag{3.16}$$

Hence if we choose z_0, $\hat{z}_0$ are close to $z=0$, then we have

$$\tilde{x}(t) = x(t) - \hat{x}(t) \to 0 \qquad \text{as } t\to\infty. \tag{3.17}$$

The size of the neighborhood of $x=0$ on which Φ is a diffeomorphism varies with the higher derivatives of β, hence a careful selection of β can enlarge the domain of the convergence of the observer. We will provide two examples to illustrate this.

In the linear case, that is, $f(x) = Fx, h(x) = Hx$, (3.11) can be reduced to the reduced-order Luenberger observer through change of coordinates. To see this, we assume

$$\begin{aligned} H &= [0^{p\times p} : I^{p\times(n-p)}] \\ F &= \begin{bmatrix} F_{11}^{(n-p)\times(n-p)} & F_{12}^{(n-p)\times p} \\ F_{21}^{p\times(n-p)} & F_{22}^{p\times p} \end{bmatrix}. \end{aligned} \tag{3.18}$$

Let $z=\theta(x)$. We need to solve the following matrix equation:

$$TF = AT + BH, \qquad \text{where } T = \frac{\partial\theta}{\partial x}(0). \tag{3.19}$$

We let

$$
\begin{aligned}
A &= F_{11} + MF_{21} \\
B &= F_{12} + MF_{22} - (F_{11} + MF_{21})M,
\end{aligned} \tag{3.20}
$$

where M is an $(n-p) \times p$ matrix. Since (H, F) is observable, so does (F_{21}, F_{11}). Thus the spectrum of A can be set arbitrarily by choice of M. Now notice that (3.19) has a solution

$$
T = [I^{(n-p)\times(n-p)} : M]. \tag{3.21}
$$

Hence in this case

$$
\Phi(x) = \begin{bmatrix} 0 & I^{p\times p} \\ I^{(n-p)\times(n-p)} & M. \end{bmatrix} x \tag{3.22}
$$

Thus

$$
\Psi(z, y) = \Phi^{-1}(z, y) = \begin{bmatrix} -M & I^{(n-p)\times(n-p)} \\ I^{p\times p} & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}. \tag{3.23}
$$

Therefore the reduced-order observer is given by

$$
\begin{aligned}
\dot{\hat{z}} &= (F_{11} + MF_{21})\hat{z} \\
&+ (F_{12} + MF_{22} - (F_{11} + MF_{12})M)y \\
\hat{x} &= \begin{bmatrix} I \\ 0 \end{bmatrix} \hat{z} + \begin{bmatrix} -M \\ I \end{bmatrix} y.
\end{aligned} \tag{3.24}
$$

Now we are ready to show Theorem 2.

Proof of Theorem 2: Since $h_1, \cdots, h_p$ are analytic according to our assumption and θ is analytic by Theorem 1, Φ is an analytic function. Without loss of generality, we can assume that

$$
H = \frac{\partial h}{\partial x}(0) = [0^{p\times p} : I^{p\times(n-p)}].
$$

Now we set β to be such that

$$
\frac{\partial \beta}{\partial x}(0) = B = F_{12} + MF_{22} - (F_{11} + MF_{21})M \tag{3.25}
$$

where

$$
\frac{\partial f}{\partial x}(0) = F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \tag{3.26}
$$

and M is an arbitrary $(n-p) \times p$ matrix, then

$$
\frac{\partial \Phi}{\partial x}(0) = \begin{bmatrix} 0 & I^{p\times p} \\ I^{(n-p)\times(n-p)} & M \end{bmatrix} \tag{3.27}
$$

which has a full rank. Therefore Φ is an invertible map in a neighborhood of the origin. □

One does not need an exact solution of the PDEs (2.9) to construct a local observer, a finite series suffices. Suppose

$$
\theta(x) = Tx + \theta^{[2]}(x) + \cdots + \theta^{[d]}(x) \tag{3.28}
$$

satisfies the following system of PDEs

$$
\frac{\partial \theta}{\partial x}(x)f(x) = A\theta(x) + \beta(h(x)) + O(x)^{d+1}, \tag{3.29}
$$

where $\theta^{[k]}(x)$ denotes a polynomial vector field that is homogeneous of degree k, and T is an $(n-p) \times n$ matrix satisfies

$$
TF = AT - BH. \tag{3.30}
$$

Then the observer (3.11) has nearly linear error dynamics in transformed coordinates

$$
\dot{\tilde{z}} = A\tilde{z} + O(z, \tilde{z})^d O(\tilde{z}). \tag{3.31}
$$

since the spectrum of A lies in the open left half plane, $\tilde{z} \to 0$ in a neighborhood of $z = 0$. Hence $x - \hat{x} \to 0$ as $t \to \infty$.

4 Linearly Unobservable systems

We consider systems (1.1) whose linear part is unobservable. Assume that the linear part of (1.1) has n_o observable modes x_o and n_u unobservable modes x_u whose linear dynamics is diagonalizable. Then after a linear change of coordinate we can assume the following form,

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_o \\ \dot{x}_u \end{bmatrix} &= \begin{bmatrix} F_o & 0 \\ 0 & F_u \end{bmatrix} \begin{bmatrix} x_o \\ x_u \end{bmatrix} + O(x_o, x_u)^2 \\
y &= \begin{bmatrix} H_o & 0 \end{bmatrix} \begin{bmatrix} x_o \\ x_u \end{bmatrix} + O(x_o, x_u)^2
\end{aligned}
$$

where F_o, H_o are in the observable form and F_u is diagonal

$$
F_u = \begin{bmatrix} \lambda_{u,1} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & \lambda_{u,n_u} \end{bmatrix}.
$$

Let T be an $(n-p) \times n$ matrix such that the following matrix

$$
\begin{bmatrix} H \\ T \end{bmatrix} \tag{4.32}
$$

is nonsingular, where $H = [H_0, 0]$.

Theorem 4 *Suppose that the system is linearly detectable, that is, the spectrum $\sigma(F_u)$ of F_u lies in the open left half plane. Then:*

1. *for any formal power series $\beta(x)$, there is a formal power series for $\theta(x)$ satisfies (2.9);*

2. *if in addition the system is real analytic and the spectrum of F_u is of type (C, ν) for $|m| > 1$ with respect to the spectrum of F then the formal power series converges.*

Proof: (a) Without loss of generality we can assume that the linear part of the change of coordinates is Tx. Thus let

$$\theta(x) := Tx + \theta_2(x)$$

where $\theta_2(x)$ is to be determined. Let

$$\theta_2(x) = \theta^{[2]}(x) + \theta^{[3]}(x) + \cdots + \theta^{[k]}(x) + \cdots.$$

Then $\theta^{[k]}(x), k \geq 2$ can be obtained from the following equation

$$\begin{aligned} &\frac{\partial \theta^{[k]}}{\partial x}(x)Fx - A\theta^{[k]}(x) \\ &= -\sum_{i=1}^{k-1} \frac{\partial \theta^{[i]}}{\partial x}(x)f^{[k-i]}(x) - \beta^{[k]}(x) \end{aligned} \tag{4.33}$$

since the eigenvalues of A_u are resonant of degree 1 with respect to the spectrum of F_u due to the linear detectability assumption.
(b) The proof is similar to the one given in [9]. □

Note that in this case, the function Φ is an invertible map near $x = 0$ since

$$\frac{\partial \Phi}{\partial x}(0) = \begin{bmatrix} H \\ T \end{bmatrix}. \tag{4.34}$$

Thus the conclusion of Theorem 3 still holds without the observable assumption.

5 Examples

First we consider a Van der Pol oscillator:

$$\begin{aligned} &\ddot{x} + (x^2 - 1)\dot{x} + x = 0 \\ &y = x \end{aligned}$$

which is equivalent to the planar system

$$\begin{aligned} &\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 \\ x_1^2 x_2 \end{bmatrix} \\ &y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

by letting $x_1 = x, x_2 = \dot{x}$. Now we have

$$\begin{aligned} &f(x) = \begin{bmatrix} x_2 \\ -x_1 + x_2 - x_1^2 x_2 \end{bmatrix}, \quad h(x) = x_1, \\ &F = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \qquad H = \begin{bmatrix} 1 & 0 \end{bmatrix}. \end{aligned}$$

In this case, $\sigma(F) = \{1/2 \pm j\sqrt{3}/2\}$. The PDE (2.9) becomes

$$x_2 \frac{\partial \theta}{\partial x_1} + (-x_1 + x_2 - x_1^2 x_2)\frac{\partial \theta}{\partial x_2} = A\theta + \beta(h(\mathrm{x})). \tag{5.35}$$

If we let

$$A = -\frac{3}{2}, \qquad \beta(y) = -\frac{19}{4}y + \frac{y^3}{2},$$

then (5.35) has an analytic solution:

$$\theta(x_1, x_2) = -\tfrac{5}{2}x_1 + \tfrac{x_1^3}{3} + x_2. \tag{5.36}$$

Thus we have

$$\Phi(x_1; x_2) = \begin{bmatrix} x_1 \\ -\frac{5}{2}x_1 + \frac{x_1^3}{3} + x_2 \end{bmatrix}. \tag{5.37}$$

The (global) reduced-order observer is then given by

$$\begin{aligned} \dot{\hat{z}} &= -\tfrac{3}{2}\hat{z} - \tfrac{19}{4}y + \tfrac{y^3}{2} \\ \hat{x}_2 &= \hat{z} + \tfrac{5}{2}y - \tfrac{y^3}{3}. \end{aligned} \tag{5.38}$$

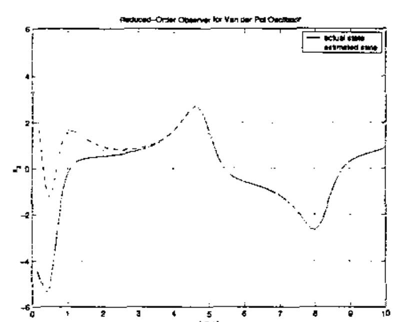

Figure 1: Observation of Van der Pol Oscillator

Next we consider the inverted pendulum driven by an armature controlled DC motor governed by the equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ \sin x_1 + x_3 \\ x_2 + x_3 \end{bmatrix} \qquad (5.39)$$

$$y = x_1.$$

In this case

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad (5.40)$$

which has eigenvalues $\{-1.2470, 0.4450, 1.8019\}$. In this case, (2.9) is in the form of

$$\begin{bmatrix} \frac{\partial \theta_1}{\partial x_1} & \frac{\partial \theta_1}{\partial x_2} & \frac{\partial \theta_1}{\partial x_3} \\ \frac{\partial \theta_2}{\partial x_1} & \frac{\partial \theta_2}{\partial x_2} & \frac{\partial \theta_2}{\partial x_3} \end{bmatrix} \begin{bmatrix} x_2 \\ \sin x_1 + x_3 \\ x_2 + x_3 \end{bmatrix} \qquad (5.41)$$
$$= A \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} + \beta(y).$$

Let us choose

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \beta(y) = \begin{bmatrix} \sin y - 10y/3 \\ y - 2\sin y \end{bmatrix}. \qquad (5.42)$$

Then (5.41) has the solution

$$\theta(x_1, x_2, x_3) = \begin{bmatrix} -\frac{5}{3}x_1 + x_2 - \frac{1}{3}x_3 \\ x_1 - 2x_2 + x_3 \end{bmatrix}. \qquad (5.43)$$

Thus we have

$$\Phi(x_1, x_2, x_3) = \begin{bmatrix} x_1 \\ -\frac{5}{3}x_1 + x_2 - \frac{1}{3}x_3 \\ x_1 - 2x_2 + x_3 \end{bmatrix}. \qquad (5.44)$$

Hence the (global) reduced-order observer is given by

$$\begin{aligned} \dot{\hat{z}}_1 &= -2\hat{z}_1 + \sin y - 10y/3 \\ \dot{\hat{z}}_2 &= -\hat{z}_2 + y - 2\sin y \\ \hat{x}_2 &= 4y + 3\hat{z}_1 + \hat{z}_2 \\ \hat{x}_3 &= 7y + 6\hat{z}_1 + 3\hat{z}_2. \end{aligned} \qquad (5.45)$$

Figure 2 shows the error dynamics of the observer, where $e_1 := x_2 - \hat{x}_2$, $e_2 := x_3 - \hat{x}_3$.

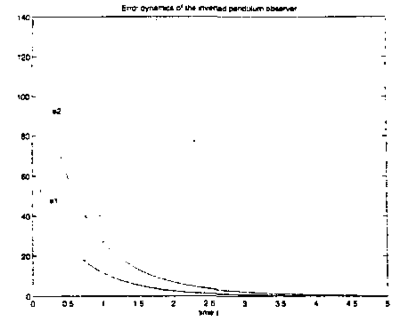

Figure 2: Error dynamics of the observer for 5 sec

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