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NORMAL FORMS AND BIFURCATIONS OF DISCRETE TIME NONLINEAR CONTROL SYSTEMS*

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Abstract. The quadratic and cubic normal forms of discrete time nonlinear control systems are presented. These are the normal forms with respect to the group of state coordinate changes and invertible state feedbacks. We introduce the concept of a control bifurcation for such systems. A control bifurcation takes place at an equilibrium where there is a loss of linear stabilizability in contrast to a classical bifurcation, which typically takes place at an equilibrium where there is a loss of linear stability. We present the analogous control bifurcations to the well-known classical bifurcations; the fold, the transcritical, the flip, and the Neimark–Sacker bifurcations. When the loop is closed, a control bifurcation can lead to a classical bifurcation.

Key words. discrete time nonlinear control system, normal forms, bifurcations, control bifurcations

AMS subject classifications. 93C10, 93C55, 37G05, 37G10, 37G15

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1. Introduction.

The theory of normal forms and bifurcations of nonlinear difference equations is well known [1], [5], [9], [13]. Briefly, it is as follows. Consider two smooth (C^4) n dimensional difference equations with equilibrium points

$$\begin{aligned} x^+ &= f(x), \\ 0 &= f(0) \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} z^+ &= g(z), \\ 0 &= g(0), \end{aligned} \tag{1.2}$$

where $x^+(t) = x(t+1)$. These are *locally diffeomorphic* if there exists a local diffeomorphism

$$\begin{aligned} z &= \phi(x), \\ 0 &= \phi(0) \end{aligned} \tag{1.3}$$

which carries (1.1) to (1.2),

$$g(\phi(x)) = \phi(f(x)).$$

Such a local diffeomorphism carries trajectories $x(t)$ in its domain onto trajectories $z(t)$ in its range,

$$z(t) = \phi(x(t));$$

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hence the two dynamics are locally smoothly equivalent.

There is a weaker notion of equivalence; (1.1) is *locally topologically conjugate* to (1.2) if there is a local homeomorphism (1.3) which carries trajectories $x(s)$ in its domain onto trajectories $z(t)$ in its range while preserving the orientation of time, but not the exact time.

The *linear approximation* of (1.1) around the fixed point $x = 0$ is

$$\delta x^+ = \frac{\partial f}{\partial x}(0)\ \delta x, \tag{1.4}$$

and this is a *hyperbolic fixed point* if $\frac{\partial f}{\partial x}(0)$ has no eigenvalues on the unit circle. The discrete time *Grobman–Hartman* theorem states that if the equilibrium $x = 0$ of (1.1) is hyperbolic, then it is locally topologically conjugate to its linear approximation (1.4). A related theorem is that two hyperbolic equilibria are locally topologically conjugate if their linear approximations have the same number of eigenvalues strictly inside the unit circle, the signs of their products are the same, and the same number of eigenavalues strictly outside and the signs of their products are the same [9].

A parametrized system

$$x^+ = f(x, \mu) \tag{1.5}$$

can have a locus of equilibria

$$x_e = f(x_e, \mu_e).$$

It undergoes a *local bifurcation* at an equilibrium $x_e,\ \mu_e$ that is not locally topologically conjugate to every nearby equilibrium. In light of the above, such a bifurcation can happen only if one or more eigenvalues of the linearized system cross the unit circle, or the sign of the product of the strictly stable eigenvalues changes, or the sign of the product of the strictly unstable eigenvalues changes.

A standard approach to analyzing the behavior of the parametrized system (1.5) around a bifurcation point is to add the parameter as an additional state with trivial dynamics

$$\mu^+ = \mu \tag{1.6}$$

and then compute the center manifold through the bifurcation point and the dynamics restricted to this manifold [3], [9]. The center manifold is an invariant manifold of the extended difference equation (1.5)–(1.6), which is tangent at the bifurcation point to the eigenspace of the eigenvalues on the unit circle. In practice, one does not compute the center manifold and its dynamics exactly; in most cases of interest, an approximation of degree two or three suffices. If the other eigenvalues are off the unit circle, then this part of the dynamics cannot affect the local topological conjugacy around the bifurcation point. If at the bifurcation point all of the eigenvalues of the linear approximation are inside or on the unit circle, then the bifurcation point will be locally asymptotically stable for the complete dynamics iff the dynamics on the center manifold is locally asymptotically stable. Of course, at some nearby equilibria the dynamics may be unstable.

The next step is to compute the Poincaré normal form of the center manifold dynamics. This is a normal form under smooth changes of coordinates

$$z = \phi(x) = Tx - \phi^{[2]}(x) - \phi^{[3]}(x) - \cdots, \tag{1.7}$$

where $\phi^{[d]}(x)$ denotes a vector field that is a homogeneous polynomial of degree d in x. The linear part of the change of coordinates T puts the linear part of the center manifold dynamics in Jordan form. The quadratic, cubic, and higher parts of the change of coordinates $\phi^{[2]}$ and $\phi^{[3]}$ simplify the quadratic, cubic, and higher parts of the center manifold dynamics by putting them in Poincaré normal form. From its normal form the bifurcation is recognized and understood. Examples are the fold (or saddle-node), the flip, and the Neimark–Sacker bifurcations. The first depends on the normal form of degree two, and the last two depend on the normal form of degree three. These are the only ones that are generic and of codimension 1, i.e., depend on a single parameter, so these are the most important.

Kang and Krener [6] developed a quadratic normal form for continuous time nonlinear systems whose linear part is controllable. This was extended to discrete time systems by Barbot, Monaco, and Normand-Cyrot [2]. These authors considered a larger group of transformations to bring the system to normal form, including invertible state feedback as well as change of state coordinates. Kang [7], [8] also developed a quadratic normal form for continuous time nonlinear systems whose linear part may have uncontrollable modes. Krener, Kang, and Chang [10], [4] described the quadratic and cubic normal forms of continuous time nonlinear control systems and also their bifurcations.

In this paper, we will develop quadratic and cubic normal forms for discrete time nonlinear control systems of the form

$$\begin{aligned} x^+ = f(x,u) = Ax + Bu + f^{[2]}(x,u) \\ +f^{[3]}(x,u) + O(x,u)^4, \end{aligned} \tag{1.8}$$

where x, u are of dimensions $n, 1$ and $f^{[d]}(x,u)$ denotes a vector field that is a homogeneous polynomial of degree d in x, u. We do not assume that the linear part of the system is controllable. Moreover, our linear and quadratic normal forms differ from that of [2] for linearly controllable systems.

We also describe some of the simplest bifurcations of discrete time nonlinear control systems. A control system does not need a parameter to bifurcate; the control can play the same role. The equilibria of a controlled difference equation,

$$x^+ = f(x,u), \tag{1.9}$$

are those values of x_e, u_e such that $f(x_e, u_e) = x_e$. The equilibria are conveniently parametrized by u or one of the state variables. Two key facts differentiate bifurcations of a control system (1.8) from that of a parametrized system (1.5). The first is that for the latter the structural stability of the equilibria is the crucial issue, but for the former the stabilizability by state feedback is the crucial issue. A control system (1.8) is linearly controllable (linearly stabilizable) at x_e, u_e if the local linear approximation

$$\delta x^+ = \frac{\partial f}{\partial x}(x_e, u_e)\ \delta x + \frac{\partial f}{\partial u}(x_e, u_e)\ \delta u$$

is controllable (stabilizable). If the linear approximation is stabilizable, then the nonlinear system is locally stabilizable. If the linear approximation is not stabilizable, then the nonlinear system may or may not be locally stabilizable, depending on higher degree terms. A *control bifurcation* of (1.8) takes place at an equilibrium where the linear approximation loses stabilizability. Notice that this is different from the bifurcation of a parametrized system (1.5), which takes place at an equilibrium where there is a loss of structural stability with respect to parameter variations. To emphasize this distinction, we shall refer to the latter as a *classical bifurcation*.

The other difference between control and classical bifurcations is that when bringing the control system into normal form, a different group of transformations is used. For classical bifurcations, we use parameter dependent change of state coordinates and change of parameter coordinates, but for control bifurcations we use change of state coordinates and state dependent change of control coordinates (invertible state feedback) to simplify the dynamics.

2. Quadratic normal form. Consider a smooth (C^3) system of the form (1.8) under the action of linear and quadratic change of state coordinates and state feedback

$$z = \phi(x) = Tx - \phi^{[2]}(x), \tag{2.1}$$

$$v = \alpha(x,u) = Kx + Lu - \alpha^{[2]}(x,u), \tag{2.2}$$

where $T,\ L$ are invertible.

It is well known that there exist a linear change of coordinates T and a linear feedback $K,\ L$ that transform the system into the linear normal form

$$\begin{aligned} \begin{bmatrix} x_1^+ \\ x_2^+ \end{bmatrix} &= \begin{bmatrix} f_1(x_1,x_2,u) \\ f_2(x_1,x_2,u) \end{bmatrix} \\ &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u \\ &\quad + \begin{bmatrix} f_1^{[2]}(x_1,x_2,u) \\ f_2^{[2]}(x_1,x_2,u) \end{bmatrix} + O(x_1,x_2,u)^3, \end{aligned} \tag{2.3}$$

where $x_1,\ x_2$ are $n_1,\ n_2$ dimensional, $n_1+n_2=n$, A_1 is in Jordan form, and A_2, B_2 are in controller (Brunovsky) form:

$$A_2 = \begin{bmatrix} 0 & 1 & \dots & 0 \\ & \ddots & \ddots & \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

The following result generalizes [11].

THEOREM 2.1. *Consider the system* (2.3)*, where A_1 is diagonal and A_2, B_2 are in Brunovsky form. There exist a quadratic change of coordinates and a quadratic feedback*

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} \phi_1^{[2]}(x_1,x_2) \\ \phi_2^{[2]}(x_1,x_2) \end{bmatrix},$$

$$v = u - \alpha^{[2]}(x_1,x_2,u)$$

which transform the system (2.3) *into the quadratic normal form*

$$\begin{aligned} \begin{bmatrix} z_1^+ \\ z_2^+ \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} v \\ &\quad + \begin{bmatrix} \tilde f_1^{[2;0]}(z_1;z_2,v) & + & \tilde f_1^{[1;1]}(z_1;z_2,v) & + & \tilde f_1^{[0;2]}(z_1;z_2,v) \\ 0 & + & 0 & + & \tilde f_2^{[0;2]}(z_1;z_2,v) \end{bmatrix} \\ &\quad + O(z_1,z_2,v)^3, \end{aligned} \tag{2.4}$$

where $\tilde{f}_i^{[d_1;d_2]}(z_1; z_2, v)$ *is a polynomial vector field homogeneous of degree* d_1 *in* z_1 *and homogeneous of degree* d_2 *in* z_2, v*. For notational convenience, we define* $z_{2,n_2+1} = v$*.*

The vector field $\tilde{f}_1^{[2;0]}$ *is in the quadratic normal form of Poincaré,*

$$\tilde{f}_1^{[2;0]} = \sum_{\lambda_i=\lambda_j\lambda_k} \beta_i^{jk}\ \mathbf{e}_1^i\ z_{1,j}\ z_{1,k}, \tag{2.5}$$

where $\lambda_1, \dots, \lambda_{n_1}$ *are the eigenvalues of* A_1*,* $\mathbf{e}_r^i$ *is the* ith *unit vector in* z_r *space, and* $z_{r,i}$ *is the* ith *component of* z_r*. The other vector fields are as follows:*

$$\begin{aligned}\tilde{f}_1^{[1;1]} = &\sum_{\lambda_i=0}\sum_{\lambda_j=0}\sum_{k=1}^{n_2+1} \gamma_i^{jk}\ \mathbf{e}_1^i\ z_{1,j}\ z_{2,k}\\ &+\sum_{\lambda_i\neq 0}\sum_{\lambda_j\neq 0} \gamma_i^{j1}\ \mathbf{e}_1^i\ z_{1,j}\ z_{2,1},\end{aligned} \tag{2.6}$$

$$\tilde{f}_1^{[0;2]} = \sum_{\lambda_i\neq 0}\sum_{k=1}^{n_2+1} \delta_i^{1k}\ \mathbf{e}_1^i\ z_{2,1}\ z_{2,k}, \tag{2.7}$$

$$\tilde{f}_2^{[0;2]} = \sum_{i=1}^{n_2-1}\sum_{k=i+2}^{n_2+1} \epsilon_i^{1k}\ \mathbf{e}_2^i\ z_{2,1}\ z_{2,k}. \tag{2.8}$$

The normal form is unique; that is, each system (2.3) *can be transformed into only one such normal form* (2.4)–(2.8) *by a quadratic change of coordinates* (2.1) *and quadratic feedback* (2.2)*. This follows from the fact that the numbers in the above,* $\beta_i^{jk}, \gamma_i^{jk}, \delta_i^{1k}, \epsilon_i^{1k}$ *for the indicated indices, are moduli, i.e., continuous invariants of the system* (2.3) *under a quadratic change of coordinates and quadratic feedback. Let* $\sigma_{jk} = 2$ *if* $j = k$ *and* $\sigma_{jk} = 1$ *otherwise. The moduli are defined as follows:*

$$\beta_i^{jk} = \frac{1}{\sigma_{jk}}\ \frac{\partial^2 f_{1,i}}{\partial x_{1,j}\partial x_{1,k}}(0,0,0) \quad \textit{for } 1 \le i,\ j,k \le n_1,\ \textit{and } \lambda_i = \lambda_j\lambda_k, \tag{2.9}$$

$$\gamma_i^{jk} = \frac{\partial^2 f_{1,i}}{\partial x_{1,j}\partial x_{2,k}}(0,0,0) \quad \textit{for } 1 \le i,\ j \le n_1,\ 1 \le k \le n_2+1,\ \textit{and } \lambda_i = \lambda_j = 0, \tag{2.10}$$

$$\gamma_i^{j1} = \sum_{l=0}^{n_2}\left(\frac{\lambda_i}{\lambda_j}\right)^l \frac{\partial^2 f_{1,i}}{\partial x_{1,j}\partial x_{2,l+1}}(0,0,0) \quad \textit{for } 1 \le i,\ j \le n_1,\ \textit{and } \lambda_i\lambda_j \neq 0, \tag{2.11}$$

$$\delta_i^{1k} = \frac{1}{\sigma_{1k}}\sum_{l=0}^{n_2-k+1}\lambda_i^l \frac{\partial^2 f_{1,i}}{\partial x_{2,1+l}\partial x_{2,k+l}}(0,0,0) \quad \textit{for } 1 \le i \le n_1,\ 1 \le k \le n_2+1,\ \textit{and } \lambda_i \neq 0, \tag{2.12}$$

$$\epsilon_i^{1k} = \sum_{l=0}^{n_2-k+1} \frac{\partial^2 f_{2,i+l}}{\partial x_{2,1+l}\partial x_{2,k+l}}(0,0,0) \quad \textit{for } 1 \le i \le n_2-1 \textit{ and } i+2 \le k \le n_2+1. \tag{2.13}$$

Remarks. If some of the eigenvalues of A_1 are complex, then a linear complex change of coordinates is required to bring it to Jordan form. In this case, some of the coordinates of z_1 are complex conjugate pairs, and some of the coefficients in the normal form are complex. These complex coefficients occur in conjugate pairs so that the real dimension of the coefficient space of the normal form is unchanged.

In the normal form of Poincaré (2.5), the eigenvalues satisfying $\lambda_i = \lambda_j\lambda_k$ are said to be in quadratic resonance.

We defer the proof to a later section as it is quite lengthy.

3. Cubic normal form. We present the cubic normal form of a system that is already in linear and quadratic normal form.

THEOREM 3.1. *Consider a smooth* (C^4) *system*

$$\begin{aligned}\begin{bmatrix} x_1^+ \\ x_2^+ \end{bmatrix} &= \begin{bmatrix} f_1(x_1,x_2,u) \\ f_2(x_1,x_2,u) \end{bmatrix} \\ &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u \\ &\quad + \begin{bmatrix} f_1^{[2;0]}(x_1;x_2,u) \\ 0 \end{bmatrix} + \begin{bmatrix} f_1^{[1;1]}(x_1;x_2,u) \\ 0 \end{bmatrix} + \begin{bmatrix} f_1^{[0;2]}(x_1;x_2,u) \\ f_2^{[0;2]}(x_1;x_2,u) \end{bmatrix} \\ &\quad + \begin{bmatrix} f_1^{[3]}(x_1;x_2,u) \\ f_2^{[3]}(x_1;x_2,u) \end{bmatrix} + O(x_1,x_2,u)^4, \end{aligned} \tag{3.1}$$

where A_1 is diagonal, A_2, B_2 are in Brunovsky form, and the quadratic terms are in the normal form of Theorem 2.1. *There exist a cubic change of coordinates and a cubic feedback*

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} \phi_1^{[3]}(x_1,x_2) \\ \phi_2^{[3]}(x_1,x_2) \end{bmatrix},$$

$$v = u - \alpha^{[3]}(x_1,x_2,u)$$

which transform the system (3.1) *into the cubic normal form*

$$\begin{aligned}\begin{bmatrix} z_1^+ \\ z_2^+ \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} v \\ &\quad + \begin{bmatrix} f_1^{[2;0]}(z_1;z_2,v) \\ 0 \end{bmatrix} + \begin{bmatrix} f_1^{[1;1]}(z_1;z_2,v) \\ 0 \end{bmatrix} + \begin{bmatrix} f_1^{[0;2]}(z_1;z_2,v) \\ f_2^{[0;2]}(z_1;z_2,v) \end{bmatrix} \\ &\quad + \begin{bmatrix} \tilde{f}_1^{[3;0]}(z_1;z_2,v) \\ 0 \end{bmatrix} + \begin{bmatrix} \tilde{f}_1^{[2;1]}(z_1;z_2,v) \\ 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} \tilde{f}_1^{[1;2]}(z_1;z_2,v) \\ \tilde{f}_2^{[1;2]}(z_1;z_2,v) \end{bmatrix} + \begin{bmatrix} \tilde{f}_1^{[0;3]}(z_1;z_2,v) \\ \tilde{f}_2^{[0;3]}(z_1;z_2,v) \end{bmatrix} + O(z_1,z_2,v)^4. \end{aligned} \tag{3.2}$$

The vector field $\tilde{f}_1^{[3;0]}$ is in the cubic normal form of Poincaré,

$$\tilde{f}_1^{[3;0]} = \sum_{\lambda_i=\lambda_j\lambda_k\lambda_l} \beta_i^{jkl}\ \mathbf{e}_1^i\ z_{1,j}\ z_{1,k}\ z_{1,l}, \tag{3.3}$$

and the other vector fields are as follows:

$$\tilde{f}_1^{[2;1]} = \sum_{\lambda_i=0} \sum_{\lambda_j\lambda_k=0} \sum_{l=1}^{n_2+1} \gamma_i^{jkl}\ \mathbf{e}_1^i\ z_{1,j}\ z_{1,k}\ z_{2,l} \tag{3.4}$$
$$+ \sum_{\lambda_i\neq 0} \sum_{\lambda_j\lambda_k\neq 0} \gamma_i^{jk1}\ \mathbf{e}_1^i\ z_{1,j}\ z_{1,k}\ z_{2,1},$$

$$\tilde{f}_1^{[1;2]} = \sum_{\lambda_i=0} \sum_{\lambda_j=0} \sum_{k=1}^{n_2+1} \sum_{l=k}^{n_2+1} \delta_i^{jkl}\ \mathbf{e}_1^i\ z_{1,j}\ z_{2,k}\ z_{2,l} \tag{3.5}$$
$$+ \sum_{\lambda_i\neq 0} \sum_{\lambda_j\neq 0} \sum_{l=1}^{n_2+1} \delta_i^{j1l}\ \mathbf{e}_1^i\ z_{1,j}\ z_{2,1}\ z_{2,l},$$

$$\tilde{f}_1^{[0;3]} = \sum_{\lambda_i\neq 0} \sum_{k=1}^{n_2+1} \sum_{l=k}^{n_2+1} \epsilon_i^{1kl}\ \mathbf{e}_1^i\ z_{2,1}\ z_{2,k}\ z_{2,l}, \tag{3.6}$$

$$\tilde{f}_2^{[1;2]} = \sum_{i=1}^{n_2-1} \sum_{\lambda_j\neq 0} \sum_{l=i+2}^{n_2+1} \zeta_i^{j1l}\ \mathbf{e}_2^i\ z_{1,j}\ z_{2,1}\ z_{2,l}, \tag{3.7}$$

$$\tilde{f}_2^{[0;3]} = \sum_{i=1}^{n_2-1} \sum_{l=i+2}^{n_2+1} \sum_{k=1}^{l} \eta_i^{1kl}\ \mathbf{e}_2^i\ z_{2,1}\ z_{2,k}\ z_{2,l}. \tag{3.8}$$

The normal form is unique; that is, each system (3.1) *can be transformed into only one such normal form* (3.2)–(3.8)*. This follows from the fact that the numbers in the above,* $\beta_i^{jkl}, \gamma_i^{jkl}, \delta_i^{1kl}, \epsilon_i^{1kl}, \zeta_i^{j1l}, \eta_i^{1kl}$ *for the indicated indices, are moduli of the system* (2.3) *under a cubic change of coordinates and cubic feedback. Let* $\sigma_{jkl} = 6$ *if* $j = k = l$ *and* $\sigma_{jkl} = \sigma_{jk}\sigma_{kl}\sigma_{jl}$ *otherwise. These moduli are defined as follows:*

$$\beta_i^{jkl} = \frac{1}{\sigma_{jkl}}\ \frac{\partial^3 f_{1,i}}{\partial x_{1,j}\partial x_{1,k}\partial x_{1,l}}(0,0,0) \tag{3.9}$$
for $1 \le i,\ j,k,l \le n_1,$ *and* $\lambda_i = \lambda_j\lambda_k\lambda_l,$

$$\gamma_i^{jkl} = \frac{1}{\sigma_{jk}}\ \frac{\partial^3 f_{1,i}}{\partial x_{1,j}\partial x_{1,k}\partial x_{2,l}}(0,0,0) \tag{3.10}$$
for $1 \le i \le n_1,\ 1 \le j \le k \le n_1,\ 1 \le l \le n_2+1,$
and $\lambda_i = \lambda_j\lambda_k = 0,$

$$\gamma_i^{jk1} = \frac{1}{\sigma_{jk}} \sum_{r=0}^{n_2-k+1} \left(\frac{\lambda_i}{\lambda_j\lambda_k}\right)^r \frac{\partial^3 f_{1,i}}{\partial x_{1,j}\partial x_{1,k}\partial x_{2,r+1}}(0,0,0) \tag{3.11}$$
for $1 \le i \le n_1,\ 1 \le j \le k \le n_1,$ *and* $\lambda_i\lambda_j\lambda_k \neq 0,$

$$\delta_i^{jkl} = \frac{1}{\sigma_{kl}}\ \frac{\partial^3 f_{1,i}}{\partial x_{1,j}\partial x_{2,k}\partial x_{2,l}}(0,0,0) \tag{3.12}$$
for $1 \le i,\ j \le n_1,\ 1 \le k \le l \le n_2+1,$ *and* $\lambda_i = \lambda_j = 0,$

$$\delta_i^{j1l} = \frac{1}{\sigma_{1l}} \sum_{r=0}^{n_2-l+1} \left(\frac{\lambda_i}{\lambda_j}\right)^l \frac{\partial^3 f_{1,i}}{\partial x_{1,j}\partial x_{2,1+r}\partial x_{2,l+r}}(0,0,0) \tag{3.13}$$
for $1 \le i,\ j \le n_1,\ 1 \le k \le n_2+1,$ *and* $\lambda_i\lambda_j\lambda_k \neq 0,$

$$\epsilon_i^{1kl} = \frac{1}{\sigma_{1kl}} \sum_{r=0}^{n_2-l+1} \lambda_i^r \frac{\partial^3 f_{1,i}}{\partial x_{2,1+r}\partial x_{2,k+r}\partial x_{2,l+r}}(0,0,0) \tag{3.14}$$

$$\textit{for } 1 \le i \le n_1,\ 1 \le k \le l,\ i+2 \le l \le n_2+1,\ \textit{and } \lambda_i \ne 0,$$

$$\zeta_i^{j1l} = \frac{1}{\sigma_{1l}} \sum_{r=0}^{n_2-l+1} \lambda_j^{-r} \frac{\partial^3 f_{2,i+r}}{\partial x_{1,j} \partial x_{2,1+r} \partial x_{2,l+r}}(0,0,0) \tag{3.15}$$

$$\textit{for } 1 \le i \le n_2-1,\ i+2 \le l \le n_2+1,\ \textit{and } \lambda_j \ne 0,$$

$$\eta_i^{1kl} = \frac{1}{\sigma_{1kl}} \sum_{r=0}^{n_2-l+1} \frac{\partial^3 f_{2,i+r}}{\partial x_{2,1+r} \partial x_{2,k+r} \partial x_{2,l+r}}(0,0,0) \tag{3.16}$$

$$\textit{for } 1 \le i \le n_2-1,\ 1 \le k \le l,\ \textit{and } i+2 \le l \le n_2+1.$$

Remarks. Once again, if some of the eigenvalues of A_1 are complex, then a linear complex change of coordinates is required to bring it to Jordan form. In this case, some of the coordinates of z_1 are complex conjugate pairs, and some of the coefficients in the normal form are complex. These complex coefficients occur in conjugate pairs so that the real dimension of the coefficient space of the normal form is unchanged.

In the normal form of Poincaré (3.3), the eigenvalues satisfying $\lambda_i = \lambda_j \lambda_k \lambda_l$ are said to be in cubic resonance.

We defer the proof to a later section as it is quite lengthy.

4. Control bifurcations. In the above theorems, there are many more details than are necessary to understand the types of bifurcations that are possible. Recall that, in the bifurcation theory of a parametrized system of difference equations, the interesting part of the dynamics is that restricted to the center manifold. This leads to a great reduction in the dimension of space that must be explored. A similar fact holds true when studying control bifurcations. In most applications, one will ultimately use state feedback in an attempt to stabilize the system so the coordinates that are linearly stabilizable can be ignored to a large extent. If there are modes which are neutrally stable and are not linearly stabilizable, then the particular choice of feedback will influence the shape of the center manifold of the closed loop system and the dynamics thereon. It might be possible to achieve asymptotically stable center manifold dynamics by the proper choice of feedback although it will not be exponentially stable. We now discuss some important bifurcations of control systems.

4.1. Fold control bifurcation. Just as with classical bifurcations of discrete time dynamical systems, the simplest control bifurcation is the fold. The uncontrollable part is one dimensional and unstable with $A_1 > 1$. Because the linearly controllable part of the quadratic normal form (2.4) is in Brunovsky form, the equilibria z_e, v_e are conveniently parametrized by $\mu = v_e$. The equilibria $z_e(\mu), v_e(\mu)$ are given by

$$z_{e1} = \mu^2(1-A_1)^{-1}\tilde{\delta} + O(\mu)^3, \tag{4.1}$$

$$z_{e2,i} = \mu + O(\mu)^2, \qquad i = 1, \dots, n_2, \tag{4.2}$$

$$v_e = \mu, \tag{4.3}$$

where

$$\tilde{\delta} = \sum_{k=1}^{n_2+1} \delta_1^{1k}.$$

The local linearization around z_e, v_e is

$$
\begin{aligned}
\begin{bmatrix} \tilde{z}_1^+ \\ \tilde{z}_2^+ \end{bmatrix} &= \left(\begin{bmatrix} A_1 + \mu\gamma_1^{11} & \mu\Delta \\ 0 & A_2 \end{bmatrix} + O(\mu)^2 \right) \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix} \\
&\quad + \left(\begin{bmatrix} \mu B_1 \\ B_2 \end{bmatrix} + O(\mu)^2 \right) \tilde{v},
\end{aligned} \tag{4.4}
$$

where $\tilde{z} = z - z_e(\mu)$, $\tilde{v} = v - v_e(\mu)$, and

$$
\begin{aligned}
\Delta &= \begin{bmatrix} \tilde{\delta} + \delta_1^{11} & \delta_1^{12} & \dots & \delta_1^{1n_2} \end{bmatrix}, \\
B_1 &= \delta_1^{1,n_2+1}.
\end{aligned}
$$

If the transversality condition

$$
\tilde{\delta} + \delta_1^{11} + A_1\delta_1^{12} + \dots + A_1^{n_2}\delta_1^{1,n_2+1} \neq 0 \tag{4.5}
$$

is satisfied, then the system is linearly controllable and hence stabilizable about any equilibrium except $\mu = 0$. Consider a parametrized family of feedbacks

$$
\begin{aligned}
v &= \kappa(z,\mu), \\
\tilde{v} &= K_1(\mu)\tilde{z}_1 + K_2(\mu)\tilde{z}_2.
\end{aligned} \tag{4.6}
$$

Ideally one would like to find a smooth family of feedbacks that makes the family of equilibria asymptotically stable, i.e., for each small μ, the closed loop system

$$
z^+ = \tilde{f}(z, \kappa(z,\mu))
$$

is asymptotically stable to $z_e(\mu)$. Notice that the lowest degree terms of more general smooth feedbacks will be like (4.6). We restrict our attention to smooth feedbacks for practical and mathematical reasons. Smooth feedbacks are easy to implement, and they allow an analysis of the closed loop system based on the low degree terms.

Clearly the z_2 subsystem is stabilizable for all μ by the proper choice of K_2, and this feedback gain can be chosen independent of μ. The question is: Can we find $K_1(\mu)$ which stabilizes the z_1 coordinate?

Since the linear approximations are stabilizable for $\mu \neq 0$, it is certainly possible to find a stabilizing feedback at each such μ. The linear approximation at $\mu = 0$ has an uncontrollable, unstable mode, so it is not possible to stabilize it. But is it possible to stabilize the approximations for $\mu \neq 0$ with a feedback that is bounded through $\mu = 0$? The answer is no for systems with a fold control bifurcation. For any bounded feedback, the closed loop system will be unstable in some neighborhood of $\mu = 0$.

To see this, note that the closed loop linear approximation

$$
\begin{bmatrix} \tilde{z}_1^+ \\ \tilde{z}_2^+ \end{bmatrix} = \left(\begin{bmatrix} A_1 + \mu\left(\gamma_1^{11} + B_1K_1\right) & \mu\left(\Delta + B_1K_2\right) \\ B_2K_1 & A_2 + B_2K_2 \end{bmatrix} + O(\mu)^2 \right) \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix} \tag{4.7}
$$

is clearly unstable at $\mu = 0$ since it has an eigenvalue $A_1 > 1$. Furthermore, if the feedback $v = K_2(\mu)z_2$ stabilizes the z_2 subsystem, then A_1 is a simple root of the characteristic polynomial of the closed loop system when $\mu = 0$. Hence there is a simple root near A_1 of the characteristic polynomial for all small $|\mu|$.

By using larger and larger gain, it is possible to stabilize the system closer and closer to $\mu = 0$. But if the feedback gain is continuous, at best it will stabilize the closed loop system for only some small but not too small $\mu > 0$ or only some small but not too small $\mu < 0$. The controllability of z_1 reverses direction (folds) at $\mu = 0$, so a continuous choice of feedback gain cannot stabilize on both sides of $\mu = 0$. If a smooth family of feedbacks (4.6) does stabilize the system for some small $\mu > 0$, the parametrized closed loop system generically undergoes a classical fold bifurcation at some smaller $\mu > 0$. A classical fold bifurcation is also called a limit point bifurcation, a saddle-node bifurcation, or a turning point bifurcation.

We illustrate this with a simple example in normal form:

$$\begin{aligned} z_1^+ &= 2z_1 - z_2^2, \\ z_2^+ &= v. \end{aligned}$$

The equilibria are $z_{e,1} = \mu^2,\ z_{e,2} = \mu,\ v_e = \mu$. Under the feedback $v = K_1(\mu)\tilde{z}_1 + K_2(\mu)\tilde{z}_2$, the closed loop linear approximation is

$$\begin{bmatrix} \tilde{z}_1^+ \\ \tilde{z}_2^+ \end{bmatrix} = \begin{bmatrix} 2 & -2\mu \\ K_1(\mu) & K_2(\mu) \end{bmatrix} \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix},$$

where $\tilde{z} = z - z_e(\mu),\ \tilde{v} = v - v_e(\mu)$. If $K(\mu)$ is bounded, then as $\mu \to 0$ one eigenvalue converges to 2, so the system is unstable for small $|\mu|$. If we choose $K_1 = 15/2$ and $K_2 = -1/2$, then the closed loop linear approximation is stable for $|\mu| > 0.1$ and unstable for $|\mu| < 0.1$. It undergoes a fold bifurcation at $\mu = 0.1$.

To see this, consider the closed loop nonlinear system under this feedback in coordinates centered at the bifurcation $\bar{z}_1 = z_1 - 0.01,\ \bar{z}_2 = z_2 - 0.1,\ \bar{\mu} = \mu - 0.1$,

$$\begin{aligned} \bar{z}_1^+ &= 2\bar{z}_1 - 0.2\bar{z}_2 - \bar{z}_2^2, \\ \bar{z}_2^+ &= 7.5\bar{z}_1 - 0.5\bar{z}_2 - 7.5\bar{\mu}^2. \end{aligned}$$

It is convenient to reparametrize by $\nu = 7.5\bar{\mu}^2 \geq 0$. The center manifold is given by

$$\bar{z}_2 = -2\nu + 5\bar{z}_1 + 440\nu^2 - 600\nu\bar{z}_1 + 250\bar{z}_1^2 + O(\bar{z}_1, \nu)^3,$$

and the center manifold dynamics is

$$\bar{z}_1^+ = 0.4\nu + \bar{z}_1 - 92\nu^2 + 140\nu\bar{z}_1 - 75\bar{z}_1^2 + O(\bar{z}_1, \nu)^3$$

or, in the variables $\hat{z}_1 = \sqrt{75}(\bar{z}_1 - 0.9333\nu),\ \hat{\nu} = 0.4\nu - 26.667\nu^2$,

$$\hat{z}_1^+ = \hat{\nu} + \hat{z}_1 - \hat{z}_1^2 + O(\hat{z}_1, \hat{\nu})^3,$$

the familiar form of a discrete time fold bifurcation [9].

4.2. Transcritical control bifurcation. A degenerate form of the above bifurcation occurs when the uncontrollable part is one dimensional and neutrally stable, $A_1 = 1$. The equilibria z_e, v_e depend on roots z_1, μ of the quadratic form

$$0 = \beta_1^{11} z_1^2 + \gamma_1^{11} z_1\mu + \tilde{\delta}\mu^2.$$

If this form is positive or negative definite, then there is only an isolated equilibrium $z_1 = z_{2,1} = \cdots z_{2,n_2} = v = 0$.

If this form is indefinite but not degenerate, i.e., if it has a positive and a negative eigenvalue, then there are two curves of equilibria that cross at $z_1 = z_{2,1} = \cdots z_{2,n_2} =$

$v = 0$. Suppose that $z_1 = c_k\mu,\ k = 1, 2$, are the two lines of roots of the quadratic form; then the equilibria z_e, v_e are given by

$$\begin{aligned} z_1 &= c_k\mu + O(\mu)^2, && k = 1, 2,\\ z_{e2,i} &= \mu + O(\mu)^2, && i = 1, \dots, n_2,\\ v_e &= \mu. \end{aligned}$$

Suppose $z_e(\mu),\ v_e(\mu)$ is one curve of equilibria, and one chooses a parametrized family of smooth feedbacks (4.6), where $\tilde{z} = z - z_e(\mu),\ \tilde{v} = v - v_e(\mu)$. Notice that the closed loop system has a single curve of equilibria. The closed loop approximation (4.7) has $\lambda = A_1 = 1$ as an eigenvalue at $\mu = 0$. This eigenvalue is a function $\lambda = \lambda(\mu)$ and

$$\frac{d\lambda}{d\mu}(0) = \gamma_1^{11} + B_1 K_1(0),$$

so generically the eigenvalues pass through the unit circle at $\mu = 0$, and the closed loop system goes from stable to unstable through a classical fold bifurcation.

Suppose one chooses a parametrized family of smooth feedbacks that preserves both curves of equilibria,

$$\begin{aligned} v &= \kappa(z, \mu),\\ \tilde{v} &= K_1(\mu)(z_1 - c_1\mu + O(\mu)^2)(z_1 - c_2\mu + O(\mu)^2) + K_2(\mu)\tilde{z}_2. \end{aligned}$$

Then generically the closed loop system undergoes a classical transcritical bifurcation.

If the quadratic form is degenerate, then the locus or loci of equilibria may depend on cubic and higher terms.

4.3. Flip control bifurcation. The next simplest control bifurcation of a discrete time system is the flip. The uncontrollable part is again one dimensional and unstable, but now $A_1 \leq -1$. The equilibria z_e, v_e are conveniently parametrized by $\mu = v_e$. The equilibria $z_e(\mu), v_e(\mu)$ are given by (4.1), and the local linearizations are given by (4.4). If the transversality condition (4.5) is satisfied, then these are controllable when $\mu \neq 0$ but unstabilizable when $\mu = 0$.

One can find a parametrized family of smooth feedbacks (4.6) which will stabilize the z_2 modes for all μ and the z_1 mode for some range of $\mu \neq 0$. If $A_1 < -1$, then as $\mu \to 0$ it requires larger and larger gain to stabilize the latter. To see this, note that the closed loop linear approximation (4.7) is clearly unstable at $\mu = 0$ since $A_1 < -1$. Furthermore, if the feedback $v = K_2(\mu)z_2$ stabilizes the z_2 subsystem, then A_1 is a simple root of the characteristic polynomial of the closed loop system when $\mu = 0$. Hence there is a simple root near A_1 of the characteristic polynomial for all small $|\mu|$.

By using larger and larger gain, it is possible to stabilize the system closer and closer to $\mu = 0$. But if the feedback gain is bounded, at best it will stabilize the closed loop system only for some small but not too small $\mu > 0$ and/or only some small but not too small $\mu < 0$. If a smooth family of feedbacks (4.6) does stabilize the system for some small $\mu > 0$, the parametrized closed loop system generically undergoes a classical flip bifurcation at some smaller $\mu > 0$.

We illustrate this with an example:

$$\begin{aligned} z_1^+ &= -2z_1 + z_2^2,\\ z_2^+ &= v. \end{aligned}$$

The equilibria are $z_{e,1} = \frac{1}{3}\mu^2,\ z_{e,2} = \mu,\ v_e = \mu$. Under the feedback $v = K_1(\mu)\tilde{z}_1 + K_2(\mu)\tilde{z}_2$, the closed loop linear approximation is

$$\begin{bmatrix} \tilde{z}_1^+ \\ \tilde{z}_2^+ \end{bmatrix} = \begin{bmatrix} -2 & 2\mu \\ K_1(\mu) & K_2(\mu) \end{bmatrix} \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix},$$

where $\tilde{z} = z - z_e(\mu),\ \tilde{v} = v - v_e(\mu)$. If $K(\mu)$ is bounded, then as $\mu \to 0$ one eigenvalue converges to -2, so the system is unstable for small $|\mu|$. If we choose $K_1 = -15/2$ and $K_2 = 1/2$, then the closed loop linear approximation is stable for $|\mu| > 0.1$ and unstable for $|\mu| < 0.1$. It undergoes a classical flip bifurcation at $\mu = 0.1$.

To see this, consider the closed loop nonlinear system under this feedback in coordinates centered at the the bifurcation $\bar{z}_1 = z_1 - 1/300,\ \bar{z}_2 = z_2 - 1/10,\ \bar{\mu} = \mu - 1/10$,

$$\begin{aligned} \bar{z}_1^+ &= -2\bar{z}_1 + 0.2\bar{z}_2 + \bar{z}_2^2, \\ \bar{z}_2^+ &= -7.5\bar{z}_1 + 0.5\bar{z}_2 + \bar{\mu} + 2.5\bar{\mu}^2. \end{aligned}$$

It is convenient to reparametrize by $\nu = \bar{\mu} + 2.5\bar{\mu}^2$. The center manifold is given by

$$\bar{z}_2 = 0.67\nu + 5\bar{z}_1 - 10.37\nu^2 + 111.11\nu\bar{z}_1 - 83.33\bar{z}_1^2 + O(\bar{z}_1, \nu)^3,$$

and the center manifold dynamics is

$$\begin{aligned} \bar{z}_1^+ = {} & 0.13\nu - \bar{z}_1 - 1.63\nu^2 + 2.89\nu\bar{z}_1 + 8.33\bar{z}_1^2 \\ & -123.12\nu^3 + 1763.0\nu^2\bar{z}_1 - 111.11\nu\bar{z}_1^2 - 1944.4\bar{z}_1^3 + O(\bar{z}_1, \nu)^4, \end{aligned}$$

or, in the variables

$$\begin{aligned} \hat{\nu} &= 3\nu + 172.50\nu^2, \\ \hat{z}_1 &= -2.87\nu + 43.30\bar{z}_1 - 8.02\nu^2 + 24.06\nu\bar{z}_1 - 180.42\bar{z}_1^2 \\ &\quad +48.11\nu^3 - 360.84\nu^2\bar{z}_1 + 2706.3\nu\bar{z}_1, \end{aligned}$$

the parametrized closed loop system is

$$\begin{aligned} \hat{\nu}^+ &= \hat{\nu}, \\ \hat{z}_1^+ &= -\hat{z}_1 + \hat{\nu}\hat{z}_1 - \hat{z}_1^3 + O(\hat{\nu}, \hat{z}_1)^4, \end{aligned}$$

a familiar form of a discrete time flip bifurcation [9].

4.4. Neimark–Sacker control bifurcation. The discrete time analogue of a classical Hopf bifurcation is called a Neimark–Sacker bifurcation. We present the control analogue of this bifurcation. The uncontrollable modes are a nonzero complex conjugate pair,

$$A_1 = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix},$$

where $\lambda = \rho e^{i\theta},\ \bar{\lambda} = \rho e^{-i\theta},\ \theta \neq 0, \pi/2, \pi, 3\pi/2$. The equilibria $z_e(\mu), v_e(\mu)$ are given by

$$\begin{bmatrix} z_{e1,1} \\ z_{e1,2} \end{bmatrix} = \mu^2 (I - A_1)^{-1} \begin{bmatrix} \tilde{\delta}_1 \\ \tilde{\delta}_2 \end{bmatrix} + O(\mu)^3, \tag{4.8}$$

$$z_{e2,i} = \mu + O(\mu)^2, \qquad i = 1, \dots, n_2, \tag{4.9}$$

$$v_e = \mu, \tag{4.10}$$

where

$$\tilde{\delta}_i = \sum_{k=1}^{n_2+1} \delta_i^{1k}.$$

The local linearization around z_e, v_e is

$$\begin{bmatrix} \tilde{z}_1^+ \\ \tilde{z}_2^+ \end{bmatrix} = \left(\begin{bmatrix} A_1 + \mu\Gamma & \mu\Delta \\ 0 & A_2 \end{bmatrix} + O(\mu)^2 \right) \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix} + \left(\begin{bmatrix} \mu B_1 \\ B_2 \end{bmatrix} + O(\mu)^2 \right) \tilde{v}, \tag{4.11}$$

where $\tilde{z} = z - z_e(\mu)$, $\tilde{v} = v - v_e(\mu)$, and

$$\Gamma = \begin{bmatrix} \gamma_1^{11} & \gamma_1^{21} \\ \gamma_2^{11} & \gamma_2^{21} \end{bmatrix},$$
$$\Delta = \begin{bmatrix} \tilde{\delta}_1 + \delta_1^{11} & \delta_1^{12} & \dots & \delta_1^{1n_2} \\ \tilde{\delta}_2 + \delta_2^{11} & \delta_2^{12} & \dots & \delta_2^{1n_2} \end{bmatrix},$$
$$B_1 = \begin{bmatrix} \delta_1^{1,n_2+1} \\ \delta_2^{1,n_2+1} \end{bmatrix}.$$

If the transversality condition

$$\begin{bmatrix} \tilde{\delta}_1 + \delta_1^{11} \\ \tilde{\delta}_2 + \delta_2^{11} \end{bmatrix} + A_1 \begin{bmatrix} \delta_1^{12} \\ \delta_2^{12} \end{bmatrix} + \cdots + A_1^{n_2} \begin{bmatrix} \delta_1^{1,n_2+1} \\ \delta_2^{1,n_2+1} \end{bmatrix} \neq 0 \tag{4.12}$$

is satisfied, then the system is linearly controllable and hence stabilizable about any equilibrium except $\mu = 0$. Consider a parametrized family of feedbacks (4.6).

If $\rho < 1$, then the system is stabilizable about any equilibrium, but if $\rho \geq 1$, then the system is not stabilizable when $\mu = 0$. The case $\rho \geq 1$ is called a Neimark–Sacker control bifurcation. We distinguish two subcases, $\rho > 1$ and $\rho = 1$.

If $\rho > 1$, then it requires larger and larger gain to stabilize the system closer and closer to $\mu = 0$. Since the feedback (4.6) is smooth, it will stabilize only for some small $\mu > 0$ or for some small $\mu < 0$ but not both. At $\mu = 0$, the poles of the closed loop system are $\lambda, \bar{\lambda}$ and the poles of $A_2 + B_2K_2(0)$. The latter can be made stable, but the former are unstable. Since the feedback is bounded, as $\mu \to 0$ the poles converge to these. The system is controllable for $\mu \neq 0$, so the poles can be placed arbitrarily by feedback. The poles associated primarily with the z_2 subsystem can be kept stable, but the two poles associated primarily with the z_1 subsystem will leave the unit disk at some small value(s) of μ. Depending on the choice of feedback, they will leave one at a time as real poles, leave together through ± 1, or leave together as a nonzero complex conjugate pair. If they leave separately as real poles, then generically the closed loop system undergoes a fold or flip bifurcation as the first pole leaves through ± 1. If they leave together as a complex conjugate pair that is neither real nor imaginary, then generically the system undergoes a Neimark–Sacker bifurcation. If they leave together through ± 1, the situation can be quite complicated and will not be discussed here.

If $\rho = 1$ and the feedback (4.6) is continuous, then generically the system undergoes a Neimark–Sacker bifurcation at $\mu = 0$ provided that $e^{ik\theta} \neq 1$ for $k = 1, 2, 3, 4$. We illustrate this with an example:

$$z_{1,1}^+ = e^{i\pi/4} z_{1,1} + z_2^2,$$

$$
\begin{aligned}
z_{1,2}^{+} &= e^{-i\pi/4} z_{1,2} + z_2^2, \\
x_2 &= u.
\end{aligned}
$$

The equilibria are

$$
\begin{aligned}
z_{e1,1} &= c\mu^2, \\
z_{e1,2} &= \bar{c}\mu^2, \\
x_{e2} &= \mu, \\
u_e &= \mu,
\end{aligned}
$$

where $c = \left(1 - e^{i\pi/4}\right)^{-1}$. The linear approximations are

$$
\begin{aligned}
\tilde{z}_{1,1}^{+} &= e^{i\pi/4} \tilde{z}_{1,1} + 2\mu \tilde{z}_2, \\
\tilde{z}_{1,2}^{+} &= e^{-i\pi/4} \tilde{z}_{1,2} + 2\mu \tilde{z}_2, \\
\tilde{x}_2^{+} &= \tilde{u},
\end{aligned}
$$

where $\tilde{z}_{1,1} = z_{1,1} - c\mu^2$, $\tilde{z}_{1,2} = z_{1,2} - \bar{c}\mu^2$, $\tilde{x}_z = z_2 - \mu$, $\tilde{u} = u - \mu$. The linear approximations are controllable except at $\mu = 0$.

The feedback

$$
u = \mu + 0.5(z_{1,1} - c\mu^2) + 0.5(z_{1,2} - \bar{c}\mu^2) + 0.5(x_2 - \mu)
$$

places the poles of the closed loop system inside the open unit disk at $0.7953 \pm 0.5743i$, 0.3957 at $\mu = 0.1$. A pair of poles leaves the unit disk at $e^{\pm i\pi/4}$ when $\mu = 0$.

The closed loop dynamics undergoes a Neimark–Sacker classical bifurcation at $\mu = 0$. The discrete time analogue of the first Lyapunov coefficient is found in Kuznetsov [9, p. 186, formula (5.74)]. For this example, its value is 46.8, which indicates that the system undergoes a subcritical Neimark–Sacker bifurcation at $\mu = 0$. For small $\mu > 0$, the equilibrium is exponentially stable, but there is an unstable invariant closed curve nearby. For small $\mu < 0$, the equilibrium is unstable as is the bifurcation equilibrium $\mu = 0$.

5. Proof of the quadratic normal form. We can expand the change of coordinates and feedback as follows:

$$
\begin{aligned}
\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &- \begin{bmatrix} \phi_1^{[2;0]}(x_1; x_2) \\ \phi_2^{[2;0]}(x_1; x_2) \end{bmatrix} \\
&- \begin{bmatrix} \phi_1^{[1;1]}(x_1; x_2) \\ \phi_2^{[1;1]}(x_1; x_2) \end{bmatrix} - \begin{bmatrix} \phi_1^{[0;2]}(x_1; x_2) \\ \phi_2^{[0;2]}(x_1; x_2) \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
v = u &- \alpha^{[2;0]}(x_1; x_2, u) \\
&- \alpha^{[1;1]}(x_1; x_2, u) - \alpha^{[0;2]}(x_1; x_2, u).
\end{aligned}
$$

These do not change the linear part of the dynamics. The quadratic part of the dynamics is changed to

$$
\begin{aligned}
\tilde{f}_i^{[d_1;d_2]}(z_1; z_2, v) = f_i^{[d_1;d_2]}&(z_1; z_2, v) \\
&- \phi_i^{[d_1;d_2]}(A_1 z_1; A_2 z_2) \\
&+ A_i \phi_i^{[d_1;d_2]}(z_1; z_2) \\
&- B_i \alpha^{[d_1;d_2]}(z_1; z_2, v),
\end{aligned}
$$

where $B_1 = 0$, so the proof splits into six cases, $i = 1, 2;\ d_1 = 0, 1, 2;\ d_2 = 2 - d_1$.

The normal form of $\tilde{f}_2^{[0;2]}(z_1; z_2, v)$. We start by showing that $\tilde{f}_2^{[0;2]}(z_1; z_2, v)$ can be brought into the above form. There are two basic operations, *pull up* and *push down*, which are used to achieve this. Consider a part of the dynamics

$$\begin{aligned} z_{2,i-1}^{+} &= z_{2,i} + \cdots, \\ z_{2,i}^{+} &= z_{2,i+1} + c z_{2,j} z_{2,k} + \cdots, \\ z_{2,i+1}^{+} &= z_{2,i+2} + \cdots, \end{aligned}$$

where $1 < i \leq n_2$, $1 \leq j \leq k \leq n_2 + 1$; recall that $z_{2,n+1} = v$.

If $1 < j$, we can *pull up* the quadratic term by defining

$$\bar{z}_{2,i} = z_{2,i} - c z_{2,j-1} z_{2,k-1},$$

and then the dynamics becomes

$$\begin{aligned} z_{2,i-1}^{+} &= \bar{z}_{2,i} + c z_{2,j-1} z_{2,k-1} + \cdots, \\ \bar{z}_{2,i}^{+} &= z_{2,i+1} + \cdots, \\ z_{2,i+1}^{+} &= z_{2,i+2} + \cdots, \end{aligned}$$

and all the other quadratic terms remain the same. Notice that if $i = 1$, we can still pull up, and the term disappears. By pulling up all the quadratic terms until $j = 1$, we obtain

$$z_{2,i}^{+} = z_{2,i+1} + c z_{2,1} z_{2,k} + \cdots. \tag{5.1}$$

The other operation on the dynamics is *push down*. If $k \leq n_2$, define

$$\bar{z}_{2,i+1} = z_{2,i+1} + c z_{2,j} z_{2,k}$$

yielding

$$\begin{aligned} z_{2,i-1}^{+} &= z_{2,i} + \cdots, \\ z_{2,i}^{+} &= \bar{z}_{2,i+1} + \cdots, \\ \bar{z}_{2,i+1}^{+} &= z_{2,i+2} + c z_{2,j+1} z_{2,k+1} + \cdots, \end{aligned}$$

and all the other quadratic terms remain unchanged. Notice that if $i + 1 = n_2$, then we can absorb the quadratic term into the control using feedback. From (5.1) we push down every term where $k \leq i + 1$. These terms can be pushed all the way down and absorbed in the control. The result is (2.8).

Next we show that the number ϵ_i^{1k} (2.13) is an invariant. Clearly ϵ_i^{1k} is potentially changed only by $\phi_2^{[0;2]}(x_1; x_2)$ and $\alpha_2^{[0;2]}(x_1; x_2, u)$. Therefore, we need only consider coordinate changes of the form

$$\bar{x}_{2,\rho} = x_{2,\rho} + c x_{2,\sigma} x_{2,\tau},$$

where $1 \leq \rho \leq n_2$, $1 \leq \sigma \leq \tau \leq n_2$, and feedbacks of the form

$$\bar{u} = u + c x_{2,\sigma} x_{2,\tau},$$

where $1 \leq \sigma \leq \tau \leq n_2 + 1$ with $x_{2,n+1} = u$. More general coordinate changes and feedbacks are just compositions of these. The coordinate change affects only a piece of the dynamics (2.3),

$$x^+_{2,\rho-1} = x_{2,\rho} + f^{[2]}_{2,\rho-1}(x_1, x_2, u) + O(x_1, x_2, u)^3,$$
$$x^+_{2,\rho} = x_{2,\rho+1} + f^{[2]}_{2,\rho}(x_1, x_2, u) + O(x_1, x_2, u)^3$$

is transformed to

$$x^+_{2,\rho-1} = \bar{x}_{2,\rho} + f^{[2]}_{2,\rho-1}(x_1, x_2, u) - cx_{2,\sigma}x_{2,\tau} + O(x_1, x_2, u)^3,$$
$$\bar{x}^+_{2,\rho} = x_{2,\rho+1} + f^{[2]}_{2,\rho}(x_1, x_2, u) + cx_{2,\sigma+1}x_{2,\tau+1} + O(x_1, x_2, u)^3,$$

and ϵ^{1k}_i is unchanged. The feedback affects only

$$x^+_{2,n_2} = u + f^{[2]}_{2,n_2}(x_1, x_2, u) + O(x_1, x_2, u)^3,$$

transforming it into

$$x^+_{2,n_2} = \bar{u} + f^{[2]}_{2,n_2}(x_1, x_2, u) - cx_{2,\sigma}x_{2,\tau} + O(x_1, x_2, u)^3,$$

and again ϵ^{1k}_i is unchanged because $i + l \leq n_2 - 1$.

The normal form of $\tilde{f}^{[1;1]}_2(z_1; z_2, v)$. The two basic operations, pull up and push down, are slightly different. Consider a part of the dynamics

$$z^+_{2,i-1} = z_{2,i} + \cdots,$$
$$z^+_{2,i} = z_{2,i+1} + cz_{1,j}z_{2,k} + \cdots,$$
$$z^+_{2,i+1} = z_{2,i+2} + \cdots,$$

where $1 < i \leq n_2$, $1 \leq j \leq n_1$, $1 \leq k \leq n_2 + 1$.

If $\lambda_j \neq 0$ and $1 < k$, we can pull up the quadratic term by defining

$$\bar{z}_{2,i} = z_{2,i} - \frac{c}{\lambda_j} z_{1,j}z_{2,k-1};$$

then the dynamics becomes

$$z^+_{2,i-1} = \bar{z}_{2,i} + \frac{c}{\lambda_j} z_{1,j}z_{2,k-1} + \cdots,$$
$$\bar{z}^+_{2,i} = z_{2,i+1} + \cdots,$$
$$z^+_{2,i+1} = z_{2,i+2} + \cdots,$$

and all the other quadratic terms remain the same. Again, if $i = 1$, we can still pull up, and the term disappears. So by pulling up all quadratic terms where $\lambda_j \neq 0$ until $k = 1$, we obtain

$$z^+_{2,i} = z_{2,i+1} + cz_{1,j}z_{2,1} + \cdots.$$

Repeated pushing down eliminates this term. Define

$$\bar{z}_{2,i+1} = z_{2,i+1} + cz_{1,j}z_{2,1},$$

yielding

$$\begin{aligned} z_{2,i-1}^+ &= z_{2,i} + \cdots, \\ z_{2,i}^+ &= \bar{z}_{2,i+1} + \cdots, \\ \bar{z}_{2,i+1}^+ &= z_{2,i+2} + c\lambda_j z_{1,j} z_{2,2} + \cdots, \end{aligned}$$

and all the other quadratic terms remain unchanged. If $\lambda_j = 0$, then the term drops out. If $\lambda_j \neq 0$, then we can continue to push down until $i+1 = n_2$ and the quadratic term can be absorbed into the control using feedback. The result is $\tilde{f}_2^{[1;1]}(z_1; z_2, v) = 0$.

The normal form of $\tilde{f}_2^{[2;0]}(z_1; z_2, v)$. Consider a part of the dynamics

$$\begin{aligned} z_{2,i}^+ &= z_{2,i+1} + c z_{1,j} z_{1,k} + \cdots, \\ z_{2,i+1}^+ &= z_{2,i+2} + \cdots, \end{aligned}$$

where $1 \leq i \leq n_2$, $1 \leq j \leq k \leq n_1$.

Pushing down one or more times eliminates this term. Define

$$\bar{z}_{2,i+1} = z_{2,i+1} + c z_{1,j} z_{1,k},$$

yielding

$$\begin{aligned} z_{2,i-1}^+ &= z_{2,i} + \cdots, \\ z_{2,i}^+ &= \bar{z}_{2,i+1} + \cdots, \\ \bar{z}_{2,i+1}^+ &= z_{2,i+2} + c\lambda_j \lambda_k z_{1,j} z_{1,k} + \cdots, \end{aligned}$$

and all the other quadratic terms remain unchanged. If $\lambda_j \lambda_k = 0$, then the term drops out. Otherwise, the quadratic term can be pushed down repeatedly until it is absorbed in the control. The result is $\tilde{f}_2^{[2;0]}(z_1; z_2, v) = 0$.

The normal form of $\tilde{f}_1^{[2;0]}(z_1; z_2, v)$. This is just the quadratic normal form of Poincaré as described in the introduction, and β_i^{jk} are the invariants. See [1], [5], [9], or [13]. Consider a part of the dynamics

$$z_{1,i}^+ = \lambda_i z_{1,i} + c z_{1,j} z_{1,k} + \cdots,$$

where $1 \leq i \leq n_1$, $1 \leq j \leq k \leq n_1$.

If $\lambda_i \neq \lambda_j \lambda_k$, then define

$$\bar{z}_{1,i} = z_{1,i} - \frac{c}{(\lambda_j \lambda_k - \lambda_i)} z_{1,j} z_{1,k}$$

so that

$$\bar{z}_{1,i}^+ = \lambda_i \bar{z}_{1,i} + \cdots.$$

Next we show that the numbers β_i^{jk} (2.9) are invariants. Clearly β_i^{jk} is potentially changed only by $\phi_1^{[2;0]}(x_1; x_2)$. Therefore, we need only consider coordinate changes of the form

$$\bar{x}_{1,\rho} = x_{1,\rho} + c x_{1,\sigma} x_{1,\tau},$$

where $1 \le \rho \le n_1$, $1 \le \sigma \le \tau \le n_1$, because more general ones are just compositions of these. This coordinate change affects only a piece of the dynamics (2.3), and

$$x_{1,\rho}^{+} = \lambda_\rho x_{1,\rho} + f_{1,\rho}^{[2]}(x_1, x_2, u) + O(x_1, x_2, u)^3$$

is transformed to

$$\bar{x}_{1,\rho}^{+} = \lambda_\rho \bar{x}_{1,\rho} + f_{1,\rho}^{[2]}(x_1, x_2, u) + c(\lambda_\sigma \lambda_\tau - \lambda_\rho) x_{1,\sigma} x_{1,\tau} + O(x_1, x_2, u)^3.$$

Clearly, if $\lambda_\rho = \lambda_\sigma \lambda_\tau$, then β_i^{jk} (2.9) is unchanged.

The normal form of $\tilde{f}_1^{[1;1]}(z_1; z_2, v)$. Consider a part of the dynamics

$$z_{1,i}^{+} = \lambda_i z_{1,i} + c z_{1,j} z_{2,k} + \cdots,$$

where $1 \le i \le n_1$, $1 \le j \le n_1$, $1 \le k \le n_2 + 1$.

If $\lambda_j \ne 0$ and $k > 1$, then we can pull up by defining

$$\bar{z}_{1,i} = z_{1,i} - \frac{c}{\lambda_j} z_{1,j} z_{2,k-1}$$

so that

$$\bar{z}_{1,i}^{+} = \lambda_i \bar{z}_{1,i} + \frac{c\lambda_i}{\lambda_j} z_{1,j} z_{2,k-1} + \cdots.$$

If $\lambda_i = 0$, then the term disappears; otherwise, we can continue to pull up until $k = 1$.

If $\lambda_i \ne 0$, then we can push down by defining

$$\bar{z}_{1,i} = z_{1,i} + \frac{c}{\lambda_i} z_{1,j} z_{2,k};$$

then

$$\bar{z}_{1,i}^{+} = \lambda_i \bar{z}_{1,i} + \frac{c\lambda_j}{\lambda_i} z_{1,j} z_{2,k}^{+} + \cdots.$$

If $\lambda_j = 0$, then the term disappears.

If $\lambda_i = \lambda_j = 0$, then we cannot pull up or push down. The result is (2.7).

Next we show that the numbers γ_i^{jk} (2.10)–(2.11) are invariants. Clearly γ_i^{jk} is potentially changed only by $\phi_1^{[1;1]}(x_1; x_2)$. Therefore, we need only consider coordinate changes of the form

$$\bar{x}_{1,\rho} = x_{1,\rho} + c x_{1,\sigma} x_{2,\tau},$$

where $1 \le \rho, \sigma \le n_1$, $1 \le \tau \le n_2$, because more general ones are just compositions of these. This coordinate change affects only a piece of the dynamics (2.3), and

$$x_{1,\rho}^{+} = \lambda_\rho x_{1,\rho} + f_{1,\rho}^{[2]}(x_1, x_2, u) + O(x_1, x_2, u)^3$$

is transformed to

$$\bar{x}_{1,\rho}^{+} = \lambda_\rho \bar{x}_{1,\rho} + f_{1,\rho}^{[2]}(x_1, x_2, u) + c\lambda_\sigma x_{1,\sigma} x_{2,\tau+1} - c\lambda_\rho x_{1,\sigma} x_{2,\tau} + O(x_1, x_2, u)^3.$$

Clearly, if $\lambda_\rho = \lambda_\sigma = 0$, then γ_i^{jk} (2.10) is unchanged. A simple calculation shows that if $\lambda_\rho \lambda_\sigma \ne 0$, then γ_i^{j1} (2.11) is unchanged.

The normal form of $\tilde{f}_1^{[0;2]}(z_1;z_2,v)$. Consider a part of the dynamics

$$z_{1,i}^+ = \lambda_i z_{1,i} + c z_{2,j} z_{2,k} + \cdots,$$

where $1 \le i \le n_1$, $1 \le j \le k \le n_2$.

If $j > 1$, then we can pull up by defining

$$\bar{z}_{1,i} = z_{1,i} - c z_{2,j-1} z_{2,k-1};$$

then

$$\bar{z}_{1,i}^+ = \lambda_i \bar{z}_{1,i} + c\lambda_i z_{2,j-1} z_{2,k-1} + \cdots.$$

If $\lambda_i = 0$, then the term disappears; otherwise, we can continue to pull up until $j = 1$. The result is (2.7).

Finally, we show that the numbers δ_i^{1k} (2.12) are invariants. Clearly, δ_i^{1k} is potentially changed only by $\phi_1^{[0;2]}(x_1;x_2)$. Therefore, we need only consider coordinate changes of the form

$$\bar{x}_{1,\rho} = x_{1,\rho} + c x_{2,\sigma} x_{2,\tau},$$

where $1 \le \rho \le n_1$, $1 \le \sigma \le \tau \le n_2$, because more general ones are just compositions of these. This change of coordinates affects only a piece of the dynamics (2.3), and

$$x_{1,\rho}^+ = \lambda_\rho x_{1,\rho} + f_{1,\rho}^{[2]}(x_1,x_2,u) + O(x_1,x_2,u)^3$$

is transformed to

$$\bar{x}_{1,\rho}^+ = \lambda_\rho \bar{x}_{1,\rho} + f_{1,\rho}^{[2]}(x_1,x_2,u) + c x_{2,\sigma+1} x_{2,\tau+1} - c\lambda_\rho x_{2,\sigma} x_{2,\tau} + O(x_1,x_2,u)^3.$$

Clearly, if $\lambda_\rho \ne 0$, then δ_i^{1k} (2.12) is unchanged.

6. Proof of the cubic normal form. Cubic changes of coordinates and cubic feedbacks do not change the linear and quadratic parts of the system. Their effect on the cubic part of the system splits into cases, this time eight cases, $i = 1, 2;\ d_1 = 0, 1, 2, 3;\ d_2 = 3 - d_1$.

The normal form of $\tilde{f}_2^{[0;3]}(z_1;z_2,v)$. We again use the two basic operations pull up and push down. Consider a part of the dynamics

$$\begin{aligned} z_{2,i-1}^+ &= z_{2,i} + f_{2,i-1}^{[2]}(z_1,z_2,v) + \cdots, \\ z_{2,i}^+ &= z_{2,i+1} + f_{2,i}^{[2]}(z_1,z_2,v) + c z_{2,j} z_{2,k} z_{2,l} + \cdots, \\ z_{2,i+1}^+ &= z_{2,i+2} + f_{2,i+1}^{[2]}(z_1,z_2,v) + \cdots, \end{aligned}$$

where $1 < i \le n_2$, $1 \le j \le k \le l \le n_2 + 1$; recall that $z_{2,n+1} = v$.

If $1 < j$, we can *pull up* the cubic term by defining

$$\bar{z}_{2,i} = z_{2,i} - c z_{2,j-1} z_{2,k-1} z_{2,l-1};$$

then the dynamics becomes

$$\begin{aligned} z_{2,i-1}^+ &= \bar{z}_{2,i} + f_{2,i-1}^{[2]}(z_1,z_2,v) + c z_{2,j-1} z_{2,k-1} z_{2,l-1} + \cdots, \\ \bar{z}_{2,i}^+ &= z_{2,i+1} + f_{2,i}^{[2]}(z_1,z_2,v) + \cdots, \\ z_{2,i+1}^+ &= z_{2,i+2} + f_{2,i+1}^{[2]}(z_1,z_2,v) + \cdots, \end{aligned}$$

and all the other cubic terms remain the same. Notice that if $i = 1$, we can still pull up, and the term disappears. By pulling up all cubic terms until $j = 1$, we obtain that

$$z_{2,i}^{+} = z_{2,i+1} + f_{2,i}^{[2]}(z_1, z_2, v) + cz_{2,1}z_{2,k}z_{2,l} + \cdots. \tag{6.1}$$

The other operation on the dynamics is *push down.* If $l \leq n_2$, define

$$\bar{z}_{2,i+1} = z_{2,i+1} + cz_{2,j}z_{2,k}z_{2,l},$$

yielding

$$\begin{aligned}
z_{2,i-1}^{+} &= z_{2,i} + f_{2,i-1}^{[2]}(z_1, z_2, v) + \cdots, \\
z_{2,i}^{+} &= \bar{z}_{2,i+1} + f_{2,i}^{[2]}(z_1, z_2, v) + \cdots, \\
\bar{z}_{2,i+1}^{+} &= z_{2,i+2} + f_{2,i+1}^{[2]}(z_1, z_2, v) + cz_{2,j+1}z_{2,k+1}z_{2,l+1} + \cdots,
\end{aligned}$$

and all the other cubic terms remain unchanged. Notice that if $i + 1 = n_2$, then we can absorb the cubic term into the control. From (6.1) we push down every term where $l \leq i + 1$. These terms can be pushed all the way down and absorbed in the control. The result is (3.8).

Next we show that the number η_i^{1kl} (3.16) is an invariant. Clearly η_i^{1kl} is potentially changed only by $\phi_2^{[0;3]}(x_1; x_2)$ and $\alpha_2^{[0;3]}(x_1; x_2, u)$. Therefore, we need only consider coordinate changes of the form

$$\bar{x}_{2,\rho} = x_{2,\rho} + cx_{2,\sigma}x_{2,\tau}x_{2,\upsilon},$$

where $1 \leq \rho \leq n_2$, $1 \leq \sigma \leq \tau \leq \upsilon \leq n_2$, and feedbacks of the form

$$\bar{u} = u + cx_{2,\sigma}x_{2,\tau}x_{2,\upsilon},$$

where $1 \leq \rho \leq n_2, 1 \leq \sigma \leq \tau \leq \upsilon \leq n_2 + 1$ with $x_{2,n+1} = u$ because more general ones are just compositions of these. The coordinate change affects only a piece of the dynamics (3.1),

$$\begin{aligned}
x_{2,\rho-1}^{+} &= x_{2,\rho} + f_{2,\rho-1}^{[2]}(x_1, x_2, u) + f_{2,\rho-1}^{[3]}(x_1, x_2, u), \\
x_{2,\rho}^{+} &= x_{2,\rho+1} + f_{2,\rho}^{[2]}(x_1, x_2, u) + f_{2,\rho}^{[3]}(x_1, x_2, u)
\end{aligned}$$

is transformed to

$$\begin{aligned}
x_{2,\rho-1}^{+} &= \bar{x}_{2,\rho} + f_{2,\rho-1}^{[2]}(x_1, x_2, u) + f_{2,\rho-1}^{[3]}(x_1, x_2, u) - cx_{2,\sigma}x_{2,\tau}x_{2,\upsilon}, \\
\bar{x}_{2,\rho}^{+} &= x_{2,\rho+1} + f_{2,\rho}^{[2]}(x_1, x_2, u) + f_{2,\rho}^{[3]}(x_1, x_2, u) + cx_{2,\sigma+1}x_{2,\tau+1}x_{2,\upsilon+1},
\end{aligned}$$

and η_i^{1kl} is unchanged. The feedback affects only

$$x_{2,n_2}^{+} = u + f_{2,n_2}^{[2]}(x_1, x_2, u) + f_{2,n_2}^{[3]}(x_1, x_2, u),$$

transforming it into

$$x_{2,n_2}^{+} = \bar{u} + f_{2,n_2}^{[2]}(x_1, x_2, u) + f_{2,n_2}^{[3]}(x_1, x_2, u) - cx_{2,\sigma}x_{2,\tau}x_{2,\upsilon},$$

and again η_i^{1kl} is unchanged because $i + r \leq n_2 - 1$.

The normal form of $\tilde{f}_2^{[1;2]}(z_1; z_2, v)$. The two basic operations, pull up and push down, are slightly different. Consider a part of the dynamics

$$\begin{aligned} z_{2,i-1}^+ &= z_{2,i} + f_{2,i-1}^{[2]}(z_1, z_2, v) + \cdots, \\ z_{2,i}^+ &= z_{2,i+1} + f_{2,i}^{[2]}(z_1, z_2, v) + cz_{1,j}z_{2,k}z_{2,l} + \cdots, \\ z_{2,i+1}^+ &= z_{2,i+2} + f_{2,i}^{[2]}(z_1, z_2, v) + \cdots, \end{aligned}$$

where $1 < i \leq n_2$, $1 \leq j \leq n_1$, $1 \leq k \leq l \leq n_2 + 1$.

If $\lambda_j \neq 0$ and $1 < k$, we can pull up the cubic term by defining

$$\bar{z}_{2,i} = z_{2,i} - \frac{c}{\lambda_j} z_{1,j} z_{2,k-1} z_{2,l-1};$$

then the dynamics becomes

$$\begin{aligned} z_{2,i-1}^+ &= \bar{z}_{2,i} + \frac{c}{\lambda_j} z_{1,j} z_{2,k-1} z_{2,l-1} + \cdots, \\ \bar{z}_{2,i}^+ &= z_{2,i+1} + \cdots, \\ z_{2,i+1}^+ &= z_{2,i+2} + \cdots, \end{aligned}$$

and all the other cubic terms remain the same. Again, if $i = 1$, we can still pull up, and the term disappears. So by pulling up all cubic terms where $\lambda_j \neq 0$ until $k = 1$, we obtain

$$z_{2,i}^+ = z_{2,i+1} + f_{2,i}^{[2]}(z_1, z_2, v) + cz_{1,j} z_{2,1} z_{2,l} + \cdots.$$

If $l \leq n_2$, we can also push down by defining

$$\bar{z}_{2,i+1} = z_{2,i+1} + cz_{1,j} z_{2,1} z_{2,l},$$

yielding

$$\begin{aligned} z_{2,i-1}^+ &= z_{2,i} + f_{2,i-1}^{[2]}(z_1, z_2, v) + \cdots, \\ z_{2,i}^+ &= \bar{z}_{2,i+1} + f_{2,i}^{[2]}(z_1, z_2, v) + \cdots, \\ \bar{z}_{2,i+1}^+ &= z_{2,i+2} + f_{2,i+1}^{[2]}(z_1, z_2, v) + c\lambda_j z_{1,j} z_{2,2} z_{2,l+1} + \cdots, \end{aligned}$$

and all the other cubic terms remain unchanged. If $\lambda_j = 0$, then the term drops out. If $\lambda_j \neq 0$ and $l \leq i + 1$, then the cubic term can be pushed down repeatedly and absorbed in the control. The result is (3.7).

Next we show that the number ζ_i^{j1l} (3.15) is an invariant. Clearly ζ_i^{j1l} is potentially changed only by $\phi_2^{[1;2]}(x_1; x_2)$ and $\alpha_2^{[1;2]}(x_1; x_2, u)$. Therefore, we need only consider coordinate changes of the form

$$\bar{x}_{2,\rho} = x_{2,\rho} + cx_{1,\sigma} x_{2,\tau} x_{2,\upsilon},$$

where $1 \leq \rho \leq n_2$, $1 \leq \sigma \leq n_1$, $1 \leq \tau \leq \upsilon \leq n_2$ and feedbacks of the form

$$\bar{u} = u + cx_{1,\sigma} x_{2,\tau} x_{2,\upsilon},$$

where $1 \le \sigma \le n_1$, $1 \le \tau \le \upsilon \le n_2+1$ with $x_{2,n+1} = u$ because more general ones are just compositions of these. The coordinate change affects only a piece of the dynamics (3.1),

$$\begin{aligned} x^+_{2,\rho-1} &= x_{2,\rho} + f^{[2]}_{2,\rho-1}(x_1,x_2,u) + f^{[3]}_{2,\rho-1}(x_1,x_2,u),\\ x^+_{2,\rho} &= x_{2,\rho+1} + f^{[2]}_{2,\rho}(x_1,x_2,u) + f^{[3]}_{2,\rho}(x_1,x_2,u) \end{aligned}$$

is transformed to

$$\begin{aligned} x^+_{2,\rho-1} &= \bar{x}_{2,\rho} + f^{[2]}_{2,\rho-1}(x_1,x_2,u) + f^{[3]}_{2,\rho-1}(x_1,x_2,u) - c x_{1,\sigma} x_{2,\tau} x_{2,\upsilon},\\ \bar{x}^+_{2,\rho} &= x_{2,\rho+1} + f^{[2]}_{2,\rho}(x_1,x_2,u) + f^{[3]}_{2,\rho}(x_1,x_2,u) + c\lambda_\sigma x_{1,\sigma} x_{2,\tau+1} x_{2,\upsilon+1}, \end{aligned}$$

and ζ^{j1l}_i is unchanged. The feedback affects only

$$x^+_{2,n_2} = u + f^{[2]}_{2,n_2}(x_1,x_2,u) + f^{[3]}_{2,n_2}(x_1,x_2,u),$$

transforming it into

$$x^+_{2,n_2} = \bar{u} + f^{[2]}_{2,n_2}(x_1,x_2,u) + f^{[3]}_{2,n_2}(x_1,x_2,u) - c x_{1,\sigma} x_{2,\tau} x_{2,\upsilon},$$

and again ζ^{j1l}_i is unchanged.

The normal form of $\tilde{f}^{[2;1]}_2(z_1; z_2, v)$. Consider a part of the dynamics

$$\begin{aligned} z^+_{2,i-1} &= z_{2,i} + f^{[2]}_{2,i-1}(z_1,z_2,v) + \cdots,\\ z^+_{2,i} &= z_{2,i+1} + f^{[2]}_{2,i}(z_1,z_2,v) + c z_{1,j} z_{1,k} z_{2,l} + \cdots,\\ z^+_{2,i+1} &= z_{2,i+2} + f^{[2]}_{2,i+1}(z_1,z_2,v) + \cdots, \end{aligned}$$

where $1 < i \le n_2$, $1 \le j \le k \le n_1$, $1 \le l \le n_2+1$.

If $\lambda_j\lambda_k \ne 0$ and $1 < l$, we can pull up the cubic term by defining

$$\bar{z}_{2,i} = z_{2,i} - \frac{c}{\lambda_j\lambda_k} z_{1,j} z_{1,k} z_{2,l-1};$$

then the dynamics becomes

$$\begin{aligned} z^+_{2,i-1} &= \bar{z}_{2,i} + f^{[2]}_{2,i-1}(z_1,z_2,v) + \frac{c}{\lambda_j\lambda_k} z_{1,j} z_{1,k} z_{2,l-1} + f^{[2]}_{2,i}(z_1,z_2,v) + \cdots,\\ \bar{z}^+_{2,i} &= z_{2,i+1} + f^{[2]}_{2,i}(z_1,z_2,v) + \cdots,\\ z^+_{2,i+1} &= z_{2,i+2} + f^{[2]}_{2,i+1}(z_1,z_2,v) + \cdots, \end{aligned}$$

and all the other cubic terms remain the same. Again, if $i = 1$, we can still pull up, and the term disappears. So by pulling up all cubic terms where $\lambda_j\lambda_k \ne 0$ until $l = 1$, we obtain

$$z^+_{2,i} = z_{2,i+1} + c z_{1,j} z_{1,k} z_{2,1} + \cdots.$$

Pushing down eliminates this term and any term with $\lambda_j\lambda_k = 0$. Define

$$\bar{z}_{2,i+1} = z_{2,i+1} + c z_{1,j} z_{1,k} z_{2,1},$$

yielding

$$\begin{aligned} z_{2,i-1}^{+} &= z_{2,i} + f_{2,i-1}^{[2]}(z_1, z_2, v) + \cdots, \\ z_{2,i}^{+} &= \bar{z}_{2,i+1} + f_{2,i}^{[2]}(z_1, z_2, v) + \cdots, \\ \bar{z}_{2,i+1}^{+} &= z_{2,i+2} + f_{2,i+1}^{[2]}(z_1, z_2, v) + c\lambda_j\lambda_k z_{1,j} z_{1,k} z_{2,2} + \cdots, \end{aligned}$$

and all the other cubic terms remain unchanged. If $\lambda_j\lambda_k = 0$, then the term drops out. If $\lambda_j\lambda_k \neq 0$, then we can push down repeatedly until the cubic term is absorbed in the control. The result is $\tilde{f}_2^{[2;1]}(z_1; z_2, v) = 0$.

The normal form of $\tilde{f}_2^{[3;0]}(z_1; z_2, v)$. Consider a part of the dynamics

$$\begin{aligned} z_{2,i}^{+} &= z_{2,i+1} + f_{2,i}^{[2]}(z_1, z_2, v) + c z_{1,j} z_{1,k} z_{1,l} + \cdots, \\ z_{2,i+1}^{+} &= z_{2,i+2} + f_{2,i+1}^{[2]}(z_1, z_2, v) + \cdots, \end{aligned}$$

where $1 \leq i \leq n_2$, $1 \leq j \leq k \leq l \leq n_1$.

Pushing down one or more times eliminates this term. Define

$$\bar{z}_{2,i+1} = z_{2,i+1} + c z_{1,j} z_{1,k} z_{1,l},$$

yielding

$$\begin{aligned} z_{2,i}^{+} &= \bar{z}_{2,i+1} + f_{2,i}^{[2]}(z_1, z_2, v) + \cdots, \\ \bar{z}_{2,i+1}^{+} &= z_{2,i+2} + f_{2,i+1}^{[2]}(z_1, z_2, v) + c\lambda_j\lambda_k\lambda_l z_{1,j} z_{1,k} z_{1,l} + \cdots, \end{aligned}$$

and all the other cubic terms remain unchanged. If $\lambda_j\lambda_k\lambda_l = 0$, then the term drops out. Otherwise, the term can be pushed down repeatedly until it is absorbed in the control. The result is $\tilde{f}_2^{[3;0]}(z_1; z_2, v) = 0$.

The normal form of $\tilde{f}_1^{[3;0]}(z_1; z_2, v)$. This is just the cubic normal form and invariants of Poincaré (see [1], [5], [9], and [13]). Consider a part of the dynamics

$$z_{1,i}^{+} = \lambda_i z_{1,i} + f_{1,i}^{[2]}(z_1, z_2, v) + c z_{1,j} z_{1,k} z_{1,l} + \cdots,$$

where $1 \leq i \leq n_1$, $1 \leq j \leq k \leq l \leq n_1$.

If $\lambda_i \neq \lambda_j\lambda_k\lambda_l$, then define

$$\bar{z}_{1,i} = z_{1,i} - \frac{c}{(\lambda_j\lambda_k\lambda_l - \lambda_i)} z_{1,j} z_{1,k} z_{1,l}$$

so that

$$\bar{z}_{1,i}^{+} = \lambda_i \bar{z}_{1,i} + f_{1,i}^{[2]}(z_1, z_2, v) + \cdots.$$

Next we show that the numbers β_i^{jkl} (3.9) are invariants. Clearly, β_i^{jkl} is potentially changed only by $\phi_1^{[3;0]}(x_1; x_2)$. Therefore, we need only consider coordinate changes of the form

$$\bar{x}_{1,\rho} = x_{1,\rho} + c x_{1,\sigma} x_{1,\tau} x_{1,\upsilon},$$

where $1 \leq \rho \leq n_1$, $1 \leq \sigma \leq \tau \leq \upsilon \leq n_1$ because more general ones are just compositions of these. This coordinate change affects only a piece of the dynamics (3.1), and

$$x_{1,\rho}^{+} = \lambda_\rho x_{1,\rho} + f_{1,\rho}^{[2]}(x_1, x_2, u) + f_{1,\rho}^{[3]}(x_1, x_2, u) + O(x_1, x_2, u)^3$$

is transformed to

$$\bar{x}^+_{1,\rho} = \lambda_\rho \bar{x}_{1,\rho} + f^{[2]}_{1,\rho}(x_1,x_2,u) + f^{[3]}_{1,\rho}(x_1,x_2,u) + c(\lambda_\sigma\lambda_\tau\lambda_\upsilon - \lambda_\rho)x_{1,\sigma}x_{1,\tau} + O(x_1,x_2,u)^3.$$

Clearly, if $\lambda_\rho = \lambda_\sigma\lambda_\tau\lambda_\upsilon$, then β_i^{jkl} (3.9) is unchanged.

The normal form of $\tilde{f}_1^{[2;1]}(z_1;z_2,v)$. Consider a part of the dynamics

$$z^+_{1,i} = \lambda_i z_{1,i} + f^{[2]}_{1,i}(z_1,z_2,v) + cz_{1,j}z_{1,k}z_{2,l} + \cdots,$$

where $1 \le i \le n_1$, $1 \le j \le k \le n_1$, $1 \le l \le n_2$.

If $\lambda_j\lambda_k \neq 0$ and $l > 1$, we can pull up by defining

$$\bar{z}_{1,i} = z_{1,i} - \frac{c}{\lambda_j\lambda_k} z_{1,j}z_{1,k}z_{2,l-1}$$

so that

$$\bar{z}^+_{1,i} = \lambda_i \bar{z}_{1,i} + f^{[2]}_{1,i}(z_1,z_2,v) + \frac{c\lambda_i}{\lambda_j\lambda_k} z_{1,j}z_{1,k}z_{2,l-1} + \cdots.$$

If $\lambda_i = 0$, then the term disappears; otherwise, we can continue to pull up until $l = 1$.

If $\lambda_i \neq 0$ and $\lambda_j\lambda_k = 0$, then the term disappears by pushing down

$$\bar{z}_{1,i} = z_{1,i} + \frac{c}{\lambda_i} z_{1,j}z_{1,k}z_{2,l}$$

so that

$$\begin{aligned}\bar{z}^+_{1,i} &= \lambda_i \bar{z}_{1,i} + f^{[2]}_{1,i}(z_1,z_2,v) + \frac{c\lambda_j\lambda_k}{\lambda_i} z_{1,j}z_{1,k}z^+_{2,l} + \cdots, \\ &= \lambda_i \bar{z}_{1,i} + f^{[2]}_{1,i}(z_1,z_2,v) + \cdots.\end{aligned}$$

If $\lambda_i = \lambda_j\lambda_k = 0$, then we cannot pull up or push down. The result is (3.4).

Next we show that the numbers γ_i^{jkl} (3.10)–(3.11) are invariants. Clearly, γ_i^{jkl} is potentially changed only by $\phi_1^{[2;1]}(x_1;x_2)$. Therefore, we need only consider coordinate changes of the form

$$\bar{x}_{1,\rho} = x_{1,\rho} + cx_{1,\sigma}x_{1,\tau}x_{2,\upsilon},$$

where $1 \le \rho \le n_1$, $1 \le \sigma \le \tau \le n_1$, $1 \le \upsilon \le n_2$ because more general ones are just compositions of these. This coordinate change affects only a piece of the dynamics (3.1), and

$$x^+_{1,\rho} = \lambda_\rho x_{1,\rho} + f^{[2]}_{1,\rho}(x_1,x_2,u) + f^{[3]}_{1,\rho}(x_1,x_2,u) + O(x_1,x_2,u)^4$$

is transformed to

$$\begin{aligned}\bar{x}^+_{1,\rho} = {} & \lambda_\rho \bar{x}_{1,\rho} + f^{[2]}_{1,\rho}(x_1,x_2,u) + f^{[3]}_{1,\rho}(x_1,x_2,u) \\ & + c\lambda_\sigma\lambda_\tau x_{1,\sigma}x_{1,\tau}x_{2,\upsilon+1} - c\lambda_\rho x_{1,\sigma}x_{1,\tau}x_{2,\upsilon} + O(x_1,x_2,u)^4.\end{aligned}$$

Clearly, if $\lambda_\rho = \lambda_\sigma\lambda_\tau = 0$, then γ_i^{jkl} (3.10) is unchanged. A simple calculation shows that if $\lambda_\rho\lambda_\sigma\lambda_\tau \neq 0$, then γ_i^{jk1} (3.11) is unchanged.

The normal form of $\tilde{f}_1^{[1;2]}(z_1; z_2, v)$. Consider a part of the dynamics

$$z_{1,i}^+ = \lambda_i z_{1,i} + f_{1,i}^{[2]}(z_1, z_2, v) + c z_{1,j} z_{2,k} z_{2,l} + \cdots,$$

where $1 \leq i \leq n_1$, $1 \leq j \leq n_1$, $1 \leq k \leq l \leq n_2$.

If $\lambda_j \neq 0$ and $k > 1$, then we can pull up by defining

$$\bar{z}_{1,i} = z_{1,i} - \frac{c}{\lambda_j} z_{1.j} z_{2,k-1} z_{2,l-1};$$

then

$$\bar{z}_{1,i}^+ = \lambda_i \bar{z}_{1,i} + f_{1,i}^{[2]}(z_1, z_2, v) + \frac{c\lambda_i}{\lambda_j} z_{1,j} z_{2,k-1} z_{2,l-1} + \cdots.$$

If $\lambda_i = 0$, then the term disappears; otherwise, we can continue to pull up until $k = 1$.

If $\lambda_i \neq 0$ and $\lambda_j = 0$, then the term disappears by pushing down

$$\bar{z}_{1,i} = z_{1,i} + \frac{c}{\lambda_i} z_{1,j} z_{2,k} z_{2,l};$$

then

$$\begin{aligned}\bar{z}_{1,i}^+ &= \lambda_i \bar{z}_{1,i} + f_{1,i}^{[2]}(z_1, z_2, v) + \frac{c\lambda_j}{\lambda_i} z_{1,j} z_{1,k}^+ z_{2,l}^+ + \cdots, \\ &= \lambda_i \bar{z}_{1,i} + f_{1,i}^{[2]}(z_1, z_2, v) + \cdots.\end{aligned}$$

If $\lambda_i = \lambda_j = 0$, then we cannot pull up or push down. The result is (3.5).

Next we show that the numbers δ_i^{jkl} (3.12)–(3.13) are invariants. Clearly, δ_i^{jkl} is potentially changed only by $\phi_1^{[1;2]}(x_1; x_2)$. Therefore, we need only consider coordinate changes of the form

$$\bar{x}_{1,\rho} = x_{1,\rho} + c x_{1,\sigma} x_{2,\tau} x_{2,\upsilon},$$

where $1 \leq \rho, \sigma \leq n_1$, $1 \leq \tau \leq \upsilon \leq n_2$ because more general ones are just compositions of these. This coordinate change affects only a piece of the dynamics (3.1), and

$$x_{1,\rho}^+ = \lambda_\rho x_{1,\rho} + f_{1,\rho}^{[2]}(x_1, x_2, u) + f_{1,\rho}^{[3]}(x_1, x_2, u) + O(x_1, x_2, u)^4$$

is transformed to

$$\begin{aligned}\bar{x}_{1,\rho}^+ = \lambda_\rho \bar{x}_{1,\rho} &+ f_{1,\rho}^{[2]}(x_1, x_2, u) + f_{1,\rho}^{[3]}(x_1, x_2, u) \\ &+ c\lambda_\sigma \lambda_\tau x_{1,\sigma} x_{1,\tau} x_{2,\upsilon+1} - c\lambda_\rho x_{1,\sigma} x_{1,\tau} x_{2,\upsilon} + O(x_1, x_2, u)^4.\end{aligned}$$

Clearly, if $\lambda_\rho = \lambda_\sigma \lambda_\tau = 0$, then δ_i^{jkl} (3.12) is unchanged. A simple calculation shows that if $\lambda_\rho \lambda_\sigma \lambda_\tau \neq 0$, then δ_i^{j1l} (3.13) is unchanged.

The normal form of $\tilde{f}_1^{[0;3]}(z_1; z_2, v)$. Consider a part of the dynamics

$$z_{1,i}^+ = \lambda_i z_{1,i} + f_{1,i}^{[2]}(z_1, z_2, v) + c z_{2,j} z_{2,k} z_{2,l} + \cdots,$$

where $1 \leq i \leq n_1$, $1 \leq j \leq k \leq l \leq n_2$.

If $j > 1$, we can pull up by defining

$$\bar{z}_{1,i} = z_{1,i} - c z_{2,j-1} z_{2,k-1} z_{2,l-1};$$

then

$$\bar{z}_{1,i}^{+} = \lambda_i \bar{z}_{1,i} + f_{1,i}^{[2]}(z_1, z_2, v) + c\lambda_i z_{2,j-1} z_{2,k-1} z_{2,l-1} + \cdots .$$

If $\lambda_i = 0$, then the term disappears; otherwise, we can continue to pull up until $j = 1$. The result is (3.6).

Finally, we show that the numbers ϵ_i^{1kl} (3.14) are invariants. Clearly, ϵ_i^{1kl} is potentially changed only by $\phi_1^{[0;3]}(x_1; x_2)$. Therefore, we need only consider coordinate changes of the form

$$\bar{x}_{1,\rho} = x_{1,\rho} + c x_{2,\sigma} x_{2,\tau} x_{2,\upsilon},$$

where $1 \le \rho \le n_1$, $1 \le \sigma \le \tau \le \upsilon \le n_2$ because more general ones are just compositions of these. This change of coordinates affects only a piece of the dynamics (2.3), and

$$x_{1,\rho}^{+} = \lambda_\rho x_{1,\rho} + f_{1,\rho}^{[2]}(x_1, x_2, u) + f_{1,\rho}^{[3]}(x_1, x_2, u)$$

is transformed to

$$\begin{aligned} \bar{x}_{1,\rho}^{+} = \lambda_\rho \bar{x}_{1,\rho} &+ f_{1,\rho}^{[2]}(x_1, x_2, u) + f_{1,\rho}^{[3]}(x_1, x_2, u) \\ &+ c x_{2,\sigma+1} x_{2,\tau+1} x_{2,\upsilon+1} - c\lambda_\rho x_{2,\sigma} x_{2,\tau} x_{2,\upsilon}. \end{aligned}$$

Clearly, if $\lambda_\rho \neq 0$, then ϵ_i^{1kl} (3.14) is unchanged.

7. Conclusion. We have developed a theory of quadratic and cubic normal forms for discrete time control systems. To avoid notational difficulties, we have restricted our attention to scalar input systems whose uncontrollable part is diagonalizable. But the basic operations of pull up and push down extend to more general systems. We have also shown the uniqueness of the normal forms.

We have introduced the concept of control bifurcation and have exhibited some simple examples.

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