Goal:

We motivate the Arzelà-Ascoli Theorem and the contraction mapping principle. Both theorems exploit topological property of continuity and the structure of the underlying metric space. Moreover, both play a central role in approximation theory, especially for PDEs.

**Arzelà-Ascoli Theorem:** If a family $(f_\alpha) \subset C([0,1])$ is uniformly bounded and equicontinuous, then there exists a subsequence $(f_{\alpha_k})_k$ that uniformly converges.

**Proof.** The main proof idea employs the diagonalization argument to apply the “$\varepsilon/3$-trick:"

\[
|f_{\alpha_m}(x) - f_{\alpha_n}(x)| \leq |f_{\alpha_m}(x) - f_{\alpha_m}(x_i)| + |f_{\alpha_m}(x_i) - f_{\alpha_n}(x_i)| + |f_{\alpha_n}(x_i) - f_{\alpha_n}(x)| < \varepsilon.
\]

**Contraction Mapping Principle:** If $T: X \to X$ is a (strict) contraction on a complete metric space $X$ then there exists exactly one fixed point $x \in X$ of the map $T$.

**Proof.** Take $x_0 \in X$ arbitrary and let $x_n = Tx_{n-1}$ for $n \geq 1$. Take $0 \leq c < 1$ the contractive constant of $T$. Without loss of generality suppose $n \geq m$. Then

\[
d(x_n, x_m) \leq d(T^m x_0, T^m x_0) \\
\leq c^m d(T^{n-m} x_0, x_0) \\
\leq c^n [d(T^{n-m} x_0, T^{n-m-1} x_0) + d(T^{n-m-1} x_0, T^{n-m-2} x_0) + \cdots + d(T x_0, x_0)] \\
\leq \left( \frac{c^m}{1-c} \right) d(x_1, x_0).
\]

Thus as $m \to \infty$ the distance $d(x_n, x_m) \to 0$. By completeness of $X$, let $x^* = \lim_n x_n$. Then

\[Tx^* = T \lim_n x_n = \lim_n Tx_n = \lim_n x_{n+1} = x^*.
\]

Uniqueness follows from the fact that if $y^*$ is another fixed point then

\[0 \leq d(x^*, y^*) = d(Tx^*, Ty^*) = cd(x^*, y^*).
\]

**Exercise & Important Points:**

1. (Tao) The Arzelà-Ascoli theorem is often used to solve some equation $F(u) = f$, for some $u: X \to \mathbb{R}^m$ by considering approximate solutions $F(u_n) \to f$ as $n \to \infty$ in some suitable sense. If one can show the $u_n$ are equicontinuous and point wise bounded then the AA theorem allows us to extract a uniformly convergent subsequence (If $u \in L^1_{loc}(\Omega)$ for $\Omega \subset \mathbb{R}^n$ is weakly harmonic with $\phi \in C_0^2(\Omega)$ then $u$ is harmonic.). This method is called the *compactness method*. However, has the drawback that it is difficult to establish uniqueness and regularity of limits.
2. The AA theorem only describes the family of continuous functions (topological property) in the uniform norm, and does not give higher-order differentiability (analytic property) unless more about the family is known. The AA theorem helps preserve properties of differentiation and integration under limits.

3. Fixed point theorems pervade mathematics: Brouwer’s fixed point theorem, Picard-Lindelhof theorem, Schauder fixed point theorem, Lefschetz fixed point theorem, Hensel’s lemma, etc. Unlike topological fixed point theorems (where proofs are based on contradiction, are nonconstructive, and produce no computational approximations) the contraction mapping theorem is constructive.

4. The AA theorem generalizes to $\sigma$-compact locally compact Hausdorff spaces though convergences is only uniform on compact subsets. The proof proceeds as in the original proof but also diagonalizes over increasing compact sets.

5. Both theorems illustrate the idea that convergence in a strong topology requires a uniform control on the regularity or decay of functions in addition to uniform bounds.

6. Note that the strict contraction property provides an additional “regularity” property (Lipschitz continuity). Unlike compactness results, this implies the set $(T^n x)$ itself converges, as opposed to a subsequence of it (by invoking compactness).

7. Both methods are nonlinear.

Exercises:

1. Exercise: show that if any of the hypotheses of the Arzelà-Ascoli theorem are omitted the result fails.

2. Exercise: show that if the hypotheses of the contraction mapping theorem are omitted the result fails.

3. Exercise: show that the hypothesis the family $(f_n)_n$ is uniformly bounded can be weakened to being pointwise bounded. Use compactness of $[0, 1]$.

4. Exercise (Baby Rudin Ex. 7.15): Suppose $f$ is a real continuous function on $\mathbb{R}$ with $f_n(x) = f(nx)$ for $n = 1, 2, 3, \ldots$, and $(f_n)_n$ is equicontinuous on $[0, 1]$. What conclusion can you draw about $f$?

5. Exercise (Baby Rudin Ex. 7.18): Let $(f_n)$ be a uniformly bounded sequence of functions which are Riemann-integrable on $[a, b]$, and put $F_n(x) = \int_a^x f_n(x)dx$. Prove that there exists a subsequence of $(F_n)$ that converges uniformly on $[a, b]$. 

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