SOLUTIONS FOR MAT 218A PROJECT

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Problem 5.5 In this problem, we examine if the specification of the divergence and curl of a vector field with periodic boundary conditions is sufficient to uniquely determine the vector field. We will see that up to a constant vector, such specifications do uniquely determine a vector field.

Let \( u : \mathbb{T}^3 \to \mathbb{R}^3 \) be a vector-valued function defined on \( \mathbb{T}^3 \).

1. Show that \( \text{curl}(\text{curl}(u)) = -\Delta u + \text{D}(\text{div}(u)) \) for all \( u \in C^2(\mathbb{T}^3) \).

1.1. Solution: To demonstrate that the above identity holds we will expand into component functions. Let \( u(x) = u^1(x)e_1 + u^2(x)e_2 + u^3(x)e_3 \) for the vector field, where \( e_1 \) is the standard basis vector. Expanding the right hand side,

\[
D(\text{div}(u)) = (u_{11}^1 + u_{22}^1 + u_{33}^1)e_1 + (u_{12}^1 + u_{22}^2 + u_{33}^2)e_2 + (u_{13}^1 + u_{23}^2 + u_{33}^3)e_3 - (u_{11}^1 + u_{22}^2 + u_{33}^3)e_1 - (u_{11}^2 + u_{22}^2 + u_{33}^3)e_2 - (u_{11}^3 + u_{22}^3 + u_{33}^3)e_3 = \\
(u_{21}^1 - u_{12}^1 - u_{33}^1 + u_{31}^3)e_1 + (u_{32}^1 - u_{22}^2 - u_{11}^3 + u_{12}^3)e_2 + (u_{13}^1 - u_{31}^1 - u_{22}^3 + u_{33}^3)e_3
\]

Now we expand the left hand side,

\[
\text{curl}(\text{curl}(u)) = \text{curl}((u_{32}^1 - u_{23}^1 - u_{13}^1)e_1 + (u_{31}^2 - u_{23}^2 - u_{12}^2)e_2 + (u_{31}^3 - u_{22}^3 - u_{32}^3)e_3 = \\
(u_{12}^2 - u_{12}^2 - u_{33}^1 + u_{31}^3)e_1 + (u_{32}^3 - u_{23}^3 - u_{12}^3)e_2 + (u_{13}^1 - u_{31}^1 - u_{22}^3 + u_{33}^3)e_3
\]

Now \( u \in C^2(\mathbb{T}^3) \) so that the mixed derivatives are equal, and then we see by inspection that \( \text{curl}(\text{curl}(u)) = -\Delta u + D(\text{div}(u)) \).

2. Show for each \( u \in H^1(\mathbb{T}^3) \) that the following holds:

\[
||Du||^2_{L^2(\mathbb{T}^3)} = ||\text{div}(u)||^2_{L^2(\mathbb{T}^3)} + ||\text{curl}(u)||^2_{L^2(\mathbb{T}^3)}, \quad \text{where} \quad ||Du||^2_{L^2(\mathbb{T}^3)} := \sum_{i,j=1}^{3} ||u^i_j||^2_{L^2(\mathbb{T}^3)}.
\]

2.1. Solution: We give a proof by the technique of Fourier Series. For \( i = 1, 2, 3 \) write \( u^i(x) = (2\pi)^{-3/2} \sum_{n \in \mathbb{Z}^3} \hat{u}^i_n e^{in \cdot x} \). Then, since \( u \in H^1(\mathbb{T}^3) \) we have \( u^i_j(x) = (2\pi)^{-3/2} \sum_{n \in \mathbb{Z}^3} in_i \hat{u}^i_n e^{in \cdot x} \)
and \( u'_1(x) = (2\pi)^{-3/2} \sum_{n \in \mathbb{Z}^3} i n_3 \hat{x} n e^{i n \cdot x} \). Now we expand into components and use Parseval’s to see that,

\[
\| \text{div}(u) \|_{L^2(T^3)}^2 + \| \text{curl}(u) \|_{L^2(T^3)}^2 = \| u_1 + u_2^2 + u_3^3 \|_{L^2(T^3)}^2 + \| (u_3^3 - u_2^2) e_1 + (u_2^2 - u_1^1)e_2 + (u_1^1 - u_2^2)e_3 \|_{L^2(T^3)}^2.
\]

\[
= \| (2\pi)^{-3/2} \sum_{n \in \mathbb{Z}^3} i(n_1 \hat{x} n_1 + n_2 \hat{x} n_2 + n_3 \hat{x} n_3) e^{i n \cdot x} \|_{L^2(T^3)}^2 + \| (2\pi)^{-3/2} \sum_{n \in \mathbb{Z}^3} i(n_3 \hat{x} n_1^1 - n_1 \hat{x} n_1^1) e^{i n \cdot x} \|_{L^2(T^3)}^2 + \| (2\pi)^{-3/2} \sum_{n \in \mathbb{Z}^3} i(n_2 \hat{x} n_2^2 - n_2 \hat{x} n_2^2) e^{i n \cdot x} \|_{L^2(T^3)}^2 + \| (2\pi)^{-3/2} \sum_{n \in \mathbb{Z}^3} i(n_3 \hat{x} n_3^3 - n_3 \hat{x} n_3^3) e^{i n \cdot x} \|_{L^2(T^3)}^2 + \| (2\pi)^{-3/2} \sum_{n \in \mathbb{Z}^3} i(n_2 \hat{x} n_2^2 - n_2 \hat{x} n_2^2) e^{i n \cdot x} \|_{L^2(T^3)}^2 + \| (2\pi)^{-3/2} \sum_{n \in \mathbb{Z}^3} i(n_3 \hat{x} n_3^3 - n_3 \hat{x} n_3^3) e^{i n \cdot x} \|_{L^2(T^3)}^2 \]

Therefore, we have that \( \| D_3 u \|_{L^2(T^3)}^2 = \| \text{div}(u) \|_{L^2(T^3)}^2 + \| \text{curl}(u) \|_{L^2(T^3)}^2 \).

3. Given \( f \in H^{-1}(T^3) \) and \( g \in H^{-1}(T^3; \mathbb{R}^3) \), find a vector field \( u \in L^2(T^3; \mathbb{R}^3) \) with \( \text{div}(u) = f \) and \( \text{curl}(u) = g \). Is this \( u \) unique? (Hint: Write \( u \) in the form \( u = Dp + \text{curl}(w) \) for some \( p \in H^1(T^3) \) and \( w \in H^1_{\text{div}}(T^3; \mathbb{R}^3) \). )

3.1. Solution: We use two points of view to look at this problem, both of which lead to the same result.

I. Existence and Uniqueness of Weak Solution of Two Elliptic Equations

First we show that

(1) \( \text{div}Dp = f \)

has a unique weak solution in \( H^1(T^3)/\mathbb{R} \). Note that the equation holds in the dual space of \( H^1(T^3)/\mathbb{R} \), denoted by \( H^{-1}(T^3) \).

Definition \( p \in H^1(T^3)/\mathbb{R} \) is called a weak solution of (1) if

(2) \( \int_{T^3} Dp \cdot Dv \ dx = \langle f, v \rangle_{H^1(T^3)/\mathbb{R}} \)

for all \( v \in H^1(T^3)/\mathbb{R} \).

The boundary integral vanishes because both \( v \) and \( p \) have periodic boundary values. This definition is thereby a correct weak formulation of equation (1).

The boundedness and coercivity of the operator \( \text{div}D \) can be established in the same way as in Theorem 5.27 of the lecture notes. Thus, from Lax-Milgram Theorem, there exists a unique weak solution \( p \in H^1(T^3)/\mathbb{R} \) to (1).

Next we show that

(3) \( \text{curl} \text{curl} w = g \)

has a unique weak solution in \( H^1_{\text{div}}(T^3; \mathbb{R}^3)/\mathbb{R}^3 \). Note that the equation holds in the dual space of \( H^1_{\text{div}}(T^3; \mathbb{R}^3)/\mathbb{R}^3 \), denoted by \( H^{-1}(T^3; \mathbb{R}^3) \).
Definition \( w \in H^1_{\text{div}}(\mathbb{T}^3; \mathbb{R}^3)/\mathbb{R}^3 \) is called a weak solution of (3) if

\[
\int_{\mathbb{T}^3} \text{curl} \ w \cdot \text{curl} \ \phi \ dx = \langle g, \phi \rangle_{H^1_{\text{div}}(\mathbb{T}^3; \mathbb{R}^3)/\mathbb{R}^3}
\]

for all \( \phi \in H^1_{\text{div}}(\mathbb{T}^3; \mathbb{R}^3)/\mathbb{R}^3 \).

Again, the boundary integral vanishes because either \( w \) or \( \phi \) has periodic boundary value. This definition is thereby a reasonable weak formulation of equation (3).

Use integration by parts and the definition of \( H^1_{\text{div}}(\mathbb{T}^3; \mathbb{R}^3) \), we have shown that

\[
\int_{\mathbb{T}^3} \sum_{i=1}^3 D w^i \cdot D \phi^i \ dx = \int_{\mathbb{T}^3} \text{curl} \ w \cdot \text{curl} \ \phi \ dx.
\]

The details will be shown in the presentation. So we conclude that the variational formulation (4) is equivalent to

\[
\int_{\mathbb{T}^3} \sum_{i=1}^3 D w^i \cdot D \phi^i \ dx = \langle g, \phi \rangle_{H^1_{\text{div}}(\mathbb{T}^3; \mathbb{R}^3)/\mathbb{R}^3}.
\]

Thus, the boundedness of the operator \( \text{curl} \text{curl} \) can be proved in the same way as the Step 1 of Theorem 5.32. The coercivity is guaranteed by Korn’s second inequality (Lemma 5.25) which implies that

\[
\langle \text{curl} \text{curl} \ w, w \rangle_{H^1_{\text{div}}(\mathbb{T}^3; \mathbb{R}^3)} \geq \| D w \|_{L^2(\mathbb{T}^3)} \geq \| \epsilon \|_{L^2(\mathbb{T}^3)} \geq \frac{1}{C} \| w \|_{H^1_{\text{div}}(\mathbb{T}^3; \mathbb{R}^3)}.
\]

Note that here we use the fact that \( H^1_{\text{div}}(\mathbb{T}^3; \mathbb{R}^3)/\mathcal{S} = H^1_{\text{div}}(\mathbb{T}^3; \mathbb{R}^3)/\mathbb{R}^3 \), i.e., the space of periodic functions does not contain any first-order linear functions. Therefore, from Lax-Milgram Theorem, There exists a unique weak solution \( w \in H^1_{\text{div}}(\mathbb{T}^3; \mathbb{R}^3)/\mathbb{R}^3 \) to (3).

Now let \( u = D p + \text{curl} \ w \), where \( p \in H^1(\mathbb{T}^3; \mathbb{R}) \) and \( w \in H^1_{\text{div}}(\mathbb{T}^3; \mathbb{R}^3)/\mathbb{R}^3 \) are the unique weak solutions to (1) and (3), respectively. Apparently, \( u \) is a vector-valued function in \( L^2(\mathbb{T}^3; \mathbb{R}^3) \). It can be easily shown by a density argument that \( \text{div} \text{curl} \ w = 0 \) in \( H^{-1}(\mathbb{T}^3) \) and \( \text{curl} D p = 0 \) in \( H^{-1}(\mathbb{T}^3; \mathbb{R}^3) \). Thus, \( \text{div} u = f \) and \( \text{curl} u = g \) and \( u \) is the desired vector field.

Lastly, we answer the question whether such a vector field is unique. Suppose that there are two vector fields, \( u_1 \) and \( u_2 \), satisfying the \( \text{div} \) and \( \text{curl} \) equations. Let \( u = u_1 - u_2 \) satisfies \( \text{div} u = 0 \) and \( \text{curl} u = 0 \). Take the curl of each side of the second equation, and obtain

\[
\text{curl} \text{curl} u = -\Delta u + D \text{div} u = -\Delta u = 0,
\]

where all the equalities hold in \( H^{-2}(\mathbb{T}^3; \mathbb{R}^3) \), and the first equality follows from Part 1 and a density argument. Since any \( L^2(\mathbb{T}^3; \mathbb{R}^3) \) solution of the equation \( \Delta u = 0 \) (in \( H^{-2}(\mathbb{T}^3; \mathbb{R}^3) \)) can only be a vector field with all constant components, we conclude that the vector field is unique up to a constant vector. \( \square \)
II. Fourier Analysis
We give an explicit representation of the vector field (up to a constant) in term of its Fourier series. The calculation of the Fourier coefficients involves solving a $3 \times 3$ linear system, the solvability of which is guaranteed by the fact that $g$ is divergence-free. We will present this part in the talk if time permits.