

A User's Guide for `LattE integrale` v1.5 (Linux Release) *



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1 Introduction

1.1 What is LattE?

The name “**LattE**” is an abbreviation for “**L**attice point **E**numeration.” **LattE** was developed in 2001 to count lattice points contained in convex polyhedra defined by linear equations and inequalities with integer coefficients. The polyhedra can be of any (reasonably small) dimension. In 2007, **LattE macchiato** was released and contained many algorithmic improvements. The newest edition, **LattE integrale**, developed in 2010 can compute integrals of polynomials and volumes of rational polytopes. All these algorithms run in polynomial time for fixed dimension. To learn more about the exact details of our implementation for lattice point enumeration, the interested reader can consult [7, 9] and the references listed therein. For learning the algorithmic details of integration, see [2, 8]. Here we give a rather short description of the mathematical objects used by **LattE**. Note that all our computations are done over the integers or the rationals exactly. **LattE** cannot compute with floating-point numbers.

1.1.1 Counting lattice points: Barvinok’s Rational Functions

Given a convex polyhedron $P = \{u \in \mathbb{R}^d : Au \leq b\}$, where A and b are integral, the fundamental object that we compute is a short representation of the infinite power series:

$$f(P; x) = \sum_{\alpha \in P \cap \mathbb{Z}^d} x_1^{\alpha_1} x_2^{\alpha_2} \dots x_d^{\alpha_d}.$$

Here each lattice point is given by one monomial. Note that this can be a rather long sum, in fact for a polyhedral cone it can be infinite, but the good news is that it admits short representations.

Example: Let P be the quadrangle with vertices $V_1 = (0, 0)$, $V_2 = (5, 0)$, $V_3 = (4, 2)$, and $V_4 = (0, 2)$, see Figure 1.

$$f(P; x, y) = x^5 + x^4y + x^4 + x^4y^2 + yx^3 + x^3 + x^3y^2 + yx^2 + x^2 + x^2y^2 + xy + x + xy^2 + y + 1 + y^2$$

The fundamental theorem of Barvinok (circa 1993, see [3]) says that you can write $f(P; x)$ as a sum of short rational functions, in polynomial time when the dimension of the polyhedron is fixed. In our running example we easily see that the 16 monomial polynomial can be written as shorter rational function sum:

$$f(P; x, y) = f(K_{V_1}; x, y) + f(K_{V_2}; x, y) + f(K_{V_3}; x, y) + f(K_{V_4}; x, y)$$

where

$$\begin{aligned} f(K_{V_1}; x, y) &= \frac{1}{(1-x)(1-y)} & f(K_{V_2}; x, y) &= \frac{(x^5 + x^4y)}{(1-x^{-1})(1-y^2x^{-1})} \\ f(K_{V_3}; x, y) &= \frac{(x^4y^2 + x^4)}{(1-x^{-1})(1-xy^{-2})} & f(K_{V_4}; x, y) &= \frac{y^2}{(1-y^{-1})(1-x)} \end{aligned}$$

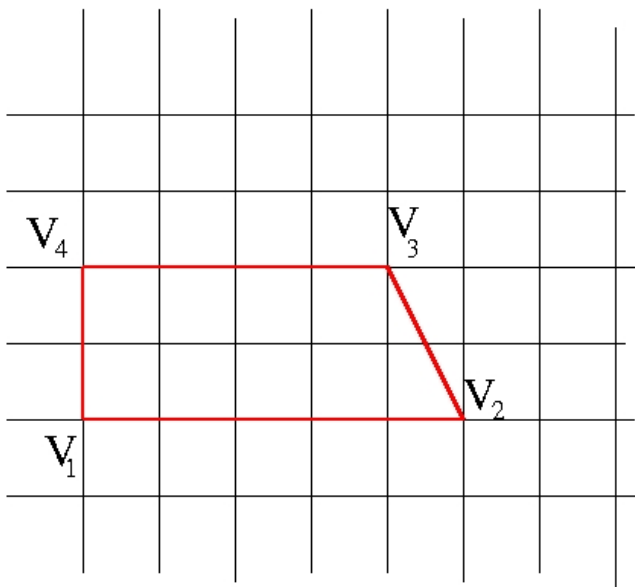


Figure 1: Quadrangle with vertices $V_1 = (0,0)$, $V_2 = (5,0)$, $V_3 = (4,2)$, and $V_4 = (0,2)$.

$$f(P; 1, 1) = 16$$

Counting the lattice points in convex polyhedra is a powerful tool which allows many applications in areas such as Combinatorics, Statistics, Optimization, and Number Theory.

For details of how the computations are done, see [7, 9] .

1.1.2 Integration

LattE integrale has two different integration algorithms for integrating a rational polynomial $p \in \mathbb{R}[x_1, \dots, x_d]$ over a d dimensional rational polytope. The first one, called the triangulation method, triangulates the polytope into simplices and integrates over each simplex. The other method, called the cone decomposition method, integrates over each tangent cone of the polytope. In order to do this, each tangent cone is triangulated into simple cones. This is the main trade off between the two integration algorithms: you can do one (possibly) large triangulation, or (possibly) many small tangent cone triangulations.

We decompose polynomials into finite sums of powers of linear forms because integrating powers of linear forms can be done in polynomial time [2]. A decomposition of a polynomial as a sum of powers of linear forms is known as the polynomial Waring problem.

See [8] for a detailed explanation on why the next example gives the correct integral.

As an example, let us integrate the polynomial $x_1 + x_2$ over the unit square with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$. The polynomial is already a power of a linear form so let $\ell = (1, 1)$. To integrate $\int (x_1 + x_2)^M dx$ over the square, we need to compute

$$\frac{M!}{(M+d)!} |\det(u_1, \dots, u_d)| \frac{(\langle \ell, s \rangle)^{M+d}}{\prod_{i=1}^d \langle -\ell, u_i \rangle}$$

at each vertex s where the u_i are the rays from the tangent cone at s , and d is the dimension of the polytope.

Vertex $s_1 = (0, 0)$: Because $\langle \ell, s_1 \rangle^{1+2} = 0$ the valuation on this cone is zero.

Vertex $s_2 = (1, 1)$:

$$\frac{M!}{(M+d)!} |\det(u_1, \dots, u_d)| \frac{(2)^{1+2}}{(-1)(-1)} = \frac{1!}{(1+2)!} \times 1 \times 8 = 4/3$$

Vertex $s_3 = (1, 0)$:

$$\frac{M!}{(M+d)!} |\det(u_1, \dots, u_d)| \frac{(1)^{1+2}}{(1)(-1)} = \frac{1!}{(1+2)!} \times 1 \times -1 = -1/6$$

Vertex $s_4 = (0, 1)$:

$$\frac{M!}{(M+d)!} |\det(u_1, \dots, u_d)| \frac{(1)^{1+2}}{(1)(-1)} = \frac{1!}{(1+2)!} \times 1 \times -1 = -1/6$$

The integral $\int_{x_1=0}^{x_1=1} \int_{x_2=0}^{x_2=1} x_1 + x_2 dx_1 dx_2 = 0 + 4/3 - 1/6 - 1/6 = 1$ as it should be.

1.2 What can LatTE compute?

LatTE contains three key executables:

count counts lattice points, computes Ehrhart polynomials and Ehrhart series of polytopes. This executable has replaced **ehrhart**, but **ehrhart** is still included for backwards compatibility.

integrate integrates polynomials over polytopes.

max/minimize perform linear integer optimization.

The other executables in latte are drivers, converters, and other small utility function described in Section 6.3.

2 Input Files

A polytope can be defined from a list of vertices (a v-representation) or a list of hyperplane inequalities (h-representation) and so **LattE** can start from either representation in different formats. Here are four common file formats:

1. **LattE** style vertex file
2. **LattE** style hyperplane file
3. CDD style vertex file
4. CDD style hyperplane file

Users of **Polymake** will notice that **Polymake**'s facets and vertices are printed in a format that is easily converted to a **LattE** style h- or v-representation.

We now explore the file syntax of each.

2.1 LattE h-representation

2.1.1 Inequality Description

Let P be a polytope described by a system of inequalities $Ax \leq b$, where $A \in \mathbb{Z}^{m \times d}$, $A = (a_{ij})$, and $b \in \mathbb{Z}^m$. Note that any hyperplane representation with rational coefficients can be brought into this form; for example $x + 1/2y \leq 5/9$ should be written as $18x + 9y \leq 10$. With $P = \{x : Ax \leq b\}$, the input file is;

```
m d+1
b -A
```

Example: Let $P = \{(x, y) : x \leq 1, y \leq 1, x + y \leq 1, x \geq 0, y \geq 0\}$. Thus

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

and the **LattE** input file would be

```
5 3
1 -1 0
1 0 -1
1 -1 -1
0 1 0
0 0 1
```

2.1.2 Equality Constraints

By default, a constraint is an inequality of type $a^T x \leq b$. But to input an equality constraint $a^T x = b$ we need to add a keyword.

Example: Let P be as in the previous example, but require $x + y = 1$ instead of $x + y \leq 1$, thus, $P = \{(x, y) : x \leq 1, y \leq 1, x + y = 1, x \geq 0, y \geq 0\}$. Then the LattE input file that describes P would be as such:

```
5 3
1 -1 0
1 0 -1
1 -1 -1
0 1 0
0 0 1
linearity 1 3
```

The last line states that among the 5 inequalities one is to be considered an equality, the third one.

In general, the linearity syntax is :

`linearity <number of equations> <row index of constraint, start counting from 1>`

2.1.3 Nonnegativity Constraints

For bigger examples it quickly becomes cumbersome to state all nonnegativity constraints for the variables one by one. Instead, you may use another shorthand.

Example: Let P be as in the previous example, then the LattE input file that describes P could also be described as such:

```
3 3
1 -1 0
1 0 -1
1 -1 -1
linearity 1 3
nonnegative 2 1 2
```

The last line states that there are two nonnegativity constraints and that the first and second variables are required to be nonnegative. **NOTE** that the first line reads “3 3” and not “5 3” as above!

In general, the nonnegative syntax is :

`nonnegative <number of variables in list> <variable index, start counting from 1>`

2.1.4 Cost Vector

The functions maximize and minimize solve the integer linear programs

$$\max\{c^\top x : x \in P \cap \mathbb{Z}^d\}$$

and

$$\min\{c^\top x : x \in P \cap \mathbb{Z}^d\}.$$

Besides a description of the polyhedron P , these functions need a linear objective function given by a certain cost vector $c \in \mathbb{Z}^d$, where the input style is very similar to a **LattE** h-representation file.

Example: If the polyhedron is given in the file “fileName”

```
4 4
1 -1 0 0
1 0 -1 0
1 0 0 -1
1 -1 -1 -1
linearity 1 4
nonnegative 3 1 2 3
```

the cost vector must be given in the file “fileName.cost”, as for example in the following three-dimensional problem:

```
1 3
2 4 7
```

The first two entries state the size of a $1 \times n$ matrix (encoding the cost vector), followed by the $1 \times n$ matrix itself. Assuming that we call maximize, this whole data encodes the integer program

$$\max\{2x_1 + 4x_2 + 7x_3 : x_1 + x_2 + x_3 = 1, x_1, x_2, x_3 \in \{0, 1\}\}.$$

2.2 LattE v-representation

LattE can start from a homogenized v-representation of the polytope. To homogenize a vertex, simply add an leading 1 to the vertex. This has the effect of lifting the polytope to a cone in one dimension higher such that the original polytope can be extracted by intersecting the cone with the $x_1 = 1$ plane. For example, take a triangle in the plane, then Figure 2 shows the resulting cone.

Let v_1, \dots, v_k be the vertices of a polytope $P \subseteq \mathbb{R}^n$, then the **LattE** v-representation file format is:



Example: Note, like `LattE` h-representations files, a rational-vertex polytope with can be written with integer data by scaling each homogenized vertex. Below are the vertices of a rectangle $(0, 0), (2/3, 0), (0, 1/4), (2/3, 1/4)$:

2.3 CDD Input Files

10

```

H-representation
begin
4 4 integer
2 -2 4 -1
3 -2 -2 3
6 2 -4 -3
1 2 2 1
end

```

For a complete description of CDD file syntax, see the CDD manual [6].

2.4 Non-full dimensional polytopes

When the input polytope is not full dimensional, **LattE** projects that polytope such that it becomes full dimensional. This transformation preserves the lattice count and volume of the input polytope. The current **LattE integrale** cannot integrate non-full dimensional polytopes.

2.5 LattE vs. CDD file formats

There are a few key differences between **LattE** and CDD file formats.

1. CDD used the file extension *.ine for h-representation files, and *.ext for v-representation files. However, **LattE** makes no assumption on the file extensions of files. We recommend *.vrep.latte and *.hrep.latte for **LattE** style files, but you are free to name your files anything.
2. CDD also requires “H-representation” or “V-representation” keywords in the file. Forgetting about the “linearity” and “nonnegative” keywords, there is no difference between a **LattE** v- and h-representation file.

2.6 Polynomials and linear forms

LattE integrale can also integrate polynomials and in particular sums of powers of linear forms. Powers of linear forms are the fundamental structure used to integrate. Next, we describe the syntax of polynomials and linear forms

- A polynomial is represented as a list of its monomials in the form

$$[monomial_1, monomial_2, \dots],$$

where $monomial_i$ is represented by

$$[coefficient, [exponent-vector]].$$

For example, $3x_0^2x_1^4x_2^6 + 7x_1^3x_2^5$ is input as `[[3,[2,4,6]], [7,[0,3,5]]]`.

- To deal directly with sums of powers of linear forms, a fundamental data structure in `LattE integrale`, the input format is

`[linear-term1, linear-term2, ...],`

where `linear-termi` is represented by

`[coefficient,[power,[coefficient-vector]]]`.

For example, $3(2x_0+4x_1+6x_2)^{10} + 7(3x_1+5x_2)^{12}$ is input as `[[3,[10,[2,4,6]]], [7,[12,[0,3,5]]]]`.

The reason this is useful is because any polynomial can be written as a sum of powers of linear forms, see [2].

3 Running LattE

3.1 How to use count

`count` has a nice help menu, to view it, run

```
./count --help
```

The following options control what `count` computes.

- Count the number of lattice points in polytope P , where P is given in a file named “`fileName.hrep.latte`” in different file formats.

```
./count fileName.hrep.latte
./count --vrep fileName.vrep.latte
./count --cdd fileName.inc
```

- Count the number of lattice points in nP , the dilation of P by the integer factor n .

```
./count --dilation=n fileName.hrep.latte
```

- Use the homogenized Barvinok algorithm [5] to count the number of lattice points in the polytope P . Use if number of vertices of P is big compared to the number of constraints.

```
./count --homog fileName.hrep.latte
```

- Compute the number of lattice points (default)

```
./count --count-lattice-points fileName.hrep.latte
```

- Compute the multivariate generating function of the set of lattice points of the polyhedron

```
./count --multivariate-generating-function fileName.hrep.latte
```

For unbounded polyhedra, one needs to combine this with `--compute-vertex-cones=4ti2`, as other methods in `LattE` currently refuse to handle unbounded polyhedra. For example,

```
count --compute-vertex-cones=4ti2 --multivariate-generating-function fileName
```

writes the multivariate generating function (in Maple notation) to “file-Name.rat.”

- Compute the Ehrhart polynomial of an *integral* polytope

```
./count --ehrhart-polynomial fileName.hrep.latte
```

- Compute the unsimplified Ehrhart series as a univariate rational function

```
./count --ehrhart-series fileName.hrep.latte
```

- Compute the simplified Ehrhart series as a univariate rational function (needs `Maple`).

```
./count --simplified-ehrhart-series fileName.hrep.latte
```

- Compute the first N terms of the Ehrhart series

```
./count --ehrhart-taylor=N fileName.hrep.latte
```

The following options relate to the Barvinok algorithm and were introduced by Matthias Köppe in `LattE macchiato`, see [7].

- Triangulate and signed-decompose in the dual space (traditional method, default)

```
./count --dual fileName.hrep.latte
```

- Triangulate in the dual space, signed-decompose in the primal space using irrationalization

```
./count --irrational-primal fileName.hrep.latte
```

- Triangulate and signed-decompose in the primal space using irrationalization

```
./count --irrational-all-primal fileName.hrep.latte
```

This gives a new method for computing Ehrhart polynomials of integral polytopes in the primal space

```
./count --all-primal --ehrhart-polynomial fileName.hrep.latte
```

- Decompose cones down to an index (determinant) of N instead down to unimodular cones (which have an index of 1).

```
./count --maxdet=N fileName.hrep.latte
```

- Do not signed-decompose simplicial cones

```
./count --no-decomposition fileName.hrep.latte
```

- Use polynomial substitution for specialization (traditional method, default)

```
./count --polynomial fileName.hrep.latte
```

- Use exponential substitution for specialization (recommended for maxdet larger than 1)

```
./count --exponential fileName.hrep.latte
```

REMARK The functionality of the LattE v1.2 `ehrhart` command has been merged into `count`:

```
count --ehrhart-series FILENAME
```

(replaces: `ehrhart FILENAME`)

```
count --simplified-ehrhart-series FILENAME
```

(replaces: `ehrhart simplify FILENAME`)

```
count --ehrhart-taylor=N FILENAME
```

(replaces: `ehrhart N FILENAME`)

The `ehrhart` program is still available, but it does not accept the new command-line options of `count`.

3.2 How to use integrate

Like `count`, `integrate` has a help menu. To view the menu, run

```
./integrate --help
```

There are two different integration (and volume) algorithms. The triangulation method triangulates the entire polytope and integrates over each simplex. In the cone decomposition method we integrate over each cone, possibly triangulating it first. Unlike other integration software, `LattE` integrates polynomials and powers of linear forms in exact arithmetic.

- Integrates using the cone-decomposition method.

```
--cone-decompose
```

- Integrates using the triangulation method.

```
--triangulate
```

- Sets what you want to compute: a volume or an integral.

```
--valuation=integrate  
--valuation=volume
```

- Prints the cone-decomposition rational function for volumes

```
--valuation=volume --cone-decompose --print-cone-decompose-function
```

- Sets the file that contains the polynomial or powers of linear forms. If this option is not set, and the valuation is integration, the polynomial or powers of linear forms will be read from stdin.

```
--monomials=FILE  
--linear-forms=FILE
```

Example: Let us view a few examples of the above options

- Integrates a polynomial in file “FILE” using the triangulation method.

```
./integrate --valuation=integral --triangulate --monomials=FILE fileName.hrep.latte
```

- Find a volume using the cone decomposition method from a `LattE` v-representation file.

```
./integrate --valuation=volume --cone-decompose --vrep fileName.vrep.latte
```

- If an integration method is not given, `LattE integrale` computes the integral with *both* methods. This can also be done by the `--all` option. The next two commands do the same thing: find a volume using both methods from a `LattE` v-representation file.

```
./integrate --valuation=volume --vrep fileName.vrep.latte
./integrate --valuation=volume --all --vrep fileName.vrep.latte
```

3.3 Options common to both count and integrate

A common subproblem in counting lattice points and integration requires finding triangulations and tangent cones. Also, there are many different software tools available to do this. Instead of reinventing the wheel, `LattE` links with other software tools to compute these basic objects. In this section, we describe how you can control which software tool is used.

- The `4ti2` program can be used instead of `cddlib` and `CDD+` to compute the vertex cones of polytopes, triangulations, and duals of cones. In many cases, `4ti2` is faster.

```
--compute-vertex-cones={cdd,4ti2}
--triangulation={cddlib,4ti2}
--dualization={cdd,4ti2}
```

- By default, `LattE` assumes the h-representation may contain redundant hyperplanes and tries to find and remove them. You can control how much more `LattE` should spend checking the input h-representation with the following option.

```
--redundancy-check={none,cddlib,full-cddlib}.
```

- “full-cddlib” (the default) uses `cddlib` to compute an irredundant system of linear equations and inequalities describing the polyhedron. This corresponds to the traditional `LattE` behavior; it can be expensive.
- “cddlib” (used to be the default in the 1.2+mk-0.9.x series) uses `cddlib` to compute some implicit linearities only; it often fails but is faster than full-cddlib.
- “none” does nothing, the input description of the polytope should be irredundant.

3.4 Optimization

LattE can also optimize over the integer points of a polytope. However, this part of the software is not as stable as the rest of the code. The optimization executables **require** a cost vector specified in “fileName.cost” if the polytope file is named “fileName.”

- Maximizes/Minimizes a given linear cost function over the lattice points in the polytope. The Digging algorithm [5] is used. Optimal point and optimal value is returned.

```
./latte-maximize fileName  
./latte-minimize fileName
```

- Maximizes/Minimizes a given linear cost function over the lattice points in the polytope. The Binary search algorithm is used. Only optimal value is returned.

```
./latte-maximize bbs fileName  
./latte-minimize bbs fileName
```

4 Downloading and Installing LattE

LattE is downloadable from the following website:

<http://www.math.ucdavis.edu/~latte/>

LattE uses the GNU Autoconf and Automake tools. Please see the **README** file in the **LattE** directory for detailed directions for installing **LattE**.

5 A Brief Tutorial

In this section we invite the reader to follow along a few examples that show how to use **LattE** and also how to counter-check results.

5.1 Counting Magic Squares

Our first example deals with counting magic 4×4 squares. We call a 4×4 array of nonnegative numbers a magic square if the sums of the 4 entries along each row, along each column and along the two main diagonals equals the same number s , the magic constant. Let us start with counting magic 4×4 squares that have the magic constant 1. Associating variables x_1, \dots, x_{16} with the 16 entries, the conditions of a magic 4×4 square of magic sum 1 can be encoded into the following input file “EXAMPLES/magic4x4” for **LattE**.

```

10 17
1 -1 -1 -1 -1 0 0 0 0 0 0 0 0 0 0 0
1 0 0 0 0 -1 -1 -1 -1 0 0 0 0 0 0 0
1 0 0 0 0 0 0 0 0 -1 -1 -1 -1 0 0 0
1 0 0 0 0 0 0 0 0 0 0 0 0 -1 -1 -1
1 -1 0 0 0 -1 0 0 0 -1 0 0 0 -1 0 0
1 0 -1 0 0 0 -1 0 0 0 -1 0 0 0 -1 0
1 0 0 -1 0 0 0 -1 0 0 0 -1 0 0 0 -1
1 0 0 0 -1 0 0 0 -1 0 0 0 -1 0 0 0
1 -1 0 0 0 0 -1 0 0 0 0 -1 0 0 0 0
1 0 0 0 -1 0 0 -1 0 0 -1 0 0 0 0
linearitiy 10 1 2 3 4 5 6 7 8 9 10
nonnegative 16 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

```

Now we simply invoke the counting function of `LattE` by typing:

```
./count EXAMPLES/magic4x4
```

The last couple of lines that `LattE` prints to the screen look as follows:

```

Total Unimodular Cones: 418
Maximum number of simplicial cones in memory at once: 27

***** Total number of lattice points: 8 *****

Computation done.
Time: 1.24219 sec

```

Therefore, there are exactly 8 magic 4×4 squares that have the magic constant 1. This is not yet impressive, as we could have done that by hand. Therefore, let us try and find the corresponding number for the magic constant 12. Since this problem is a dilation (by factor 12) of the original problem, we do not have to create a new file. Instead, we use the option “dilation” to indicate that we want to count the number of lattice points of a dilation of the given polytope:

```
./count --dilation=12 EXAMPLES/magic4x4
```

The last couple of lines that `LattE` prints to the screen look as follows:

```

Total Unimodular Cones: 418
Maximum number of simplicial cones in memory at once: 27

***** Total number of lattice points: 225351 *****

Computation done.
Time: 1.22656 sec

```

Therefore, there are exactly 225351 magic 4×4 squares that have the magic constant 12. (We would NOT want to do THAT one by hand, would we?!)

Here is some amazing observation: the running time of **LatTE** is roughly the same for counting magic squares of sum 1 and of sum 12. This phenomenon is due to the fact that the main part of the computation, the creation of the generating function that encodes all lattice points in the polytope, is nearly identical in both cases.

Although we may be already happy with these simple counting results, let us be a bit more ambitious and let us find a counting formula that, for given magic sum s , returns the number of magic 4×4 squares that have the magic constant s .

For this, simply type (note that **LatTE** invokes **Maple** to simplify intermediate expressions):

```
./count --simplified-ehrhart-series EXAMPLES/magic4x4
```

The last couple of lines that **LatTE** prints to the screen looks as follows:

```
Rational function written to EXAMPLES/magic4x4.rat
```

```
Computation done.  
Time: 0.724609 sec
```

We are informed that this call created a file “EXAMPLES/magic4x4.rat” containing the Ehrhart series as a rational function:

```
(t^8+4*t^7+18*t^6+36*t^5+50*t^4+36*t^3+18*t^2+4*t+1)/(-1+t)^4/(-1+t^2)^4
```

Now we could use **Maple** (or your favorite computer algebra software) to find a series expansion of this expression.

$$\begin{aligned} & \frac{t^8 + 4 * t^7 + 18 * t^6 + 36 * t^5 + 50 * t^4 + 36 * t^3 + 18 * t^2 + 4 * t + 1}{(-1 + t)^4(-1 + t^2)^4} \\ = & 1 + 8t^1 + 48t^2 + 200t^3 + 675t^4 + 1904t^5 + 4736t^6 + 10608t^7 + 21925t^8 + \\ & 42328t^9 + 77328t^{10} + 134680t^{11} + 225351t^{12} + 364000t^{13} + 570368t^{14} + \\ & 869856t^{15} + O(t^{16}) \end{aligned}$$

The summands $8t$ and $225351t^{12}$ reconfirm our previous counts.

Although this rational function encodes the full Ehrhart series, it is not always as easy to compute as for magic 4×4 squares. As it turns out, adding and simplifying rational functions, although in just one variable t , can be extremely

costly due to the high powers in t and due to long integer coefficients that appear.

However, even if we cannot compute the full Ehrhart series, we can at least try and find the first couple of terms of it.

```
./count --ehrhart-taylor=15 EXAMPLES/magic4x4
```

The last couple of lines that **LatTE** prints to the screen look as follows:

```
Memory Save Mode: Taylor Expansion:
```

```
1
8t^1
48t^2
200t^3
675t^4
1904t^5
4736t^6
10608t^7
21925t^8
42328t^9
77328t^10
134680t^11
225351t^12
364000t^13
570368t^14
869856t^15
Computation done.
Time: 1.83789 sec
```

Again, our previous counts are reconfirmed.

Nice, but the more terms we want to compute the more time-consuming this task becomes. Clearly, if we could find sufficiently many terms, we could compute the full Ehrhart series expansion in terms of a rational function by interpolation.

5.2 Counting Lattice Points in the 24-Cell

Our next example deals with a well-known combinatorial object, the 24-cell. Its description is given in the file “EXAMPLES/24_cell”:

```
24 5
2 -1 1 -1 -1
1 0 0 -1 0
2 -1 1 -1 1
```

```

2 -1 1 1 1
1 0 0 0 1
1 0 1 0 0
2 1 -1 1 -1
2 1 1 -1 1
2 1 1 1 1
1 1 0 0 0
2 1 1 1 -1
2 1 1 -1 -1
2 1 -1 1 1
2 1 -1 -1 1
2 1 -1 -1 -1
1 0 0 1 0
2 -1 1 1 -1
1 0 0 0 -1
2 -1 -1 1 -1
1 0 -1 0 0
2 -1 -1 1 1
2 -1 -1 -1 1
2 -1 -1 -1 -1
1 -1 0 0 0

```

Now we invoke the counting function of LattE by typing:

```
./count EXAMPLES/24_cell
```

The last couple of lines that LattE prints to the screen look as follows:

```

Total Unimodular Cones: 240
Maximum number of simplicial cones in memory at once: 30

***** Total number of lattice points: 33 *****

Computation done.
Time: 0.429686 sec

```

Therefore, there are exactly 33 lattice points in the 24-cell. We get the same result by using the homogenized Barvinok algorithm:

```
./count --homog EXAMPLES/24_cell
```

The last couple of lines that LattE prints to the screen look as follows:

```
Memory Save Mode: Taylor Expansion:
```

```
**** Total number of lattice points is: 33 ****
```

```
Computation done.  
Time: 0.957031 sec
```

5.3 Integrating over a polytope

Let us integrate the polynomial $w^2x^2y^4z^8 - 3/8x^2$ and the power of a linear form $3(w + 2x + 4y + 6z)^{10}$ over the 24-cell.

Create a file named “even.polynomial” that has on its first line the polynomial. See Section 2.6 for a review of the syntax.

```
[[1,[2,2,4,8]], [-3/8,[0,2,0,0]]]
```

After running the integration command using the triangulation method

```
./integrate --valuation=integrate --triangulate --monomials=even.polynomial 24_cell
```

we see that the two monomials were decomposed into 406 powers of linear forms and the answer is

starting to integrate 406 linear forms.

```
Integration (using the triangulation method)  
  Answer: -110535307/170059500  
  Decimal: -0.64998019516698567266162725399052  
  Time: 1.92 sec  
Computational time (algorithms + processing + program control)  
  Total time: 2.00 sec
```

Now create a new file named “power10.linearforms” that has in its first line the power of a linear form:

```
[[3,[10,[1,2,4,6]]]]
```

Then integrate this power of a linear form over the 24_cell using the cone decomposition method with the following command:

```
./integrate --cone-decompose --linear-forms=power10.linearforms 24_cell
```

We see the answer is computed very quickly.

Integration (using the cone decomposition method)
 Answer: 59555515086/77
 Decimal: 773448247.87012987012987012987013
 Time: 0.02 sec
 Computational time (algorithms + processing + program control)
 Total time: 0.07 sec

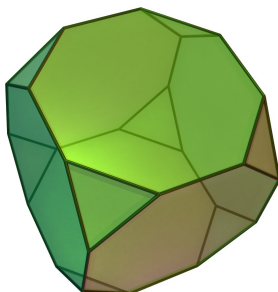


Figure 3: The truncated cube.

For the next example, consider the truncated cube in Figure 3

The vertices are

24 4
 1 3 1 1
 1 3 1 -1
 1 3 -1 1
 1 3 -1 -1
 1 -3 1 1
 1 -3 1 -1
 1 -3 -1 1
 1 -3 -1 -1
 1 1 3 1
 1 1 3 -1
 1 1 -3 1
 1 1 -3 -1
 1 -1 3 1
 1 -1 3 -1
 1 -1 -3 1
 1 -1 -3 -1
 1 1 1 3
 1 1 1 -3
 1 1 -1 3
 1 1 -1 -3
 1 -1 1 3
 1 -1 1 -3
 1 -1 -1 3

```
1 -1 -1 -3
```

This time, let us enter the polynomial $x^{40}y^{40}z^{40}$ from stdin, which will be decomposed into 68,920 powers of linear forms. Run

```
./integrate --cone-decompose --triangulation=4ti2 --vrep truncatedCube.vrep.latte
```

and type

```
p [[1,[40,40,40]]]
```

We see the exact answer is

```
93991283632941965714919247928639002510318209692293688827363993265109276641003769553256
-----
2795239135836124463932439643671211584534957465679791608181565
```

This answer displays the power of using exact rational arithmetic!

5.4 Example of Optimization with Latte

Next, let us solve the problem “cuww1” [4, 5]. Its description is given in the file “EXAMPLES/cuww1”:

```
1 6
89643482 -12223 -12224 -36674 -61119 -85569
linearity 1 1
nonnegative 5 1 2 3 4 5
```

The cost function can be found in the file “EXAMPLES/cuww1.cost”:

```
1 5
213 -1928 -11111 -2345 9123
```

Now let us maximize this cost function over the given knapsack polytope. Note that by default, the digging algorithm as described in [5] is used.

```
./latte-maximize EXAMPLES/cuww1
```

The last couple of lines that Latte prints to the screen look as follows:


```
Finished computing a rational function.  
Time: 0.158203 sec.
```

```
There is one optimal solution.
```

```
No digging.  
An optimal solution for [213 -1928 -11111 -2345 9123] is: [7334 0 0 0 0].  
The projected down opt value is: 191928257104  
The optimal value is: 1562142.  
The gap is: 7995261.806  
Computation done.  
Time: 0.203124 sec.
```

The solution (7334, 0, 0, 0, 0) is quickly found. Now let us try to find the optimal value again by a different algorithm, the binary search algorithm.

```
./latte-maximize bbs EXAMPLES/cuww1
```

The last couple of lines that **LattE** prints to the screen look as follows:

```
Total of Iterations: 26  
The total number of unimodular cones: 125562  
The optimal value: 1562142  
  
The number of optimal solutions: 1  
Time: 0.042968
```

Note that we get the same optimal value, but no optimal solution is provided.

6 Release Information

6.1 System Requirements

6.1.1 Platform

The binaries for **LattE** v1.1 as well as for **cdd** (by K. Fukuda [6]) are for platforms that run Unix, that is, Mac and Linux.

6.1.2 Memory Requirements

The memory requirements are essentially problem dependent; however, **LattE count** runs in a “memory saving mode” whenever appropriate. In this mode **count** rarely uses more than around 20 MB beyond the amount of memory needed to calculate triangulations.

6.2 Additional Maple Connection

The call

```
./count --simplified-ehrhart-series fileName
```

requires **Maple** for simplifications of expressions. It should be sufficient to have a copy of **Maple** installed on your machine, without any additional special configuration required. **LatTE** will still run even if **Maple** is not installed, but this simplification feature to “count” will not be available.

We have tested this connection with **Maple** 5.1, 8.0, and 14.0 and experienced no problem. Please let us know about any problem you experience with our connection to **Maple**.

6.3 File Descriptions

The **LatTE** Linux release consists a single archive file, **latte_v1.*.tar.gz**. The archive contains the following files:

Name	Description
count ehrhart latte-maximize latte-minimize integrate simplify2.add redcheck install cdd	Main program executables additional files used by LatTE (← modified from cdd library) external cdd software executable used by LatTE , original name in cdd : cddr+ -gmp
EXAMPLES/	Subdirectory containing sample input files
manual.pdf	User’s guide
code/	LatTE source code (including NTL , cdd and cdd lib)
gpl.txt	GNU General Public License agreement

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6.5 How to Cite Latte

Although Latte is free software, your acknowledgment is requested. If Latte is useful in your research or applications please acknowledge it by referencing this manual as

De Loera, J.A., Dutra, B., Köppe, M., Moreinis, S., Pinto, G., Wu, J.
A User's Guide for Latte integrale v1.5, 2011, software package Latte is available at <http://www.math.ucdavis.edu/~latte/>

6.6 The Latte Team

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- Prof. Nicole Berline

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- Jeremy Tauzer (`LattE v1.2`)
- Jonathan Brooks (`LattE v1.2`)
- Carol Shih (`LattE v1.2`)
- Esteban Pauli (`LattE v1.2`)
- Mike Zhang (`LattE v1.2`)

6.7 Acknowledgments

`LattE` currently uses many wonderful pieces of software. First is `cdd` [6], developed by Komei Fukuda, whose webpage can be found at:

<http://www.cs.mcgill.ca/~fukuda/>

Next, `LattE` uses `4ti2` [1] whose webpage can be found at:

<http://www.4ti2.de>

`cdd` and `4ti2` is used for finding vertices of polytopes and the triangulation of cones.

In addition, `LattE` currently uses NTL, a Library for doing Number Theory, written by Victor Shoup [10], for LLL algorithm, matrix manipulations, storing variable length integers, and floating point numbers. NTL can be found at:

<http://shoup.net/ntl/>

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Version 2, June 1991

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one line to give the program’s name and a brief idea of what it does.
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Also add information on how to contact you by electronic and paper mail.

If the program is interactive, make it output a short notice like this when it starts in an interactive mode:

```
Gnomovision version 69, Copyright (C) yyyy name of author
Gnomovision comes with ABSOLUTELY NO WARRANTY; for de-
tails type 'show w'.
This is free software, and you are welcome to redistribute it under
certain conditions; type 'show c' for details.
```

The hypothetical commands `show w` and `show c` should show the appropriate parts of the General Public License. Of course, the commands you use may be called something other than `show w` and `show c`; they could even be mouse-clicks or menu items—whatever suits your program.

You should also get your employer (if you work as a programmer) or your school, if any, to sign a “copyright disclaimer” for the program, if necessary. Here is a sample; alter the names:

```
Yoyodyne, Inc., hereby disclaims all copyright interest in the pro-
gram
'Gnomovision' (which makes passes at compilers) written by James
Hacker.
```

```
signature of Ty Coon, 1 April 1989
Ty Coon, President of Vice
```

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