A User's Guide for `LattE` v1.1 (Linux Release) *

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Contents

1 Introduction

1.1 What is `LattE`?

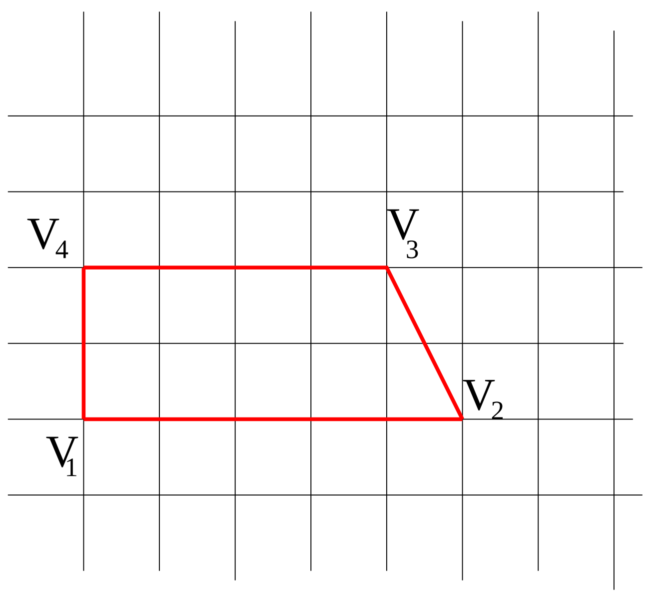
The name "`LattE`" is an abbreviation for "**Latt**ice point **E**numeration." So what exactly does `LattE` do? The software's main function is to count the lattice points contained in convex polyhedra defined by linear equations and inequalities with integer coefficients. The polyhedra can be of any (reasonably small) dimension, and `LattE` uses an algorithm that runs in polynomial time for fixed dimension: Barvinok's algorithm [1]. To learn more about the exact details of our implementation and algorithmic techniques involved, the interested reader can consult [3, 4, 5] and the references listed therein. Here we give a rather short description of the mathematical objects used by `LattE`, **Barvinok's Rational Functions**:

Given a convex polyhedron $P = \{u \in \mathbb{R}^d : Au \leq b\}$, where A and b are integral, the fundamental object that we compute is a short representation of the infinite power series:

$$f(P;x) \quad = \sum_{\alpha \in P \cap \mathbb{Z}^d} x_1^{\alpha_1} x_2^{\alpha_2} \dots x_d^{\alpha_d}.$$

Here each lattice point is given by one monomial. Note that this can be a rather long sum, in fact for a polyhedral cone it can be infinite, but the good news is that it admits short representations.

Example: Let P be the quadrangle with vertices $V_1 = (0,0)$, $V_2 = (5,0)$, $V_3 = (4,2)$, and $V_4 = (0,2)$.



$$f(P;x,y) = x^5+x^4y+x^4+x^4y^2+yx^3+x^3+x^3y^2+yx^2+x^2+x^2y^2+xy+x+xy^2+y+1+y^2$$

The fundamental theorem of Barvinok (circa 1993, see [1]) says that you can write $f(P;x)$ as a sum of short rational functions, in polynomial time when the dimension of the polyhedron is fixed. In our running example we easily see that the 16 monomial polynomial can be written as shorter rational function sum:

$$f(P;x,y) = f(K_{V_1};x,y) + f(K_{V_2};x,y) + f(K_{V_3};x,y) + f(K_{V_4};x,y)$$

where

$$f(K_{V_1};x,y) = \frac{1}{(1-x)(1-y)} \quad f(K_{V_2};x,y) = \frac{(x^5+x^4y)}{(1-x^{-1})(1-y^2x^{-1})}$$

$$f(K_{V_3};x,y) = \frac{(x^4y^2+x^4)}{(1-x^{-1})(1-xy^{-2})} \quad f(K_{V_4};x,y) = \frac{y^2}{(1-y^{-1})(1-x)}$$

$$f(P;1,1) = 16$$

Counting the lattice points in convex polyhedra is a powerful tool which allows many applications in areas such as Combinatorics, Statistics, Optimization, and Number Theory.

1.2 What can `LattE` compute?

In the following we list the operations that `LattE` v1.1 can perform on bounded convex polyhedra (more commonly referred to as *polytopes*). For the reader's convenience, we already include the basic commands to actually do the tasks. Let us assume that a description of a polytope P is given in the file "fileName" (see Section 3 for format) and that a cost vector is specified in the file "fileName.cost" (needed for the optimization part, see Section 3 for format).

Tasks performed by `LattE` v1.1:

1. Count the number of lattice points in P.

   ```
   ./count fileName
   ```

2. Count the number of lattice points in nP, the dilation of P by the integer factor n.

   ```
   ./count dil n fileName
   ```

3. Calculate a rational function that encodes the *Ehrhart series* associated with the polytope. By definition, the n-th coefficient in the Ehrhart series equals the number of lattice points in nP. For more details on Ehrhart counting functions see, for example, Chapter 4 of [9].

   ```
   ./ehrhart fileName
   ```

4. Calculate the first $n + 1$ terms of the Ehrhart series associated with the polytope.

```
./ehrhart n fileName
```

5. Maximize or minimize a given linear function of the lattice points in P.

```
./maximize fileName
./minimize fileName
```

In addition to these basic functions, there are more specific calls to `LattE`. For example to use the homogenized Barvinok algorithm instead of the original one in order to count the lattice points. These details will be explained in Section 4.

2 Downloading and Installing LattE

LattE is downloadable from the following website:

http://www.math.ucdavis.edu/~latte/downloads/

Step 1: Create directory for LattE

```
mkdir latte
```

Step 2: Download "latte_v1.1.tar.gz" to directory "latte"

```
Download ``latte_v1.1.tar.gz'' from
```

http://www.math.ucdavis.edu/~latte/downloads/

(If you have never downloaded a file from the internet: A click with your right mouse button onto the file name on the webpage should do the trick. In any case, if you do not succeed, ask your system administrator, a friend, or send us an email.)

Step 3: Change to directory for "latte"

```
cd latte
```

Step 4: Unzip and untar the archive

```
gunzip latte_v1.1.tar.gz
tar xvf latte_v1.1.tar
```

Step 5: Make "install" executable

```
chmod 700 install
```

Step 6: Install LattE

```
./install
```

2.1 Installing LattE from source code

LattE is downloadable from the following website:

http://www.math.ucdavis.edu/~latte/downloads/

Step 1: Create directory for LattE

```
mkdir latte
```

Step 2: Download "latte_v1.1.tar.gz" to directory "latte"

```
Download ``latte_v1.1.tar.gz'' from
```

http://www.math.ucdavis.edu/~latte/downloads/

(If you have never downloaded a file from the internet: A click with your right mouse button onto the file name on the webpage should do the trick. In any case, if you do not succeed, ask your system administrator, a friend, or send us an email.)

Step 3: Change to directory for "latte"

```
cd latte
```

Step 4: Unzip and untar the archive

```
gunzip latte_v1.1.tar.gz
tar xvf latte_v1.1.tar
```

Step 5: Change a directory

```
cd code
```

Step 6: Make "install" executable

```
chmod 700 install
```

Step 7: Install LattE

```
 ./install
```

Step 8: Change a directory

```
cd LattE
```

Step 9: Copy all binaries to the `LattE` directory

```
cp * ../../
```

Step 10: Go to the `LattE` directory

```
cd ../../
```

3 Input Files

3.1 LattE Input Files

3.1.1 Inequality Description

For computations involving a polytope P described by a system of inequalities $Ax \leq b$, where $A \in \mathbb{Z}^{m \times d}$, $A = (a_{ij})$, and $b \in \mathbb{Z}^m$, the LattE readable input file would be as follows:

```
m d+1
b  -A
```

EXAMPLE. Let $P = \{(x, y) : x \leq 1, y \leq 1, x + y \leq 1, x \geq 0, y \geq 0\}$. Thus

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

and the LattE input file would be as such:

```
5 3
1 -1  0
1  0 -1
1 -1 -1
0  1  0
0  0  1
```

3.1.2 Equations

In LattE, polytopes are represented by **linear constraints**, i.e. equalities or inequalities. By default a constraint is an inequality of type $ax \leq b$ unless we specify, by using a single additional line, the line numbers of constraints that are linear equalities.

EXAMPLE. Let P be as in the previous example, but require $x+y = 1$ instead of $x + y \leq 1$, thus, $P = \{(x, y) : x \leq 1, y \leq 1, x + y = 1, x \geq 0, y \geq 0\}$. Then the LattE input file that describes P would be as such:

```
5  3
1 -1  0
1  0 -1
```

```
1 -1 -1
0  1  0
0  0  1
linearity 1 3
```

The last line states that among the 5 inequalities one is to be considered an equality, the third one.

3.1.3 Nonnegativity Constraints

For bigger examples it quickly becomes cumbersome to state all nonnegativity constraints for the variables one by one. Instead, you may use another shorthand.

EXAMPLE. Let P be as in the previous example, then the `LattE` input file that describes P could also be described as such:

```
3  3
1 -1  0
1  0 -1
1 -1 -1
linearity 1 3
nonnegative 2 1 2
```

The last line states that there are two nonnegativity constraints and that the first and second variables are required to be nonnegative. **NOTE** that the first line reads "3 3" and not "5 3" as above!

3.1.4 Cost Vector

The functions maximize and minimize solve the integer linear programs

$$\max\{c^\intercal x : x \in P \cap \mathbb{Z}^d\}$$

and

$$\min\{c^\intercal x : x \in P \cap \mathbb{Z}^d\}.$$

Besides a description of the polyhedron P, these functions need a linear objective function given by a certain cost vector c. If the polyhedron is given in the file "fileName"

```
4  4
1 -1  0  0
1  0 -1  0
1  0  0 -1
```

```
1 -1 -1 -1
linearity 1 4
nonnegative 3 1 2 3
```

the cost vector must be given in the file "fileName.cost", as for example in the following three-dimensional problem:

```
1 3
2 4 7
```

The first two entries state the size of a $1 \times n$ matrix (encoding the cost vector), followed by the $1 \times n$ matrix itself. Assuming that we call maximize, this whole data encodes the integer program

$$\max\{2x_1 + 4x_2 + 7x_3 : x_1 + x_2 + x_3 = 1, x_1, x_2, x_3 \in \{0, 1\}\}.$$

4 Running LattE

4.1 Command Syntax

The basic syntax to invoke the various functions of `LattE` is:

```
./count fileName
./ehrhart fileName
./maximize fileName
./minimize fileName
```

Note that the last two functions require a cost vector specified in the file "fileName.cost"!

Additionally, a variety of options can be used. All options should be space-delimited in the command.

One option that can be set in addition to the options given below is "cdd" which tells `LattE` to read its input from a `cdd` input file. Thus, the above invocations for `cdd` input files would be

```
./count cdd fileName
./ehrhart cdd fileName
./maximize cdd fileName
./minimize cdd fileName
```

4.2 Counting

- Count the number of lattice points in polytope P, where P is given in "fileName".

  ```
  ./count fileName
  ```

- Count the number of lattice points in nP, the dilation of P by the integer factor n.

  ```
  ./count dil n fileName
  ```

- Count the number of lattice points in the interior of the polytope P, where P is given in "fileName".

  ```
  ./count int fileName
  ```

- Use the homogenized Barvinok algorithm [5] to count the number of lattice points in the polytope P, where P is given in "fileName". Use if number of vertices of P is big compared to the number of constraints.

  ```
  ./count homog fileName
  ```

4.3 Ehrhart Series

- Compute the Ehrhart series encoded as a rational function for the polytope given in “fileName”. Writes the unsimplified rational function to file “fileName.rat”.

```
./ehrhart fileName
```

- Compute the Ehrhart series encoded as a rational function for the polytope given in “fileName”. **NEEDS** Maple for simplification of terms. Writes the simplified rational function to file “fileName.rat”.

```
./ehrhart simplify fileName
```

- Compute the Taylor series expansion of Ehrhart generating function up to degree n for the polytope given in “fileName”.

```
./ehrhart n fileName
```

4.4 Optimizing

This functions **NEEDS** a cost vector specified in “fileName.cost”!!!

- Maximizes/Minimizes given linear cost function over the lattice points in the polytope given in “fileName”. Digging algorithm [5] is used. Optimal point and optimal value is returned.

```
./maximize fileName
./minimize fileName
```

- Maximizes/Minimizes given linear cost function over the lattice points in the polytope given in “fileName”. Binary search algorithm is used. Only optimal value is returned.

```
./maximize bbs fileName
./minimize bbs fileName
```

5 A Brief Tutorial

In this section we invite the reader to follow along a few examples that show how to use LattE and also how to counter-check results.

5.1 Counting Magic Squares

Our first example deals with counting magic 4×4 squares. We call a 4×4 array of nonnegative numbers a magic square if the sums of the 4 entries along each row, along each column and along the two main diagonals equals the same number s, the magic constant. Let us start with counting magic 4×4 squares that have the magic constant 1. Associating variables $x_1, \ldots, x_{16}$ with the 16 entries, the conditions of a magic 4×4 square of magic sum 1 can be encoded into the following input file "EXAMPLES/magic4x4" for LattE.

```
10 17
1 -1 -1 -1 -1  0  0  0  0  0  0  0  0  0  0  0  0
1  0  0  0  0 -1 -1 -1 -1  0  0  0  0  0  0  0  0
1  0  0  0  0  0  0  0  0 -1 -1 -1 -1  0  0  0  0
1  0  0  0  0  0  0  0  0  0  0  0  0 -1 -1 -1 -1
1 -1  0  0  0 -1  0  0  0 -1  0  0  0 -1  0  0  0
1  0 -1  0  0  0 -1  0  0  0 -1  0  0  0 -1  0  0
1  0  0 -1  0  0  0 -1  0  0  0 -1  0  0  0 -1  0
1  0  0  0 -1  0  0  0 -1  0  0  0 -1  0  0  0 -1
1 -1  0  0  0  0 -1  0  0  0  0 -1  0  0  0  0 -1
1  0  0  0 -1  0  0 -1  0  0 -1  0  0 -1  0  0  0
linearity 10 1 2 3 4 5 6 7 8 9 10
nonnegative 16 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
```

Now we simply invoke the counting function of LattE by typing:

```
./count EXAMPLES/magic4x4
```

The last couple of lines that LattE prints to the screen look as follows:

```
Total Unimodular Cones: 418
Maximum number of simplicial cones in memory at once: 27

*****  Total number of lattice points: 8  ****

Computation done.
Time: 1.24219 sec
```

Therefore, there are exactly 8 magic 4×4 squares that have the magic constant 1. This is not yet impressive, as we could have done that by hand. Therefore, let us try and find the corresponding number for the magic constant 12. Since this problem is a dilation (by factor 12) of the original problem, we do not have to create a new file. Instead, we use the option "dil" to indicate that we want to count the number of lattice points of a dilation of the given polytope:

```
./count dil 12 EXAMPLES/magic4x4
```

The last couple of lines that `LattE` prints to the screen look as follows:

```
Total Unimodular Cones: 418
Maximum number of simplicial cones in memory at once: 27

*****  Total number of lattice points: 225351  ****

Computation done.
Time: 1.22656 sec
```

Therefore, there are exactly 225351 magic 4×4 squares that have the magic constant 12. (We would NOT want to do THAT one by hand, would we?!)

Here is some amazing observation: the running time of `LattE` is roughly the same for counting magic squares of sum 1 and of sum 12. This phenomenon is due to the fact that the main part of the computation, the creation of the generating function that encodes all lattice points in the polytope, is nearly identical in both cases.

Although we may be already happy with these simple counting results, let us be a bit more ambitious and and let us find a counting formula that, for given magic sum s, returns the number of magic 4×4 squares that have the magic constant s.

For this, simply type (note that `LattE` invokes `Maple` to simplify intermediate expressions):

```
./ehrhart simplify EXAMPLES/magic4x4
```

The last couple of lines that `LattE` prints to the screen looks as follows:

```
Rational function written to EXAMPLES/magic4x4.rat

Computation done.
Time: 0.724609 sec
```

We are informed that this call created a file "EXAMPLES/magic4x4.rat" containing the Ehrhart series as a rational function:

```
(t^8+4*t^7+18*t^6+36*t^5+50*t^4+36*t^3+18*t^2+4*t+1)/(-1+t)^4/(-1+t^2)^4
```

Now we could use `Maple` (or your favorite computer algebra software) to find a series expansion of this expression.

$$\begin{aligned}&\frac{t^8+4*t^7+18*t^6+36*t^5+50*t^4+36*t^3+18*t^2+4*t+1}{(-1+t)^4(-1+t^2)^4}\\ =\;& 1+8t^1+48t^2+200t^3+675t^4+1904t^5+4736t^6+10608t^7+21925t^8+\\ & 42328t^9+77328t^{10}+134680t^{11}+225351t^{12}+364000t^{13}+570368t^{14}+\\ & 869856t^{15}+O(t^{16})\end{aligned}$$

The summands $8t$ and $225351t^{12}$ reconfirm our previous counts.

Although this rational function encodes the full Ehrhart series, it is not always as easy to compute as for magic 4×4 squares. As it turns out, adding and simplifying rational functions, although in just one variable t, can be extremely costly due to the high powers in t and due to long integer coefficients that appear.

However, even if we cannot compute the full Ehrhart series, we can at least try and find the first couple of terms of it.

```
./ehrhart 15 EXAMPLES/magic4x4
```

The last couple of lines that `LattE` prints to the screen look as follows:

```
Memory Save Mode: Taylor Expansion:
1
8t^1
48t^2
200t^3
675t^4
1904t^5
4736t^6
10608t^7
21925t^8
42328t^9
77328t^10
134680t^11
225351t^12
364000t^13
570368t^14
869856t^15
Computation done.
Time: 1.83789 sec
```

Again, our previous counts are reconfirmed.

Nice, but the more terms we want to compute the more time-consuming this task becomes. Clearly, if we could find sufficiently many terms, we could compute the full Ehrhart series expansion in terms of a rational function by interpolation.

5.2 Counting Lattice Points in the 24-Cell

Our next example deals with a well-known combinatorial object, the 24-cell. Its description is given in the file "EXAMPLES/24_cell":

```
24 5
2 -1  1 -1 -1
1  0  0 -1  0
2 -1  1 -1  1
2 -1  1  1  1
1  0  0  0  1
1  0  1  0  0
2  1 -1  1 -1
2  1  1 -1  1
2  1  1  1  1
1  1  0  0  0
2  1  1  1 -1
2  1  1 -1 -1
2  1 -1  1  1
2  1 -1 -1  1
2  1 -1 -1 -1
1  0  0  1  0
2 -1  1  1 -1
1  0  0  0 -1
2 -1 -1  1 -1
1  0 -1  0  0
2 -1 -1  1  1
2 -1 -1 -1  1
2 -1 -1 -1 -1
1 -1  0  0  0
```

Now we invoke the counting function of `LattE` by typing:

```
./count EXAMPLES/24_cell
```

The last couple of lines that `LattE` prints to the screen look as follows:

```
Total Unimodular Cones: 240
Maximum number of simplicial cones in memory at once: 30
```

```
*****  Total number of lattice points: 33  ****

Computation done.
Time: 0.429686 sec
```

Therefore, there are exactly 33 lattice points in the 24-cell. We get the same result by using the homogenized Barvinok algorithm:

```
./count homog EXAMPLES/24_cell
```

The last couple of lines that `LattE` prints to the screen look as follows:

```
Memory Save Mode: Taylor Expansion:

****  Total number of lattice points is: 33  ****

Computation done.
Time: 0.957031 sec
```

But how many of these 33 points lie in the interior of the 24-cell?

```
./count int EXAMPLES/24_cell
```

The last couple of lines that `LattE` prints to the screen look as follows:

```
Reading .ext file...


*****  Total number of lattice points: 1 ****
```

Therefore, there is only one of the 33 lattice points in the 24-cell lies in the interior.

5.3 Maximizing Over a Knapsack Polytope

Finally, let us solve the problem "cuww1" [2, 5]. Its description is given in the file "EXAMPLES/cuww1":

```
1 6
89643482 -12223 -12224 -36674 -61119 -85569
linearity 1 1
nonnegative 5 1 2 3 4 5
```

The cost function can be found in the file "EXAMPLES/cuww1.cost":

```
1 5
213 -1928 -11111 -2345 9123
```

Now let us maximize this cost function over the given knapsack polytope. Note that by default, the digging algorithm as described in [5] is used.

```
./maximize EXAMPLES/cuww1
```

The last couple of lines that `LattE` prints to the screen look as follows:

```
Finished computing a rational function.
Time: 0.158203 sec.

There is one optimal solution.

No digging.
An optimal solution for [213 -1928 -11111 -2345 9123] is: [7334 0 0 0 0].
The projected down opt value is: 191928257104
The optimal value is: 1562142.
The gap is: 7995261.806
Computation done.
Time: 0.203124 sec.
```

The solution $(7334, 0, 0, 0, 0)$ is quickly found. Now let us try to find the optimal value again by a different algorithm, the binary search algorithm.

```
./maximize bbs EXAMPLES/cuww1
```

The last couple of lines that `LattE` prints to the screen look as follows:

```
Total of Iterations: 26
The total number of unimodular cones: 125562
The optimal value: 1562142

The number of optimal solutions: 1
Time: 0.042968
```

Note that we get the same optimal value, but no optimal solution is provided.

6 Release Information

6.1 System Requirements

6.1.1 Platform

The binaries for `LattE` v1.1 as well as for `cdd` (by K. Fukuda [6]) are for platforms that run Linux. Both codes were compiled using `gcc` version 2.96. This or a higher version of `gcc` is needed to run `LattE`.

6.1.2 Memory Requirements

The memory requirements are essentially problem dependent; however, `LattE` runs in a "memory saving mode" whenever appropriate. In this mode `LattE` rarely uses more than around 20 MB beyond the amount of memory needed to calculate triangulations.

6.2 Additional `MAPLE` Connection

The call

```
./ehrhart simplify fileName
```

requires `MAPLE` for simplifications of expressions. It should be sufficient to have a copy of `MAPLE` installed on your machine, without any additional special configuration required. `LattE` will still run even if `MAPLE` is not installed, but this simplification feature to "ehrhart" will not be available.

We have tested this connection with Maple 5.1 and 8.0 and experienced no problem. Please let us know about any problem you experience with our connection to Maple.

6.3 File Descriptions

The `LattE` v1.1 Linux release consists a single archive file, **latte_v1.1.tar.gz**. The archive contains the following files:

Name	Description
count ehrhart maximize minimize	Main program executables
simplify2.add redcheck_gmp install	additional files used by `LattE` (from `cdd` library)
cdd	external `cdd` software executable used by `LattE`, original name in `cdd`: cddr+_gmp
EXAMPLES/	Subdirectory containing sample input files
manual_v1.1.ps manual_v1.1.pdf	User's guide in Postscript and PDF formats
code/	`LattE` source code (including `NTL`, `cdd` and `cdd lib`)
gpl.txt	GNU General Public License agreement

6.4 License Agreement

This program is free software; you can redistribute it and/or modify it under the terms of the version 2 of GNU General Public License as published by the Free Software Foundation.

This program is distributed in the hope that it will be useful, but WITHOUT ANY WARRANTY; without even the implied warranty of MERCHANTABILITY or FITNESS FOR A PARTICULAR PURPOSE. See the GNU General Public License for more details. (The careful user can find an install script for the source code in the directory "code/".)

You should have received a copy of the GNU General Public License along with this program; see the file COPYING. If not, write to the Free Software Foundation, Inc., 59 Temple Place - Suite 330, Boston, MA 02111-1307, USA.

We have included a copy of the GNU General Public License also at the end of this document.

6.5 How to Cite LattE

Although LattE is free software, your acknowledgment is requested. If LattE is useful in your research or applications please acknowledge it by referencing this manual as

De Loera, J.A., Haws, D., Hemmecke, R., Huggins, P., Tauzer, J., Yoshida, R. *A User's Guide for* LattE *v1.1*, 2003, software package LattE is available at http://www.math.ucdavis.edu/~latte/

6.6 The LattE Team

- **Project director**: Prof. Jesús A. De Loera
- **Postdoctoral scientist**: Dr. Raymond Hemmecke
- **Graduate Student**: Ruriko Yoshida
- **Undergraduate Students**: David Haws, Peter Huggins, Jeremy Tauzer
- **Webmaster**: Jon Brooks

6.7 Acknowledgments

LattE currently uses two wonderful pieces of software. First is cdd, developed by Komei Fukuda, whose webpage can be found at:

http://www.cs.mcgill.ca/~fukuda/

cdd is used for finding vertices of polytopes and the triangulation of cones.

In addition, LattE currently uses NTL, a Library for doing Number Theory, written by Victor Shoup [7], for LLL algorithm, matrix manipulations, storing variable length integers, and floating point numbers. NTL can be found at:

http://shoup.net/ntl/

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