Linear Algebra in Twenty Five Lectures

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Preface

These linear algebra lecture notes are designed to be presented as twenty five, fifty minute lectures suitable for sophomores likely to use the material for applications but still requiring a solid foundation in this fundamental branch of mathematics. The main idea of the course is to emphasize the concepts of vector spaces and linear transformations as mathematical structures that can be used to model the world around us. Once "persuaded" of this truth, students learn explicit skills such as Gaussian elimination and diagonalization in order that vectors and linear transformations become calculational tools, rather than abstract mathematics.

In practical terms, the course aims to produce students who can perform computations with large linear systems while at the same time understand the concepts behind these techniques. Often-times when a problem can be reduced to one of linear algebra it is "solved". These notes do not devote much space to applications (there are already a plethora of textbooks with titles involving some permutation of the words "linear", "algebra" and "applications"). Instead, they attempt to explain the fundamental concepts carefully enough that students will realize for their own selves when the particular application they encounter in future studies is ripe for a solution via linear algebra.

There are relatively few worked examples or illustrations in these notes, this material is instead covered by a series of "linear algebra how-to videos". They can be viewed by clicking on the take one icon . The "scripts" for these movies are found at the end of the notes if students prefer to read this material in a traditional format and can be easily reached via the script icon . Watch an introductory video below:

Introductory Video

The notes are designed to be used in conjunction with a set of online homework exercises which help the students read the lecture notes and learn basic linear algebra skills. Interspersed among the lecture notes are links to simple online problems that test whether students are actively reading the notes. In addition there are two sets of sample midterm problems with solutions as well as a sample final exam. There are also a set of ten online assignments which are usually collected weekly. The first assignment is designed to ensure familiarity with some basic mathematic notions (sets, functions, logical quantifiers and basic methods of proof). The remaining nine assignments are devoted to the usual matrix and vector gymnastics expected from any sophomore linear algebra class. These exercises are all available at

> http://webwork.math.ucdavis.edu/webwork2/MAT22A-Waldron-Winter-2012/

Webwork is an open source, online homework system which originated at the University of Rochester. It can efficiently check whether a student has answered an explicit, typically computation-based, problem correctly. The problem sets chosen to accompany these notes could contribute roughly 20% of a student's grade, and ensure that basic computational skills are mastered. Most students rapidly realize that it is best to print out the Webwork assignments and solve them on paper before entering the answers online. Those who do not tend to fare poorly on midterm examinations. We have found that there tend to be relatively few questions from students in office hours about the Webwork assignments. Instead, by assigning 20% of the grade to written assignments drawn from problems chosen randomly from the review exercises at the end of each lecture, the student's focus was primarily on understanding ideas. They range from simple tests of understanding of the material in the lectures to more difficult problems, all of them require thinking, rather than blind application of mathematical "recipes". Office hour questions reflected this and offered an excellent chance to give students tips how to present written answers in a way that would convince the person grading their work that they deserved full credit!

Each lecture concludes with references to the comprehensive online textbooks of Jim Hefferon and Rob Beezer:

> http://joshua.smcvt.edu/linearalgebra/

> http://linear.ups.edu/index.html

and the notes are also hyperlinked to Wikipedia where students can rapidly access further details and background material for many of the concepts. Videos of linear algebra lectures are available online from at least two sources:

- The Khan Academy,
  http://www.khanacademy.org/?video#Linear Algebra

- MIT OpenCourseWare, Professor Gilbert Strang,
  http://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/video-lectures/

There are also an array of useful commercially available texts. A non-exhaustive list includes

- "Introductory Linear Algebra, An Applied First Course", B. Kolman and D. Hill, Pearson 2001.
- "Linear Algebra and Its Applications", David C. Lay, Addison–Weseley 2011.
- "Introduction to Linear Algebra", Gilbert Strang, Wellesley Cambridge Press 2009.
- "Linear Algebra Done Right", S. Axler, Springer 1997.
- "Algebra and Geometry", D. Holten and J. Lloyd, CBRC, 1978.
- "Schaum's Outline of Linear Algebra", S. Lipschutz and M. Lipson, McGraw-Hill 2008.

A good strategy is to find your favorite among these in the University Library.

There are many, many useful online math resources. A partial list is given in Appendix I.

Students have also started contributing to these notes. Click here to see some of their work.

There are many "cartoon" type images for the important theorems and formalæ . In a classroom with a projector, a useful technique for instructors is to project these using a computer. They provide a colorful relief for students from (often illegible) scribbles on a blackboard. These can be downloaded at:

Lecture Materials

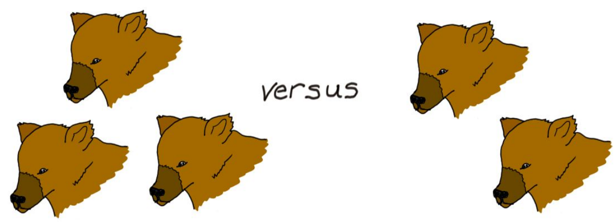
There are still many errors in the notes, as well as awkwardly explained concepts. An army of 400 students, Fu Liu, Stephen Pon and Gerry Puckett have already found many of them. Rohit Thomas has spent a great deal of time editing these notes and the accompanying webworks and has improved them immeasurably. Katrina Glaeser and Travis Scrimshaw have spent many hours shooting and scripting the how-to videos and taken these notes to a whole new level! Anne Schilling shot a great guest video. We also thank Captain Conundrum for providing us his solutions to the sample midterm and final questions. The review exercises would provide a better survey of what linear algebra really is if there were more "applied" questions. We welcome your contributions!

Andrew and Tom.

1 What is Linear Algebra?

`Video Overview`

> Three bears go into a cave, two come out. Would you go in?
>
> Brian Butterworth



Numbers are highly useful tools for surviving in the modern world, so much so that we often introduce abstract *pronumerals* to represent them:

> n bears go into a cave, $n-1$ come out. Would you go in?

A single number alone is not sufficient to model more complicated real world situations. For example, suppose I asked everybody in this room to rate the likeability of everybody else on a scale from 1 to 10. In a room full of n people (or bears *sic*) there would be n^2 ratings to keep track of (how much Jill likes Jill, how much does Jill like Andrew, how much does Andrew like Jill, how much does Andrew like Andrew, *etcetera*). We could arrange these in a square array

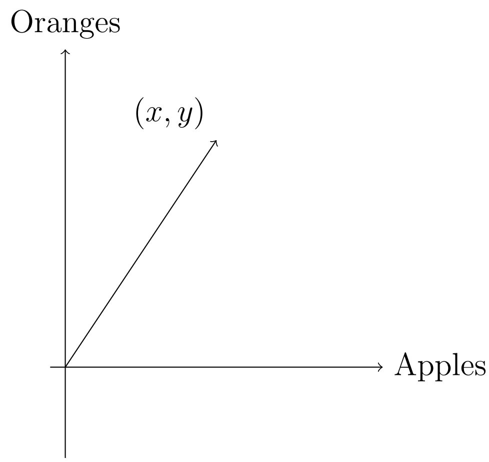
$$\begin{pmatrix} 9 & 4 & \cdots \\ 10 & 6 & \\ \vdots & & \ddots \end{pmatrix}$$

Would it make sense to replace such an array by an abstract symbol M? In the case of numbers, the pronumeral n was more than a placeholder for a particular piece of information; there exists a myriad of mathematical operations (addition, subtraction, multiplication,...) that can be performed with the symbol n that could provide useful information about the real world system at hand. The array M is often called a *matrix* and is an example of a more general abstract structure called a *linear transformation* on which many mathematical operations can also be defined. (To understand why having an abstract theory of linear transformations might be incredibly useful and even lucrative, try replacing "likeability ratings" with the number of times internet websites link to one another!) In this course, we'll learn about three main topics: Linear Systems, Vector Spaces, and Linear Transformations. Along the way we'll learn about matrices and how to manipulate them.

For now, we'll illustrate some of the basic ideas of the course in the case of two by two matrices. Everything will carefully defined later, we just want to start with some simple examples to get an idea of the things we'll be working with.

Example Suppose I have a bunch of apples and oranges. Let x be the number of apples I have, and y be the number of oranges I have. As everyone knows, apples and oranges don't mix, so if I want to keep track of the number of apples and oranges I have, I should put them in a list. We'll call this list a *vector,* and write it like this: (x, y). The order here matters! I should remember to always write the number of apples first and then the number of oranges–otherwise if I see the vector $(1, 2)$, I won't know whether I have two apples or two oranges.

This vector (x, y) in the example is just a list of two numbers, so if we want to, we can represent it with a point in the plane with the corresponding coordinates, like so:



In the plane, we can imagine each point as some combination of apples and oranges (or parts thereof, for the points that don't have integer coordinates). Then each point corresponds to some vector. The collection of all such vectors—all the points in our apple-orange plane—is an example of a *vector space*.

Example There are 27 pieces of fruit in a barrel, and twice as many oranges as apples. How many apples and oranges are in the barrel?

How to solve this conundrum? We can re-write the question mathematically as follows:

$$\begin{aligned} x + y &= 27 \\ y &= 2x \end{aligned}$$

This is an example of a *Linear System*. It's a collection of equations in which variables are multiplied by constants and summed, and no variables are multiplied together: There are no powers of x or y greater than one, no fractional or negative powers of x or y, and no places where x and y are multiplied together.

 Reading homework: problem 1.1

Notice that we can solve the system by manipulating the equations involved. First, notice that the second equation is the same as $-2x + y = 0$. Then if you subtract the second equation from the first, you get on the left side $x+y-(-2x+y)=3x$, and on the left side you get $27-0=27$. Then $3x=27$, so we learn that $x=9$. Using the second equation, we then see that $y=18$. Then there are 9 apples and 18 oranges.

Let's do it again, by working with the list of equations as an object in itself. First we rewrite the equations tidily:

$$\begin{aligned} x+y &= 27 \\ 2x-y &= 0 \end{aligned}$$

We can express this set of equations with a matrix as follows:

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 27 \\ 0 \end{pmatrix}$$

The square list of numbers is an example of a *matrix*. We can multiply the matrix by the vector to get back the linear system using the following rule for ~~multiplying matrices by vectors~~:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} \tag{1}$$

 Reading homework: problem 1.2

`A` 3×3 `matrix example`

The matrix is an example of a *Linear Transformation*, because it takes one vector and turns it into another in a "linear" way.

Our next task is to solve linear systems. We'll learn a general method called Gaussian Elimination.

References

Hefferon, Chapter One, Section 1
Beezer, Chapter SLE, Sections WILA and SSLE
Wikipedia, Systems of Linear Equations

Review Problems

1. Let M be a matrix and u and v vectors:
   $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, v = \begin{pmatrix} x \\ y \end{pmatrix}, u = \begin{pmatrix} w \\ z \end{pmatrix}$.

   (a) *Propose* a definition for $u + v$.

   (b) *Check* that your definition obeys $Mv + Mu = M(u + v)$.

2. *Matrix Multiplication:* Let M and N be matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } N = \begin{pmatrix} e & f \\ g & h \end{pmatrix},$$

   and v a vector

$$v = \begin{pmatrix} x \\ y \end{pmatrix}.$$

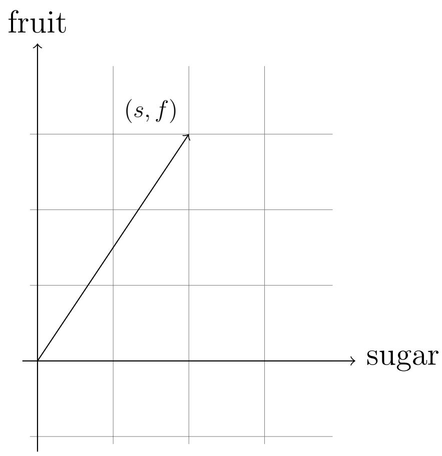
   Compute the vector Nv using the rule given above. Now multiply this vector by the matrix M, *i.e.*, compute the vector $M(Nv)$.

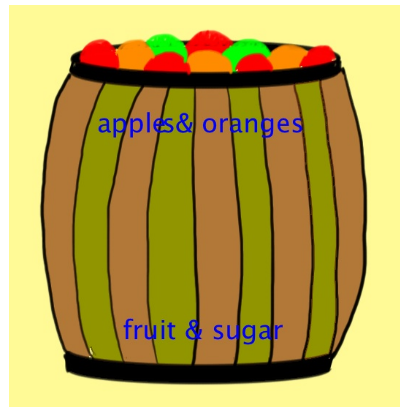
   Next recall that multiplication of ordinary numbers is associative, namely the order of brackets does not matter: $(xy)z = x(yz)$. Let us try to demand the same property for matrices and vectors, that is

$$M(Nv) = (MN)v\,.$$

   We need to be careful reading this equation because Nv is a vector and so is $M(Nv)$. Therefore the right hand side, $(MN)v$ should also be a vector. This means that MN must be a matrix; in fact it is the matrix obtained by multiplying the matrices M and N. Use your result for $M(Nv)$ to find the matrix MN.

3. Pablo is a nutritionist who knows that oranges always have twice as much sugar as apples. When considering the sugar intake of schoolchildren eating a barrel of fruit, he represents the barrel like so:





Find a linear transformation relating Pablo's representation to the one in the lecture. Write your answer as a matrix.

Hint: Let λ represent the amount of sugar in each apple.

Hint

4. There are methods for solving linear systems other than Gauss' method. One often taught in high school is to solve one of the equations for a variable, then substitute the resulting expression into other equations. That step is repeated until there is an equation with only one variable. From that, the first number in the solution is derived, and then back-substitution can be done. This method takes longer than Gauss' method, since it involves more arithmetic operations, and is also more likely to lead to errors. To illustrate how it can lead to wrong conclusions, we will use the system

$$\begin{aligned} x + 3y &= 1 \\ 2x + y &= -3 \\ 2x + 2y &= 0 \end{aligned}$$

   (a) Solve the first equation for x and substitute that expression into the second equation. Find the resulting y.

   (b) Again solve the first equation for x, but this time substitute that expression into the third equation. Find this y.

What extra step must a user of this method take to avoid erroneously concluding a system has a solution?

2 Gaussian Elimination

2.1 Notation for Linear Systems

In Lecture 1 we studied the linear system

$$\begin{array}{rcrcr} x & + & y & = & 27 \\ 2x & - & y & = & 0 \end{array}$$

and found that

$$\begin{array}{rcrcr} x & & & = & 9 \\ & & y & = & 18\,. \end{array}$$

We learned to write the linear system using a matrix and two vectors:

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 27 \\ 0 \end{pmatrix}$$

Likewise, we can write the solution as:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 18 \end{pmatrix}$$

The matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is called the *Identity Matrix*. You should check that if v is any vector, then

$$Iv = v\,.$$

A useful shorthand for a linear system is an *Augmented Matrix*, which looks like this for the linear system we've been dealing with:

$$\left(\begin{array}{cc|c} 1 & 1 & 27 \\ 2 & -1 & 0 \end{array}\right)$$

We don't bother writing the vector $\begin{pmatrix} x \\ y \end{pmatrix}$, since it will show up in any linear system we deal with. The solution to the linear system looks like this:

$$\left(\begin{array}{cc|c} 1 & 0 & 9 \\ 0 & 1 & 18 \end{array}\right)$$

Augmented Matrix Notation

Here's another example of an augmented matrix, for a linear system with three equations and four unknowns:

$$\left(\begin{array}{cccc|c} 1 & 3 & 2 & 0 & 9 \\ 6 & 2 & 0 & -2 & 0 \\ -1 & 0 & 1 & 1 & 3 \end{array}\right)$$

And finally, here's the general case. The number of equations in the linear system is the number of rows r in the augmented matrix, and the number of columns k in the matrix left of the vertical line is the number of unknowns.

$$\left(\begin{array}{cccc|c} a_1^1 & a_2^1 & \cdots & a_k^1 & b^1 \\ a_1^2 & a_2^2 & \cdots & a_k^2 & b^2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1^r & a_2^r & \cdots & a_k^r & b^r \end{array}\right)$$

Reading homework: problem 2.1

Here's the idea: Gaussian Elimination is a set of rules for taking a general augmented matrix and turning it into a very simple augmented matrix consisting of the identity matrix on the left and a bunch of numbers (the solution) on the right.

Equivalence Relations for Linear Systems

Equivalence Example

It often happens that two mathematical objects will appear to be different but in fact are exactly the same. The best-known example of this are fractions. For example, the fractions $\frac{1}{2}$ and $\frac{6}{12}$ describe the same number. We could certainly call the two fractions *equivalent*.

In our running example, we've noticed that the two augmented matrices

$$\begin{pmatrix} 1 & 1 & 27 \\ 2 & -1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 9 \\ 0 & 1 & 18 \end{pmatrix}$$

both contain the same information: $x = 9, y = 18$.

Two augmented matrices corresponding to linear systems *that actually have solutions* are said to be (row) *equivalent* if they have the *same* solutions. To denote this, we write:

$$\left(\begin{array}{cc|c} 1 & 1 & 27 \\ 2 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 9 \\ 0 & 1 & 18 \end{array}\right)$$

The symbol $\sim$ is read "is equivalent to".

A small excursion into the philosophy of mathematical notation: Suppose I have a large pile of equivalent fractions, such as $\frac{2}{4}$, $\frac{27}{54}$, $\frac{100}{200}$, and so on. Most people will agree that their favorite way to write the number represented by all these different factors is $\frac{1}{2}$, in which the numerator and denominator are relatively prime. We usually call this a *reduced fraction*. This is an example of a *canonical form*, which is an extremely impressive way of saying "favorite way of writing it down". There's a theorem telling us that every rational number can be specified by a unique fraction whose numerator and denominator are relatively prime. To say that again, but slower, *every* rational number *has* a reduced fraction, and furthermore, that reduced fraction is *unique*.

A 3×3 example

2.2 Reduced Row Echelon Form

Since there are many different augmented matrices that have the same set of solutions, we should find a canonical form for writing our augmented matrices. This canonical form is called *Reduced Row Echelon Form*, or RREF for short. RREF looks like this in general:

$$\left(\begin{array}{cccccc|c}
1 & * & 0 & * & 0 & \cdots & 0 & b^1 \\
0 & & 1 & * & 0 & \cdots & 0 & b^2 \\
0 & & 0 & & 1 & \cdots & 0 & b^3 \\
\vdots & & \vdots & & \vdots & & 0 & \vdots \\
 & & & & & & 1 & b^k \\
0 & & 0 & & 0 & \cdots & 0 & 0 \\
\vdots & & \vdots & & \vdots & & \vdots & \vdots \\
0 & & 0 & & 0 & \cdots & 0 & 0
\end{array}\right)$$

The first non-zero entry in each row is called the *pivot*. The asterisks denote arbitrary content which could be several columns long. The following properties describe the RREF.

1. In RREF, the pivot of any row is always 1.
2. The pivot of any given row is always to the right of the pivot of the row above it.
3. The pivot is the only non-zero entry in its column.

Example $\left(\begin{array}{ccc|c} 1 & 0 & 7 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Here is a NON-Example, which breaks all three of the rules:

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

The RREF is a very useful way to write linear systems: it makes it very easy to write down the solutions to the system.

Example

$$\left(\begin{array}{cccc|c} 1 & 0 & 7 & 0 & 4 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

When we write this augmented matrix as a system of linear equations, we get the following:

$$\begin{aligned} x \quad + 7z \quad &= 4 \\ y + 3z \quad &= 1 \\ w &= 2 \end{aligned}$$

Solving from the bottom variables up, we see that $w = 2$ immediately. z is not a pivot, so it is still undetermined. Set $z = \lambda$. Then $y = 1 - 3\lambda$ and $x = 4 - 7\lambda$. More concisely:

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -7 \\ -3 \\ 1 \\ 0 \end{pmatrix}$$

So we can read off the solution *set* directly from the RREF. (Notice that we use the word "set" because there is not just one solution, but one for every choice of λ.)

Reading homework: problem 2.2

You need to become very adept at reading off solutions of linear systems from the RREF of their augmented matrix. The general method is to work from the bottom up and set any non-pivot variables to unknowns. Here is another example.

Example

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Here we were not told the names of the variables, so lets just call them x_1, x_2, x_3, x_4, x_5. (There are always as many of these as there are columns in the matrix before the vertical line; the number of rows, on the other hand is the number of linear equations.)

To begin with we immediately notice that there are no pivots in the second and fourth columns so x_2 and x_4 are undetermined and we set them to

$$x_2 = \lambda_1 \,, \qquad x_4 = \lambda_2 \,.$$

(Note that you get to be creative here, we could have used λ and μ or any other names we like for a pair of unknowns.)

Working from the bottom up we see that the last row just says $0 = 0$, a well known fact! *Note that a row of zeros save for a non-zero entry after the vertical line would be mathematically inconsistent and indicates that the system has NO solutions at all.*

Next we see from the second last row that $x_5 = 3$. The second row says $x_3 = 2 - 2x_4 = 2 - 2\lambda_2$. The top row then gives $x_1 = 1 - x_2 - x_4 = 1 - \lambda_1 - \lambda_2$. Again we can write this solution as a vector

$$\begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} .$$

Observe, that since no variables were given at the beginning, we do not really need to state them in our solution. As a challenge, look carefully at this solution and make sure you can see how every part of it comes from the original augmented matrix without every having to reintroduce variables and equations.

Perhaps unsurprisingly in light of the previous discussions of RREF, we have a theorem:

Theorem 2.1. *Every augmented matrix is row-equivalent to a* unique *augmented matrix in reduced row echelon form.*

In Lecture **??**, we will see why this is true.

References

Hefferon, Chapter One, Section 1
Beezer, Chapter SLE, Section RREF
Wikipedia, Row Echelon Form

Review Problems

1. State whether the following augmented matrices are in RREF and compute their solution sets.

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right),$$

$$\left(\begin{array}{cccccc|c} 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$\left(\begin{array}{ccccccc|c} 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right).$$

2. Show that this pair of augmented matrices are row equivalent, assuming $ad - bc \neq 0$:

$$\left(\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & \frac{de-bf}{ad-bc} \\ 0 & 1 & \frac{af-ce}{ad-bc} \end{array}\right)$$

3. Consider the augmented matrix: $\left(\begin{array}{cc|c} 2 & -1 & 3 \\ -6 & 3 & 1 \end{array}\right)$

   Give a *geometric* reason why the associated system of equations has no solution. (Hint, plot the three vectors given by the columns of this augmented matrix in the plane.) Given a general augmented matrix

$$\left(\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}\right),$$

   can you find a condition on the numbers a, b, c and d that create the geometric condition you found?

4. List as many operations on augmented matrices that *preserve* row equivalence as you can. Explain your answers. Give examples of operations that break row equivalence.

5. Row equivalence of matrices is an example of an *equivalence relation*. Recall that a relation $\sim$ on a set of objects U is an equivalence relation if the following three properties are satisfied:

   - Reflexive: For any $x \in U$, we have $x \sim x$.
   - Symmetric: For any $x, y \in U$, if $x \sim y$ then $y \sim x$.
   - Transitive: For any x, y and $z \in U$, if $x \sim y$ and $y \sim z$ then $x \sim z$.

   (For a fuller discussion of equivalence relations, see Homework 0, Problem 4)

   Show that row equivalence of augmented matrices is an equivalence relation.

Hints for Questions 4 and 5

3 Elementary Row Operations

Our goal is to begin with an arbitrary matrix and apply operations that respect row equivalence until we have a matrix in Reduced Row Echelon Form (RREF). The three elementary row operations are:

- (Row Swap) Exchange any two rows.
- (Scalar Multiplication) Multiply any row by a non-zero constant.
- (Row Sum) Add a multiple of one row to another row.

`Example`

Why do these preserve the linear system in question? Swapping rows is just changing the order of the equations begin considered, which certainly should not alter the solutions. Scalar multiplication is just multiplying the equation by the same number on both sides, which does not change the solution(s) of the equation. Likewise, if two equations share a common solution, adding one to the other preserves the solution. Therefore we can define augmented matrices to be row equivalent if the are related by a sequence of elementary row operations. This definition can also be applied to augmented matrices corresponding to linear systems with no solutions at all!

There is a very simple process for row-reducing a matrix, working column by column. This process is called *Gauss–Jordan elimination* or simply Gaussian elimination.

1. If all entries in a given column are zero, then the associated variable is undetermined; make a note of the undetermined variable(s) and then ignore all such columns.
2. Swap rows so that the first entry in the first column is non-zero.
3. Multiply the first row by λ so that this pivot entry is 1.
4. Add multiples of the first row to each other row so that the first entry of every other row is zero.

5. Before moving on to step 6, add multiples of the first row any rows above that you have ignored to ensure there are zeros in the column above the current pivot entry.

6. Now ignore the first row and first column and repeat steps 2-5 until the matrix is in RREF.

Reading homework: problem 3.1

Example

$$\begin{array}{rrrl} & & 3x_3 & = 9 \\ x_1 & +5x_2 & -2x_3 & = 2 \\ \frac{1}{3}x_1 & +2x_2 & & = 3 \end{array}$$

First we write the system as an augmented matrix:

$$\left(\begin{array}{ccc|c} 0 & 0 & 3 & 9 \\ 1 & 5 & -2 & 2 \\ \frac{1}{3} & 2 & 0 & 3 \end{array}\right) \underset{\sim}{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} \frac{1}{3} & 2 & 0 & 3 \\ 1 & 5 & -2 & 2 \\ 0 & 0 & 3 & 9 \end{array}\right)$$

$$\underset{\sim}{3R_1} \left(\begin{array}{ccc|c} 1 & 6 & 0 & 9 \\ 1 & 5 & -2 & 2 \\ 0 & 0 & 3 & 9 \end{array}\right)$$

$$\underset{\sim}{R_2 = R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 6 & 0 & 9 \\ 0 & -1 & -2 & -7 \\ 0 & 0 & 3 & 9 \end{array}\right)$$

$$\underset{\sim}{-R_2} \left(\begin{array}{ccc|c} 1 & 6 & 0 & 9 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 3 & 9 \end{array}\right)$$

$$\underset{\sim}{R_1 = R_1 - 6R_2} \left(\begin{array}{ccc|c} 1 & 0 & -12 & -33 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 3 & 9 \end{array}\right)$$

$$\underset{\sim}{\tfrac{1}{3}R_3} \left(\begin{array}{ccc|c} 1 & 0 & -12 & -33 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\underset{\sim}{R_1 = R_1 + 12R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\underset{\sim}{R_2 = R_2 - 2R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Now we're in RREF and can see that the solution to the system is given by $x_1 = 3$, $x_2 = 1$, and $x_3 = 3$; it happens to be a unique solution. Notice that we kept track of the steps we were taking; this is important for checking your work!

Example

$$
\left(\begin{array}{cccc|c}
1 & 0 & -1 & 2 & -1 \\
1 & 1 & 1 & -1 & 2 \\
0 & -1 & -2 & 3 & -3 \\
5 & 2 & -1 & 4 & 1
\end{array}\right)
$$

$$
\overset{R_2-R_1;R_4-5R_1}{\sim} \quad
\left(\begin{array}{cccc|c}
1 & 0 & -1 & 2 & -1 \\
0 & 1 & 2 & -3 & 3 \\
0 & -1 & -2 & 3 & -3 \\
0 & 2 & 4 & -6 & 6
\end{array}\right)
$$

$$
\overset{R_3+R_2;R_4-2R_3}{\sim} \quad
\left(\begin{array}{cccc|c}
1 & 0 & -1 & 2 & -1 \\
0 & 1 & 2 & -3 & 3 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

Here the variables x_3 and x_4 are undetermined; the solution is not unique. Set $x_3 = \lambda$ and $x_4 = \mu$ where λ and μ are *arbitrary* real numbers. Then we can write x_1 and x_2 in terms of λ and μ as follows:

$$
\begin{aligned}
x_1 &= \lambda - 2\mu - 1 \\
x_2 &= -2\lambda + 3\mu + 3
\end{aligned}
$$

We can write the solution set with vectors like so:

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix}
$$

This is (almost) our preferred form for writing the set of solutions for a linear system with many solutions.

Worked examples of Gaussian elimination

Uniqueness of Gauss-Jordan Elimination

Theorem 3.1. *Gauss-Jordan Elimination produces a unique augmented matrix in RREF.*

Proof. Suppose Alice and Bob compute the RREF for a linear system but get different results, A and B. Working from the left, discard all columns except for the pivots and the first column in which A and B differ. By Review Problem 1b, removing columns does not affect row equivalence. Call the new, smaller, matrices $\hat{A}$ and $\hat{B}$. The new matrices should look this:

$$\hat{A} = \left(\begin{array}{c|c} I_N & a \\ 0 & 0 \end{array}\right) \text{ and } \hat{B} = \left(\begin{array}{c|c} I_N & b \\ 0 & 0 \end{array}\right),$$

where I_N is an $N \times N$ identity matrix and a and b are vectors.

Now if $\hat{A}$ and $\hat{B}$ have the same solution, then we must have $a = b$. But this is a contradiction! Then $A = B$. □

`Explanation of the proof`

References

Hefferon, Chapter One, Section 1.1 and 1.2
Beezer, Chapter SLE, Section RREF
Wikipedia, Row Echelon Form
Wikipedia, Elementary Matrix Operations

Review Problems

1. (Row Equivalence)

   (a) Solve the following linear system using Gauss-Jordan elimination:

$$\begin{aligned}
2x_1 + 5x_2 - 8x_3 + 2x_4 + 2x_5 &= 0 \\
6x_1 + 2x_2 - 10x_3 + 6x_4 + 8x_5 &= 6 \\
3x_1 + 6x_2 + 2x_3 + 3x_4 + 5x_5 &= 6 \\
3x_1 + 1x_2 - 5x_3 + 3x_4 + 4x_5 &= 3 \\
6x_1 + 7x_2 - 3x_3 + 6x_4 + 9x_5 &= 9
\end{aligned}$$

   Be sure to set your work out carefully with equivalence signs $\sim$ between each step, labeled by the row operations you performed.

(b) Check that the following two matrices are row-equivalent:

$$\left(\begin{array}{ccc|c} 1 & 4 & 7 & 10 \\ 2 & 9 & 6 & 0 \end{array}\right) \text{ and } \left(\begin{array}{ccc|c} 0 & -1 & 8 & 20 \\ 4 & 18 & 12 & 0 \end{array}\right)$$

Now remove the third column from each matrix, and show that the resulting two matrices (shown below) are row-equivalent:

$$\left(\begin{array}{cc|c} 1 & 4 & 10 \\ 2 & 9 & 0 \end{array}\right) \text{ and } \left(\begin{array}{cc|c} 0 & -1 & 20 \\ 4 & 18 & 0 \end{array}\right)$$

Now remove the fourth column from each of the original two matrices, and show that the resulting two matrices, viewed as augmented matrices (shown below) are row-equivalent:

$$\left(\begin{array}{cc|c} 1 & 4 & 7 \\ 2 & 9 & 6 \end{array}\right) \text{ and } \left(\begin{array}{cc|c} 0 & -1 & 8 \\ 4 & 18 & 12 \end{array}\right)$$

Explain why row-equivalence is never affected by removing columns.

(c) Check that the matrix $\left(\begin{array}{cc|c} 1 & 4 & 10 \\ 3 & 13 & 9 \\ 4 & 17 & 20 \end{array}\right)$ has no solutions. If you remove one of the rows of this matrix, does the new matrix have any solutions? In general, can row equivalence be affected by removing rows? Explain why or why not.

2. (Gaussian Elimination) Another method for solving linear systems is to use row operations to bring the augmented matrix to row echelon form. In row echelon form, the pivots are not necessarily set to one, and we only require that all entries left of the pivots are zero, not necessarily entries above a pivot. Provide a counterexample to show that row echelon form is not unique.

   Once a system is in row echelon form, it can be solved by "back substitution." Write the following row echelon matrix as a system of equations, then solve the system using back-substitution.

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 3 & 3 \end{array}\right)$$

3. Explain why the linear system has no solutions:

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 6 \end{array}\right)$$

For which values of k does the system below have a solution?

$$\begin{array}{rcrcrcr} x & - & 3y & & & = & 6 \\ x & & & + & 3z & = & -3 \\ 2x & + & ky & + & (3-k)z & = & 1 \end{array}$$

Hint for question 3

4 Solution Sets for Systems of Linear Equations

For a system of equations with r equations and k unknowns, one can have a number of different outcomes. For example, consider the case of r equations in three variables. Each of these equations is the equation of a plane in three-dimensional space. To find solutions to the system of equations, we look for the common intersection of the planes (if an intersection exists). Here we have five different possibilities:

1. **No solutions.** Some of the equations are contradictory, so no solutions exist.

2. **Unique Solution.** The planes have a unique point of intersection.

3. **Line.** The planes intersect in a common line; any point on that line then gives a solution to the system of equations.

4. **Plane.** Perhaps you only had one equation to begin with, or else all of the equations coincide geometrically. In this case, you have a plane of solutions, with two free parameters.

`Planes`

5. **All of $\mathbb{R}^3$.** If you start with no information, then any point in $\mathbb{R}^3$ is a solution. There are three free parameters.

In general, for systems of equations with k unknowns, there are $k + 2$ possible outcomes, corresponding to the number of free parameters in the solutions set, plus the possibility of no solutions. These types of "solution sets" are hard to visualize, but luckily "hyperplanes" behave like planes in $\mathbb{R}^3$ in many ways.

`Pictures and Explanation`

Reading homework: problem 4.1

4.1 Non-Leading Variables

Variables that are not a pivot in the reduced row echelon form of a linear system are *free*. We set them equal to arbitrary parameters $\mu_1, \mu_2, \dots$

Example $\left(\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

Here, x_1 and x_2 are the pivot variables and x_3 and x_4 are non-leading variables, and thus free. The solutions are then of the form $x_3 = \mu_1$, $x_4 = \mu_2$, $x_2 = 1 + \mu_1 - \mu_2$, $x_1 = 1 - \mu_1 + \mu_2$.

The preferred way to write a solution set is with set notation. Let S be the set of solutions to the system. Then:

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \mu_1 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Example

We have already seen how to write a linear system of two equations in two unknowns as a matrix multiplying a vector. We can apply exactly the same idea for the above system of three equations in four unknowns by calling

$$M = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \text{ and } V = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Then if we take for the product of the matrix M with the vector X of unknowns

$$MX = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + x_3 - x_4 \\ x_2 - x_3 + x_4 \\ 0 \end{pmatrix}$$

our system becomes simply

$$MX = V.$$

Stare carefully at our answer for the product MX above. First you should notice that each of the three rows corresponds to the left hand side of one of the equations in the system. Also observe that each entry was obtained by matching the entries in the corresponding row of M with the column entries of X. For example, using the second row of M we obtained the second entry of MX

$$\begin{matrix} 0 & 1 & -1 & 1 \end{matrix} \quad \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} \quad \longmapsto \; x_2 - x_3 + x_4 \,.$$

In Lecture 8 we will study matrix multiplication in detail, but you can already try to discover the main rules for yourself by working through Review Question 3 on multiplying matrices by vectors.

Given two vectors we can *add* them term-by-term:

$$\begin{pmatrix} a^1 \\ a^2 \\ a^3 \\ \vdots \\ a^r \end{pmatrix} + \begin{pmatrix} b^1 \\ b^2 \\ b^3 \\ \vdots \\ b^r \end{pmatrix} = \begin{pmatrix} a^1 + b^1 \\ a^2 + b^2 \\ a^3 + b^3 \\ \vdots \\ a^r + b^r \end{pmatrix}$$

We can also multiply a vector by a scalar, like so:

$$\lambda \begin{pmatrix} a^1 \\ a^2 \\ a^3 \\ \vdots \\ a^r \end{pmatrix} = \begin{pmatrix} \lambda a^1 \\ \lambda a^2 \\ \lambda a^3 \\ \vdots \\ \lambda a^r \end{pmatrix}$$

Then yet another way to write the solution set for the example is:

$$X = X_0 + \mu_1 Y_1 + \mu_2 Y_2$$

where

$$X_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, Y_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, Y_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

Definition Let X and Y be vectors and α and β be scalars. A function f is *linear* if

$$f(\alpha X + \beta Y) = \alpha f(X) + \beta f(Y)$$

This is called the *linearity property* for matrix multiplication.

The notion of linearity is a core concept in this course. Make sure you understand what it means and how to use it in computations!

Example Consider our example system above with

$$M = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \text{ and } Y = \begin{pmatrix} y_1 \\ y_2 \\ y^3 \\ y^4 \end{pmatrix},$$

and take for the function of vectors

$$f(X) = MX.$$

Now let us check the linearity property for f. The property needs to hold for *any* scalars α and β, so for simplicity let us concentrate first on the case $\alpha = \beta = 1$. This means that we need to compare the following two calculations:

1. First add $X + Y$, then compute $f(X + Y)$.
2. First compute $f(X)$ and $f(Y)$, then compute the sum $f(X) + f(Y)$.

The second computation is slightly easier:

$$f(X) = MX = \begin{pmatrix} x_1 + x_3 - x_4 \\ x_2 - x_3 + x_4 \\ 0 \end{pmatrix} \text{ and } f(Y) = MY = \begin{pmatrix} y_1 + y_3 - y_4 \\ y_2 - y_3 + y_4 \\ 0 \end{pmatrix},$$

(using our result above). Adding these gives

$$f(X) + f(Y) = \begin{pmatrix} x_1 + x_3 - x_4 + y_1 + y_3 - y_4 \\ x_2 - x_3 + x_4 + y_2 - y_3 + y_4 \\ 0 \end{pmatrix}.$$

Next we perform the first computation beginning with:

$$X + Y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 \end{pmatrix},$$

from which we calculate

$$f(X+Y)=\begin{pmatrix}x_1+y_2+x_3+y_3-(x_4+y_4)\\ x_2+y_2-(x_3+y_3)+x_4+y_4\\ 0\end{pmatrix}.$$

Distributing the minus signs and remembering that the order of adding numbers like $x_1, x_2, \ldots$ does not matter, we see that the two computations give exactly the same answer.

Of course, you should complain that we took a special choice of α and β. Actually, to take care of this we only need to check that $f(\alpha X) = \alpha f(X)$. It is your job to explain this in Review Question 1

Later we will show that matrix multiplication is always linear. Then we will know that:

$$M(\alpha X + \beta Y) = \alpha MX + \beta MY$$

Then the two equations $MX = V$ and $X = X_0 + \mu_1 Y_1 + \mu_2 Y_2$ together say that:

$$MX_0 + \mu_1 MY_1 + \mu_2 MY_2 = V$$

for *any* $\mu_1, \mu_2 \in \mathbb{R}$. Choosing $\mu_1 = \mu_2 = 0$, we obtain

$$MX_0 = V\,.$$

Here, X_0 is an example of what is called a *particular solution* to the system.

Given the particular solution to the system, we can then deduce that $\mu_1 MY_1 + \mu_2 MY_2 = 0$. Setting $\mu_1 = 1, \mu_2 = 0$, and recalling the particular solution $MX_0 = V$, we obtain

$$MY_1 = 0\,.$$

Likewise, setting $\mu_1 = 0, \mu_1 = 0$, we obtain

$$MY_2 = 0\,.$$

Here Y_1 and Y_2 are examples of what are called *homogeneous* solutions to the system. They *do not* solve the original equation $MX = V$, but instead its associated homogeneous system of equations $MY = 0$.

Example Consider the linear system with the augmented matrix we've been working with.

$$\begin{array}{rrrrl} x & & +z & -w & = 1 \\ & y & -z & +w & = 1 \end{array}$$

Recall that the system has the following solution set:

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \mu_1 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Then $MX_0 = V$ says that $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ solves the original system of equations, which is certainly true, but this is not the only solution.

$MY_1 = 0$ says that $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ solves the homogeneous system.

$MY_2 = 0$ says that $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$ solves the homogeneous system.

Notice how adding any multiple of a homogeneous solution to the particular solution yields another particular solution.

Definition Let M a matrix and V a vector. Given the linear system $MX = V$, we call X_0 a *particular solution* if $MX_0 = V$. We call Y a *homogeneous solution* if $MY = 0$. The linear system

$$MX = 0$$

is called the (associated) *homogeneous system*.

If X_0 is a particular solution, then the general solution to the system is[1]:

[1]The notation $S = \{X_0 + Y : MY = 0\}$ is read, "S is the set of all $X_0 + Y$ such that $MY = 0$," and means exactly that. Sometimes a pipe $|$ is used instead of a colon.

$$S = \{X_0 + Y : MY = 0\}$$

In other words, the general solution = particular + homogeneous.

Reading homework: problem 4.2

References

Hefferon, Chapter One, Section I.2
Beezer, Chapter SLE, Section TSS
Wikipedia, Systems of Linear Equations

Review Problems

1. Let $f(X) = MX$ where

$$M = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

   Suppose that α is any number. Compute the following four quantities:

$$\alpha X\,,\; f(X)\,,\; \alpha f(X) \text{ and } f(\alpha X)\,.$$

   Check your work by verifying that

$$\alpha f(X) = f(\alpha X)\,.$$

   Now explain why the result checked in the Lecture, namely

$$f(X+Y) = f(X) + f(Y)\,,$$

   and your result $f(\alpha X) = \alpha f(X)$ together imply

$$f(\alpha X + \beta Y) = \alpha f(X) + \beta f(Y)\,.$$

2. Write down examples of augmented matrices corresponding to each of the five types of solution sets for systems of equations with three unknowns.

3. Let

$$M = \begin{pmatrix} a^1_1 & a^1_2 & \cdots & a^1_k \\ a^2_1 & a^2_2 & \cdots & a^2_k \\ \vdots & \vdots & & \vdots \\ a^r_1 & a^r_2 & \cdots & a^r_k \end{pmatrix}, \qquad X = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^k \end{pmatrix}$$

Propose a rule for MX so that $MX = 0$ is equivalent to the linear system:

$$\begin{array}{rrcrl} a^1_1 x^1 & + a^1_2 x^2 & \cdots & + a^1_k x^k & = 0 \\ a^2_1 x^1 & + a^2_2 x^2 & \cdots & + a^2_k x^k & = 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a^r_1 x^1 & + a^r_2 x^2 & \cdots & + a^r_k x^k & = 0 \end{array}$$

Show that your rule for multiplying a matrix by a vector obeys the linearity property.

Note that in this problem, x^2 does not denote the square of x. Instead x^1, x^2, x^3, etc... denote different variables. Although confusing at first, this notation was invented by Albert Einstein who noticed that quantities like $a^2_1 x^1 + a^2_2 x^2 \cdots + a^2_k x^k$ could be written in summation notation as $\sum_{j=1}^k a^2_j x^j$. Here j is called a summation index. Einstein observed that you could even drop the summation sign $\sum$ and simply write $a^2_j x^j$.

Problem 3 hint

4. Use the rule you developed in the problem 3 to compute the following products

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 14 \\ 14 \\ 21 \\ 35 \\ 62 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 42 & 97 & 2 & -23 & 46 \\ 0 & 1 & 3 & 1 & 0 & 33 \\ 11 & \pi & 1 & 0 & 46 & 29 \\ -98 & 12 & 0 & 33 & 99 & 98 \\ \log 2 & 0 & \sqrt{2} & 0 & e & 23 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 & 17 & 18 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Now that you are good at multiplying a matrix with a column vector, try your hand at a product of two matrices

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 & 17 & 18 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hint, to do this problem view the matrix on the right as three column vectors next to one another.

5. The *standard basis vector* e_i is a column vector with a one in the ith row, and zeroes everywhere else. Using the rule for multiplying a matrix times a vector in problem 3, find a simple rule for multiplying Me_i, where M is the general matrix defined there.

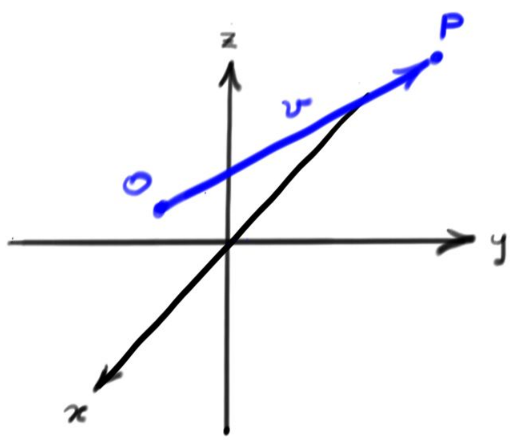
5 Vectors in Space, n-Vectors

In vector calculus classes, you encountered three-dimensional vectors. Now we will develop the notion of n-vectors and learn some of their properties.

`Overview`

We begin by looking at the space $\mathbb{R}^n$, which we can think of as the space of points with n coordinates. We then specify an *origin* O, a favorite point in $\mathbb{R}^n$. Now given any other point P, we can draw a *vector* v from O to P. Just as in $\mathbb{R}^3$, a vector has a *magnitude* and a *direction*.

If O has coordinates $(o^1, \ldots, o^n)$ and p has coordinates $(p^1, \ldots, p^n)$, then the *components* of the vector v are $\begin{pmatrix} p^1 - o^1 \\ p^2 - o^2 \\ \vdots \\ p^n - o^n \end{pmatrix}$. This construction allows us to put the origin anywhere that seems most convenient in $\mathbb{R}^n$, not just at the point with zero coordinates:



Remark A quick note on points versus vectors. We might sometimes interpret a point and a vector as the same object, but they are slightly different concepts and should be treated as such. For more details, see Appendix D

Do not be confused by our use of a superscript to label components of a vector. Here v^2 denotes the second component of a vector v, rather than a number v squared!

Most importantly, we can *add* vectors and *multiply* vectors by a scalar:

Definition Given two vectors a and b whose components are given by

$$a = \begin{pmatrix} a^1 \\ \vdots \\ a^n \end{pmatrix} \text{ and } \ b = \begin{pmatrix} b^1 \\ \vdots \\ b^n \end{pmatrix}$$

their *sum* is

$$a + b = \begin{pmatrix} a^1 + b^1 \\ \vdots \\ a^n + b^n \end{pmatrix} .$$

Given a scalar λ, the *scalar multiple*

$$\lambda a = \begin{pmatrix} \lambda a^1 \\ \vdots \\ \lambda a^n \end{pmatrix} .$$

Example Let

$$a = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \text{ and } \ b = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} .$$

Then, for example

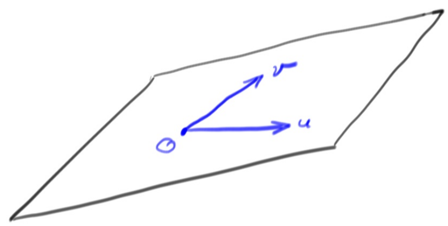
$$a + b = \begin{pmatrix} 5 \\ 5 \\ 5 \\ 5 \end{pmatrix} \text{ and } \ 3a - 2b = \begin{pmatrix} -5 \\ 0 \\ 5 \\ 10 \end{pmatrix} .$$

Notice that these are the same rules we saw in Lecture 4! In Lectures 1-4, we thought of a vector as being a list of numbers which captured information about a linear system. Now we are thinking of a vector as a magnitude and a direction in $\mathbb{R}^n$, and luckily the same rules apply.

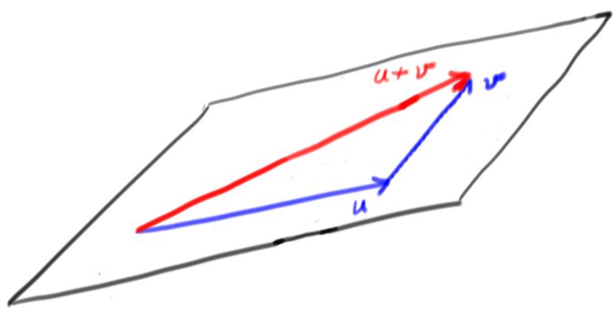
A special vector is the *zero vector* connecting the origin to itself. All of its components are zero. Notice that with respect to the usual notions of Euclidean geometry, it is the only vector with zero magnitude, and the only one which points in no particular direction. Thus, any single vector determines a line, *except* the zero-vector. Any scalar multiple of a non-zero vector lies in the line determined by that vector.

The line determined by a non-zero vector v through a point P can be written as $\{P + tv | t \in \mathbb{R}\}$. For example, $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \middle| t \in \mathbb{R} \right\}$ describes a line in 4-dimensional space parallel to the x-axis.

Given two non-zero vectors u, v, they will *usually* determine a plane,



unless both vectors are in the same line. In this case, one of the vectors can be realized as a scalar multiple of the other. The sum of u and v corresponds to laying the two vectors head-to-tail and drawing the connecting vector. If u and v determine a plane, then their sum lies in plane determined by u and v.



The plane determined by two vectors u and v can be written as

$$\{P + su + tv | s, t \in \mathbb{R}\} .$$

Example

$$\left\{ \begin{pmatrix} 3 \\ 1 \\ 4 \\ 1 \\ 5 \\ 9 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \middle| s, t \in \mathbb{R} \right\}$$

describes a plane in 6-dimensional space parallel to the xy-plane.

Parametric Notation

We can generalize the notion of a plane:

Definition A set of k vectors $v_1, \ldots, v_k$ in $\mathbb{R}^n$ with $k \leq n$ determines a k-dimensional *hyperplane*, unless any of the vectors v_i lives in the same hyperplane determined by the other vectors. If the vectors do determine a k-dimensional hyperplane, then any point in the hyperplane can be written as:

$$\{P + \sum_{i=1}^{k} \lambda_i v_i | \lambda_i \in \mathbb{R}\}$$

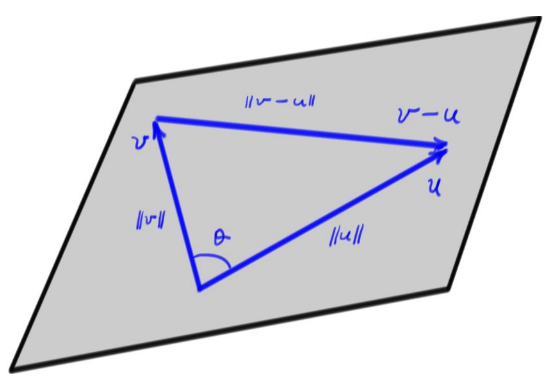
When the dimension k is not specified, one usually assumes that $k = n - 1$ for a hyperplane inside $\mathbb{R}^n$.

5.1 Directions and Magnitudes

Consider the Euclidean length of a vector:

$$\|v\| = \sqrt{(v^1)^2 + (v^2)^2 + \cdots (v^n)^2} \; = \; \sqrt{\sum_{i=1}^{n} (v^i)^2} \, .$$

Using the Law of Cosines, we can then figure out the angle between two vectors. Given two vectors v and u that span a plane in $\mathbb{R}^n$, we can then connect the ends of v and u with the vector $v - u$.



Then the Law of Cosines states that:

$$\|v-u\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\,\|v\|\cos\theta$$

Then isolate $\cos\theta$:

$$\begin{aligned}
\|v-u\|^2 - \|u\|^2 - \|v\|^2 &= (v^1-u^1)^2 + \cdots + (v^n-u^n)^2 \\
&\quad -((u^1)^2 + \cdots + (u^n)^2) \\
&\quad -((v^1)^2 + \cdots + (v^n)^2) \\
&= -2u^1v^1 - \cdots - 2u^nv^n
\end{aligned}$$

Thus,

$$\|u\|\,\|v\|\cos\theta = u^1v^1 + \cdots + u^nv^n\,.$$

Note that in the above discussion, we have assumed (correctly) that Euclidean lengths in $\mathbb{R}^n$ give the usual notion of lengths of vectors in the plane. This now motivates the definition of the dot product.

Definition The *dot product* of two vectors $u = \begin{pmatrix} u^1 \\ \vdots \\ u^n \end{pmatrix}$ and $v = \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}$ is

$$u \cdot v = u^1v^1 + \cdots + u^nv^n\,.$$

The *length* or *norm* or *magnitude* of a vector

$$\|v\| = \sqrt{v \cdot v}\,.$$

The *angle* θ between two vectors is determined by the formula

$$u \cdot v = \|u\|\|v\|\cos\theta\,.$$

The dot product has some important properties:

1. The dot product is *symmetric*, so

$$u \cdot v = v \cdot u\,,$$

2. *Distributive* so

$$u \cdot (v+w) = u \cdot v + u \cdot w\,,$$

3. *Bilinear*, which is to say, linear in both u and v. Thus

$$u \cdot (cv + dw) = c\, u \cdot v + d\, u \cdot w\,,$$

and

$$(cu + dw) \cdot v = c\, u \cdot v + d\, w \cdot v\,.$$

4. *Positive Definite*:

$$u \cdot u \geq 0\,,$$

and $u \cdot u = 0$ only when u itself is the 0-vector.

There are, in fact, many different useful ways to define lengths of vectors. Notice in the definition above that we first defined the dot product, and then defined everything else in terms of the dot product. So if we change our idea of the dot product, we change our notion of length and angle as well. The dot product determines the *Euclidean length and angle* between two vectors.

Other definitions of length and angle arise from *inner products*, which have all of the properties listed above (except that in some contexts the positive definite requirement is relaxed). Instead of writing $\cdot$ for other inner products, we usually write $\langle u, v\rangle$ to avoid confusion.

Reading homework: problem 5.1

Example Consider a four-dimensional space, with a special direction which we will call "time". The *Lorentzian inner product* on $\mathbb{R}^4$ is given by $\langle u, v\rangle = u^1v^1 + u^2v^2 + u^3v^3 - u^4v^4$. This is of central importance in Einstein's theory of special relativity, but note that it is not positive definite.

As a result, the "squared-length" of a vector with coordinates x, y, z and t is $\|v\|^2 = x^2 + y^2 + z^2 - t^2$. Notice that it is possible for $\|v\|^2 \leq 0$ for non-vanishing v!

Theorem 5.1 (Cauchy-Schwarz Inequality)**.** *For non-zero vectors u and v with an inner-product $\langle\ ,\ \rangle$,*

$$\frac{|\langle u, v\rangle|}{\|u\|\,\|v\|} \leq 1$$

Proof. The easiest proof would use the definition of the angle between two vectors and the fact that $\cos\theta \leq 1$. However, strictly speaking speaking we did not check our assumption that we could apply the Law of Cosines to the

Euclidean length in $\mathbb{R}^n$. There is, however a simple algebraic proof. Let α be any real number and consider the following positive, quadratic polynomial in α

$$0 \leq \langle u + \alpha v, u + \alpha v \rangle = \langle u, u \rangle + 2\alpha \langle u, v \rangle + \alpha^2 \langle v, v \rangle \, .$$

You should carefully check for yourself exactly which properties of an inner product were used to write down the above inequality!

Next, a tiny calculus computation shows that any quadratic $a\alpha^2 + 2b\alpha + c$ takes its minimal value $c - \frac{b^2}{a}$ when $\alpha = -\frac{b}{a}$. Applying this to the above quadratic gives

$$0 \leq \langle u, u \rangle - \frac{\langle u, v \rangle^2}{\langle v, v \rangle} \, .$$

Now it is easy to rearrange this inequality to reach the Cauchy–Schwarz one above. $\square$

Theorem 5.2 (Triangle Inequality)**.** *Given vectors u and v, we have:*

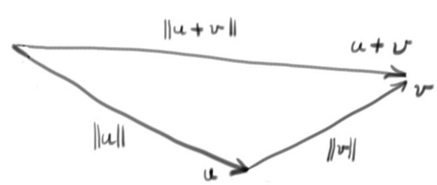
$$\|u + v\| \leq \|u\| + \|v\|$$

Proof.

$$\begin{aligned}
\|u + v\|^2 &= (u + v) \cdot (u + v) \\
&= u \cdot u + 2u \cdot v + v \cdot v \\
&= \|u\|^2 + \|v\|^2 + 2 \, \|u\| \, \|v\| \cos\theta \\
&= (\|u\| + \|v\|)^2 + 2 \, \|u\| \, \|v\| (\cos\theta - 1) \\
&\leq (\|u\| + \|v\|)^2
\end{aligned}$$

Then the square of the left-hand side of the triangle inequality is $\leq$ the right-hand side, and both sides are positive, so the result is true. $\square$

The triangle inequality is also "self-evident" examining a sketch of u, v and $u + v$



Example Let

$$a = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \text{ and } b = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix},$$

so that

$$a \cdot a = b \cdot b = 1 + 2^2 + 3^2 + 4^2 = 30$$

$$\Rightarrow \|a\| = \sqrt{30} = \|b\| \text{ and } \big(\|a\| + \|b\|\big)^2 = (2\sqrt{30})^2 = 120\,.$$

Since

$$a + b = \begin{pmatrix} 5 \\ 5 \\ 5 \\ 5 \end{pmatrix},$$

we have

$$\|a + b\|^2 = 5^2 + 5^2 + 5^2 + 5^2 = 100 < 120 = \big(\|a\| + \|b\|\big)^2$$

as predicted by the triangle inequality.

Notice also that $a \cdot b = 1.4 + 2.3 + 3.2 + 4.1 = 20 < \sqrt{30}.\sqrt{30} = 30 = \|a\|\,\|b\|$ in accordance with the Cauchy–Schwarz inequality.

Reading homework: problem 5.2

References

Hefferon: Chapter One.II
Beezer: Chapter V, Section VO, Subsection VEASM
Beezer: Chapter V, Section O, Subsections IP-N
Relevant Wikipedia Articles:

- Dot Product
- Inner Product Space
- Minkowski Metric

Review Problems

1. When he was young, Captain Conundrum mowed lawns on weekends to help pay his college tuition bills. He charged his customers according to the size of their lawns at a rate of 5¢ per square foot and meticulously kept a record of the areas of their lawns in an ordered list:

$$A = (200, 300, 50, 50, 100, 100, 200, 500, 1000, 100)\,.$$

He also listed the number of times he mowed each lawn in a given year, for the year 1988 that ordered list was

$$f = (20, 1, 2, 4, 1, 5, 2, 1, 10, 6)\,.$$

   (a) Pretend that A and f are vectors and compute $A \cdot f$.

   (b) What quantity does the dot product $A \cdot f$ measure?

   (c) How much did Captain Conundrum earn from mowing lawns in 1988? Write an expression for this amount in terms of the vectors A and f.

   (d) Suppose Captain Conundrum charged different customers different rates. How could you modify the expression in part 1c to compute the Captain's earnings?

2. (2) Find the angle between the diagonal of the unit square in $\mathbb{R}^2$ and one of the coordinate axes.

   (3) Find the angle between the diagonal of the unit cube in $\mathbb{R}^3$ and one of the coordinate axes.

   (n) Find the angle between the diagonal of the unit (hyper)-cube in $\mathbb{R}^n$ and one of the coordinate axes.

   (∞) What is the limit as $n \to \infty$ of the angle between the diagonal of the unit (hyper)-cube in $\mathbb{R}^n$ and one of the coordinate axes?

3. Consider the matrix $M = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and the vector $X = \begin{pmatrix} x \\ y \end{pmatrix}$.

   (a) Sketch X and MX in $\mathbb{R}^2$ for several values of X and θ.

   (b) Compute $\frac{||MX||}{||X||}$ for arbitrary values of X and θ.

(c) Explain your result for (b) and describe the action of M geometrically.

4. Suppose in $\mathbb{R}^2$ I measure the x direction in inches and the y direction in miles. Approximately what is the real-world angle between the vectors $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$? What is the angle between these two vectors according to the dot-product? Give a definition for an inner product so that the angles produced by the inner product are the actual angles between vectors.

5. (Lorentzian Strangeness). For this problem, consider $\mathbb{R}^n$ with the Lorentzian inner product and metric defined above.

   (a) Find a non-zero vector in two-dimensional Lorentzian space-time with zero length.

   (b) Find and sketch the collection of all vectors in two-dimensional Lorentzian space-time with zero length.

   (c) Find and sketch the collection of all vectors in three-dimensional Lorentzian space-time with zero length.

`The Story of Your Life`

6 Vector Spaces

Thus far we have thought of vectors as lists of numbers in $\mathbb{R}^n$. As it turns out, the notion of a vector applies to a much more general class of structures than this. The main idea is to define vectors based on their most important properties. Once complete, our new definition of vectors will include vectors in $\mathbb{R}^n$, but will also cover many other extremely useful notions of vectors. We do this in the hope of creating a mathematical structure applicable to a wide range of real-world problems.

The two key properties of vectors are that they can be added together and multiplied by scalars. So we make the following definition.

Definition A *vector space* (over $\mathbb{R}$) is a set V with two operations $+$ and $\cdot$ satisfying the following properties for all $u, v \in V$ and $c, d \in \mathbb{R}$:

(+i) (Additive Closure) $u + v \in V$. (Adding two vectors gives a vector.)

(+ii) (Additive Commutativity) $u + v = v + u$. (Order of addition doesn't matter.)

(+iii) (Additive Associativity) $(u + v) + w = u + (v + w)$ (Order of adding many vectors doesn't matter.)

(+iv) (Zero) There is a special vector $0_V \in V$ such that $u + 0_V = u$ for all u in V.

(+v) (Additive Inverse) For every $u \in V$ there exists $w \in V$ such that $u + w = 0_V$.

($\cdot$ i) (Multiplicative Closure) $c \cdot v \in V$. (Scalar times a vector is a vector.)

($\cdot$ ii) (Distributivity) $(c+d)\cdot v = c\cdot v + d\cdot v$. (Scalar multiplication distributes over addition of scalars.)

($\cdot$ iii) (Distributivity) $c\cdot(u+v) = c\cdot u + c\cdot v$. (Scalar multiplication distributes over addition of vectors.)

($\cdot$ iv) (Associativity) $(cd) \cdot v = c \cdot (d \cdot v)$.

($\cdot$ v) (Unity) $1 \cdot v = v$ for all $v \in V$.

Examples of each rule

Remark Don't confuse the scalar product $\cdot$ with the dot product $\bullet$. The scalar product is a function that takes a vector and a number and returns a vector. (In notation, this can be written $\cdot\colon \mathbb{R} \times V \to V$.) On the other hand, the dot product takes two vectors and returns a number. (In notation: $\bullet\colon V \times V \to \mathbb{R}$.)

Once the properties of a vector space have been verified, we'll just write scalar multiplication with juxtaposition $cv = c \cdot v$, though, to avoid confusing the notation.

Remark It isn't hard to devise strange rules for addition or scalar multiplication that break some or all of the rules listed above.

Example of a vector space

One can also find many interesting vector spaces, such as the following.

Example

$$V = \{f \mid f\colon \mathbb{N} \to \mathbb{R}\}$$

Here the vector space is the set of functions that take in a natural number n and return a real number. The addition is just addition of functions: $(f_1+f_2)(n) = f_1(n)+f_2(n)$. Scalar multiplication is just as simple: $c \cdot f(n) = cf(n)$.

We can think of these functions as infinite sequences: $f(0)$ is the first term, $f(1)$ is the second term, and so on. Then for example the function $f(n) = n^3$ would look like this:

$$f = \{0, 1, 8, 27, \ldots, n^3, \ldots\}.$$

Thinking this way, V is the space of all infinite sequences.

Let's check some axioms.

(+i) (Additive Closure) $f_1(n) + f_2(n)$ is indeed a function $\mathbb{N} \to \mathbb{R}$, since the sum of two real numbers is a real number.

(+iv) (Zero) We need to propose a zero vector. The constant zero function $g(n) = 0$ works because then $f(n) + g(n) = f(n) + 0 = f(n)$.

The other axioms that should be checked come down to properties of the real numbers.

Reading homework: problem 6.1

Example Another very important example of a vector space is the space of all differentiable functions:

$$\left\{ f \mid f \colon \mathbb{R} \to \mathbb{R},\ \frac{d}{dx} f \text{ exists} \right\}.$$

The addition is point-wise

$$(f+g)(x) = f(x) + g(x)\,,$$

as is scalar multiplication

$$c \cdot f(x) = cf(x)\,.$$

From calculus, we know that the sum of any two differentiable functions is differentiable, since the derivative distributes over addition. A scalar multiple of a function is also differentiable, since the derivative commutes with scalar multiplication ($\frac{d}{dx}(cf) = c\frac{d}{dx}f$). The zero function is just the function such that $0(x) = 0$ for every x. The rest of the vector space properties are inherited from addition and scalar multiplication in $\mathbb{R}$.

In fact, the set of functions with at least k derivatives is always a vector space, as is the space of functions with infinitely many derivatives.

Vector Spaces Over Other Fields Above, we defined vector spaces over the real numbers. One can actually define vector spaces over any *field*. A field is a collection of "numbers" satisfying a number of properties.

One other example of a field is the complex numbers,

$$\mathbb{C} = \left\{ x + iy \mid i^2 = -1, x, y \in \mathbb{R} \right\}.$$

In quantum physics, vector spaces over $\mathbb{C}$ describe all possible states a system of particles can have.

For example,

$$V = \left\{ \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \mid \lambda, \mu \in \mathbb{C} \right\}$$

describes states of an electron, where $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ describes spin "up" and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ describes spin "down". Other states, like $\begin{pmatrix} i \\ -i \end{pmatrix}$ are permissible, since the base field is the complex numbers.

Complex numbers are extremely useful because of a special property that they enjoy: every polynomial over the complex numbers factors into a product of linear polynomials. For example, the polynomial x^2+1 doesn't factor over the real numbers, but over the complex numbers it factors into $(x+i)(x-i)$. This property ends up having very far-reaching consequences: often in mathematics problems that are very difficult when working over the real numbers become relatively simple when working over the complex numbers. One example of this phenomenon occurs when diagonalizing matrices, which we will learn about later in the course.

Another useful field is the rational numbers $\mathbb{Q}$. This is field is important in computer algebra: a real number given by an infinite string of numbers after the decimal point can't be stored by a computer. So instead rational approximations are used. Since the rationals are a field, the mathematics of vector spaces still apply to this special case.

In this class, we will work mainly over the real numbers and the complex numbers, and occasionally work over $\mathbb{Z}_2 = \{0, 1\}$ where $1 + 1 = 0$. For more on fields in general, see Appendix E.3; however the full story of fields is typically covered in a class on abstract algebra or Galois theory.

References

Hefferon, Chapter One, Section I.1
Beezer, Chapter VS, Section VS
Wikipedia:

- Vector Space
- Field
- Spin $\frac{1}{2}$
- Galois Theory

Review Problems

1. Check that $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x, y \in \mathbb{R} \right\} = \mathbb{R}^2$ with the usual addition and scalar multiplication is a vector space.

2. Check that the complex numbers $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$ form a vector space over $\mathbb{C}$. Make sure you state carefully what your rules for

vector addition and scalar multiplication are. Also, explain what would happen if you used $\mathbb{R}$ as the base field (try comparing to problem 1).

3. (a) Consider the set of convergent sequences, with the same addition and scalar multiplication that we defined for the space of sequences:

$$V = \left\{ f \mid f \colon \mathbb{N} \to \mathbb{R}, \lim_{n\to\infty} f \in \mathbb{R} \right\}$$

   Is this still a vector space? Explain why or why not.

   (b) Now consider the set of divergent sequences, with the same addition and scalar multiplication as before:

$$V = \left\{ f \mid f \colon \mathbb{N} \to \mathbb{R}, \lim_{n\to\infty} f \text{ does not exist or is } \pm\infty \right\}$$

   Is this a vector space? Explain why or why not.

4. Consider the set of 2×4 matrices:

$$V = \left\{ \begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix} \mid a, b, c, d, e, f, g, h \in \mathbb{C} \right\}$$

   Propose definitions for addition and scalar multiplication in V. Identify the zero vector in V, and check that every matrix has an additive inverse.

5. Let $P_3^{\mathbb{R}}$ be the set of polynomials with real coefficients of degree three or less.

   - Propose a definition of addition and scalar multiplication to make $P_3^{\mathbb{R}}$ a vector space.
   - Identify the zero vector, and find the additive inverse for the vector $-3 - 2x + x^2$.
   - Show that $P_3^{\mathbb{R}}$ is not a vector space over $\mathbb{C}$. Propose a small change to the definition of $P_3^{\mathbb{R}}$ to make it a vector space over $\mathbb{C}$.

`Problem 5 hint`

7 Linear Transformations

Recall that the key properties of vector spaces are vector addition and scalar multiplication. Now suppose we have two vector spaces V and W and a map L between them:

$$L: V \to W$$

Now, both V and W have notions of vector addition and scalar multiplication. It would be ideal if the map L *preserved* these operations. In other words, if adding vectors and then applying L were the same as applying L to two vectors and then adding them. Likewise, it would be nice if, when multiplying by a scalar, it didn't matter whether we multiplied before or after applying L. In formulas, this means that for any $u, v \in V$ and $c \in \mathbb{R}$:

$$\begin{aligned} L(u+v) &= L(u) + L(v) \\ L(cv) &= cL(v) \end{aligned}$$

Combining these two requirements into one equation, we get the definition of a linear function or linear transformation.

Definition A function $L: V \to W$ is linear if for all $u, v \in V$ and $r, s \in \mathbb{R}$ we have

$$L(ru + sv) = rL(u) + sL(v)$$

Notice that on the left the addition and scalar multiplication occur in V, while on the right the operations occur in W. This is often called the *linearity property* of a linear transformation.

Reading homework: problem 7.1

Example Take $L: \mathbb{R}^3 \to \mathbb{R}^3$ defined by:

$$L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ y+z \\ 0 \end{pmatrix}$$

Call $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Now check linearity.

$$
\begin{aligned}
L(ru+sv) &= L\left(r\begin{pmatrix}x\\y\\z\end{pmatrix}+s\begin{pmatrix}a\\b\\c\end{pmatrix}\right)\\
&= L\left(\begin{pmatrix}rx\\ry\\rz\end{pmatrix}+\begin{pmatrix}sa\\sb\\sc\end{pmatrix}\right)\\
&= L\begin{pmatrix}rx+sa\\ry+sb\\rz+sx\end{pmatrix}\\
&= \begin{pmatrix}rx+sa+ry+sb\\ry+sb+rz+sx\\0\end{pmatrix}
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
rL(u)+sL(v) &= rL\begin{pmatrix}x\\y\\z\end{pmatrix}+sL\begin{pmatrix}a\\b\\c\end{pmatrix}\\
&= r\begin{pmatrix}x+y\\y+z\\0\end{pmatrix}+s\begin{pmatrix}a+b\\b+c\\0\end{pmatrix}\\
&= \begin{pmatrix}rx+ry\\ry+rz\\0\end{pmatrix}+\begin{pmatrix}sa+sb\\sb+sc\\0\end{pmatrix}\\
&= \begin{pmatrix}rx+sa+ry+sb\\ry+sb+rz+sx\\0\end{pmatrix}
\end{aligned}
$$

Then the two sides of the linearity requirement are equal, so L is a linear transformation.

Remark We can write the linear transformation L in the previous example using a matrix like so:

$$
L\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x+y\\y+z\\0\end{pmatrix}=\begin{pmatrix}1&1&0\\0&1&1\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}
$$

Reading homework: problem 7.2

We previously checked that matrix multiplication on vectors obeyed the rule $M(ru+sv) = rMu + sMv$, so matrix multiplication is linear. As such, our check on L was guaranteed to work. In fact, matrix multiplication on vectors is a linear transformation.

A linear and non-linear example

Example Let V be the vector space of polynomials of finite degree with standard addition and scalar multiplication.

$$V = \{a_0 + a_1 x + \cdots + a_n x^n | n \in \mathbb{N}, a_i \in \mathbb{R}\}$$

Let $L \colon V \to V$ be the derivative $\frac{d}{dx}$. For p_1 and p_2 polynomials, the rules of differentiation tell us that

$$\frac{d}{dx}(rp_1 + sp_2) = r\frac{dp_1}{dx} + s\frac{dp_2}{dx}$$

Thus, the derivative is a linear function from the set of polynomials to itself.

We can represent a polynomial as a "semi-infinite vector", like so:

$$a_0 + a_1 x + \cdots + a_n x^n \longleftrightarrow \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

Then we have:

$$\frac{d}{dx}(a_0 + a_1 x + \cdots + a_n x^n) = a_1 + 2a_2 x + \cdots + na_n x^{n-1} \longleftrightarrow \begin{pmatrix} a_1 \\ 2a_2 \\ \vdots \\ na_n \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

One could then write the derivative as an "infinite matrix":

$$\frac{d}{dx} \longleftrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Foreshadowing Dimension. You probably have some intuitive notion of what dimension means, though we haven't actually defined the idea of dimension mathematically yet. Some of the examples of vector spaces we have worked with have been finite dimensional. (For example, $\mathbb{R}^n$ will turn out to have dimension n.) The polynomial example above is an example of an infinite dimensional vector space.

Roughly speaking, dimension is the number of independent directions available. To figure out dimension, I stand at the origin, and pick a direction. If there are any vectors in my vector space that aren't in that direction, then I choose another direction that isn't in the line determined by the direction I chose. If there are any vectors in my vector space not in the plane determined by the first two directions, then I choose one of them as my next direction. In other words, I choose a collection of *independent* vectors in the vector space. The size of a minimal set of independent vectors is the dimension of the vector space.

For finite dimensional vector spaces, linear transformations can always be represented by matrices. For that reason, we will start studying matrices intensively in the next few lectures.

References

Hefferon, Chapter Three, Section II. (Note that Hefferon uses the term *homomorphism* for a linear map. 'Homomorphism' is a very general term which in mathematics means 'Structure-preserving map.' A linear map preserves the linear structure of a vector space, and is thus a type of homomorphism.)

Beezer, Chapter LT, Section LT, Subsections LT, LTC, and MLT.
Wikipedia:

- Linear Transformation
- Dimension

Review Problems

1. Show that the pair of conditions:

   (i) $L(u+v) = L(u) + L(v)$

   (ii) $L(cv) = cL(v)$

   is equivalent to the single condition:

(iii) $L(ru + sv) = rL(u) + sL(v)$.

Your answer should have two parts. Show that (i,ii)$\Rightarrow$(iii), and then show that (iii)$\Rightarrow$(i,ii).

2. Let P_n be the space of polynomials of degree n or less in the variable t. Suppose L is a linear transformation from $P_2 \to P_3$ such that $L(1) = 4, L(t) = t^3$, and $L(t^2) = t - 1$.

    - Find $L(1 + t + 2t^2)$.
    - Find $L(a + bt + ct^2)$.
    - Find all values a, b, c such that $L(a + bt + ct^2) = 1 + 3t + 2t^3$.

    `Hint`

3. Show that integration is a linear transformation on the vector space of polynomials. What would a matrix for integration look like? Be sure to think about what to do with the constant of integration.

    `Finite degree example`

4. Let $z \in \mathbb{C}$. Recall that we can express $z = a+bi$ where $a, b \in \mathbb{R}$, and we can form the *complex conjugate* of z by taking $\overline{z} = a - bi$ (note that this is unique since $\overline{\overline{z}} = z$). So we can define a function from $c\colon \mathbb{R}^2 \to \mathbb{R}^2$ which sends $(a, b) \mapsto (a, -b)$, and it is clear that c agrees with complex conjugation.

    (a) Show that c is a linear map over $\mathbb{R}$ (i.e. scalars in $\mathbb{R}$).

    (b) Show that $\overline{z}$ is not linear over $\mathbb{C}$.

8 Matrices

Definition An $r \times k$ matrix $M = (m^i_j)$ for $i = 1, \ldots, r; j = 1, \ldots, k$ is a rectangular array of real (or complex) numbers:

$$M = \begin{pmatrix} m^1_1 & m^1_2 & \cdots & m^1_k \\ m^2_1 & m^2_2 & \cdots & m^2_k \\ \vdots & \vdots & & \vdots \\ m^r_1 & m^r_2 & \cdots & m^r_k \end{pmatrix}$$

The numbers m^i_j are called *entries*. The superscript indexes the row of the matrix and the subscript indexes the column of the matrix in which m^i_j appears[2].

It is often useful to consider matrices whose entries are more general than the real numbers, so we allow that possibility.

An $r \times 1$ matrix $v = (v^r_1) = (v^r)$ is called a *column vector*, written

$$v = \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^r \end{pmatrix} .$$

A $1 \times k$ matrix $v = (v^1_k) = (v_k)$ is called a *row vector*, written

$$v = \begin{pmatrix} v_1 & v_2 & \cdots & v_k \end{pmatrix} .$$

Matrices are a very useful and efficient way to store information:

Example In computer graphics, you may have encountered image files with a .gif extension. These files are actually just matrices: at the start of the file the size of the matrix is given, and then each entry of the matrix is a number indicating the color of a particular pixel in the image.

The resulting matrix then has its rows shuffled a bit: by listing, say, every eighth row, then a web browser downloading the file can start displaying an incomplete version of the picture before the download is complete.

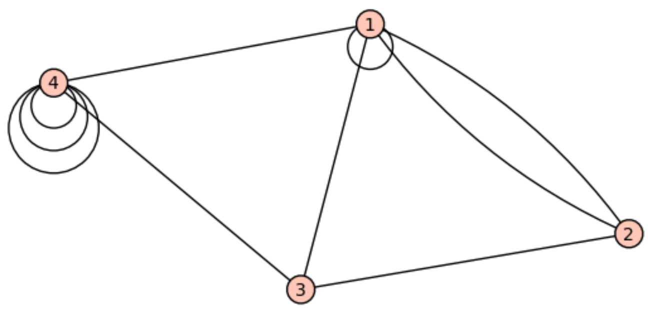
Finally, a compression algorithm is applied to the matrix to reduce the size of the file.

---

[2]This notation was first introduced by Albert Einstein.

Adjacency Matrix Example

Example Graphs occur in many applications, ranging from telephone networks to airline routes. In the subject of *graph theory*, a graph is just a collection of vertices and some edges connecting vertices. A matrix can be used to indicate how many edges attach one vertex to another.



For example, the graph pictured above would have the following matrix, where m^i_j indicates the number of edges between the vertices labeled i and j:

$$M = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 3 \end{pmatrix}$$

This is an example of a *symmetric matrix*, since $m^i_j = m^j_i$.

The space of $r \times k$ matrices M^r_k is a vector space with the addition and scalar multiplication defined as follows:

$$\begin{aligned} M + N &= (m^i_j) + (n^i_j) = (m^i_j + n^i_j) \\ rM &= r(m^i_j) = (rm^i_j) \end{aligned}$$

In other words, addition just adds corresponding entries in two matrices, and scalar multiplication multiplies every entry. Notice that $M^n_1 = \mathbb{R}^n$ is just the vector space of column vectors.

Recall that we can multiply an $r \times k$ matrix by a $k \times 1$ column vector to produce a $r \times 1$ column vector using the rule

$$MV = \big(\sum_{j=1}^{k} m^i_j v^j\big) .$$

This suggests a rule for multiplying an $r \times k$ matrix M by a $k \times s$ matrix N: our $k \times s$ matrix N consists of s column vectors side-by-side, each of dimension $k \times 1$. We can multiply our $r \times k$ matrix M by each of these s column vectors using the rule we already know, obtaining s column vectors each of dimension $r \times 1$. If we place these s column vectors side-by-side, we obtain an $r \times s$ matrix MN.

That is, let

$$N = \begin{pmatrix} n^1_1 & n^1_2 & \cdots & n^1_s \\ n^2_1 & n^2_2 & \cdots & n^2_s \\ \vdots & \vdots & & \vdots \\ n^k_1 & n^k_2 & \cdots & n^k_s \end{pmatrix}$$

and call the columns N_1 through N_s:

$$N_1 = \begin{pmatrix} n^1_1 \\ n^2_1 \\ \vdots \\ n^k_1 \end{pmatrix}, N_2 = \begin{pmatrix} n^1_2 \\ n^2_2 \\ \vdots \\ n^k_2 \end{pmatrix}, \ldots, N_s = \begin{pmatrix} n^1_s \\ n^2_s \\ \vdots \\ n^k_s \end{pmatrix} .$$

Then

$$MN = M \begin{pmatrix} | & | & & | \\ N_1 & N_2 & \cdots & N_s \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ MN_1 & MN_2 & \cdots & MN_s \\ | & | & & | \end{pmatrix}$$

A more concise way to write this rule is: If $M = (m^i_j)$ for $i = 1, \ldots, r; j = 1, \ldots, k$ and $N = (n^i_j)$ for $i = 1, \ldots, k; j = 1, \ldots, s$, then $MN = L$ where $L = (\ell^i_j)$ for $i = i, \ldots, r; j = 1, \ldots, s$ is given by

$$\ell^i_j = \sum_{p=1}^{k} m^i_p n^p_j.$$

This rule obeys linearity.

Notice that in order for the multiplication to make sense, the columns and rows must match. For an $r \times k$ matrix M and an $s \times m$ matrix N, then to make the product MN we must have $k = s$. Likewise, for the product NM, it is required that $m = r$. A common shorthand for keeping track of the sizes of the matrices involved in a given product is:

$$\Big(r \times k\Big) \times \Big(k \times m\Big) = \Big(r \times m\Big)$$

Example Multiplying a (3×1) matrix and a (1×2) matrix yields a (3×2) matrix.

$$\begin{pmatrix}1\\3\\2\end{pmatrix}\begin{pmatrix}2 & 3\end{pmatrix} = \begin{pmatrix}1\cdot 2 & 1\cdot 3\\3\cdot 2 & 3\cdot 3\\2\cdot 2 & 2\cdot 3\end{pmatrix} = \begin{pmatrix}2 & 3\\6 & 9\\4 & 6\end{pmatrix}$$

Reading homework: problem 8.1

Recall that $r \times k$ matrices can be used to represent linear transformations $\mathbb{R}^k \to \mathbb{R}^r$ via

$$MV = \sum_{j=1}^{k} m^i_j v^j ,$$

which is the same rule we use when we multiply an $r \times k$ matrix by a $k \times 1$ vector to produce an $r \times 1$ vector.

Likewise, we can use a matrix $N = (n^i_j)$ to represent a linear transformation

$$L \colon M^s_k \xrightarrow{N} M^r_k$$

via

$$L(M)^i_l = \sum_{j=1}^{s} n^i_j m^j_l .$$

This is the same as the rule we use to multiply matrices. In other words, $L(M) = NM$ is a linear transformation.

Matrix Terminology The entries m^i_i are called *diagonal*, and the set $\{m^1_1, m^2_2, \ldots\}$ is called the *diagonal of the matrix*.

Any $r \times r$ matrix is called a *square matrix*. A square matrix that is zero for all non-diagonal entries is called a diagonal matrix.

The $r \times r$ diagonal matrix with all diagonal entries equal to 1 is called the *identity matrix*, I_r, or just $\mathbb{1}$. An identity matrix looks like

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

The identity matrix is special because

$$I_r M = M I_k = M$$

for all M of size $r \times k$.

In the matrix given by the product of matrices above, the diagonal entries are 2 and 9. An example of a diagonal matrix is

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Definition The *transpose* of an $r \times k$ matrix $M = (m_j^i)$ is the $k \times r$ matrix with entries

$$M^T = (\bar{m}_j^i)$$

with $\bar{m}_j^i = m_i^j$.

A matrix M is *symmetric* if $M = M^T$.

Example $\begin{pmatrix} 2 & 5 & 6 \\ 1 & 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 2 & 1 \\ 5 & 3 \\ 6 & 4 \end{pmatrix}$

Reading homework: problem 8.2

Observations

- Only square matrices can be symmetric.
- The transpose of a column vector is a row vector, and vice-versa.
- Taking the transpose of a matrix twice does nothing. *i.e.*, $(M^T)^T = M$.

Theorem 8.1 (Transpose and Multiplication)**.** *Let M, N be matrices such that MN makes sense. Then $(MN)^T = N^T M^T$.*

The proof of this theorem is left to Review Question 2.

Sometimes matrices do not share the properties of regular numbers, watch this video to see why:

`Matrices do not Commute`

Many properties of matrices following from the same property for real numbers. Here is an example.

Example *Associativity of matrix multiplication.* We know for real numbers x, y and z that

$$x(yz) = (xy)z\,,$$

$i.e.$ the order of bracketing does not matter. The same property holds for matrix multiplication, let us show why. Suppose $M = \big(m^i_j\big)$, $N = \big(n^j_k\big)$ and $R = \big(r^k_l\big)$ are, respectively, $m \times n$, $n \times r$ and $r \times t$ matrices. Then from the rule for matrix multiplication we have

$$MN = \Big(\sum_{j=1}^{n} m^i_j n^j_k\Big) \text{ and } NR = \Big(\sum_{k=1}^{r} n^j_k r^k_l\Big)\,.$$

So first we compute

$$(MN)R = \Big(\sum_{k=1}^{r}\Big[\sum_{j=1}^{n} m^i_j n^j_k\Big] r^k_l\Big) = \Big(\sum_{k=1}^{r}\sum_{j=1}^{n}\Big[m^i_j n^j_k\Big] r^k_l\Big) = \Big(\sum_{k=1}^{r}\sum_{j=1}^{n} m^i_j n^j_k r^k_l\Big)\,.$$

In the first step we just wrote out the definition for matrix multiplication, in the second step we moved summation symbol outside the bracket (this is just the distributive property $x(y+z) = xy+xz$ for numbers) and in the last step we used the associativity property for real numbers to remove the square brackets. Exactly the same reasoning shows that

$$M(NR) = \Big(\sum_{j=1}^{n} m^i_j\Big[\sum_{k=1}^{r} n^j_k r^k_l\Big]\Big) = \Big(\sum_{k=1}^{r}\sum_{j=1}^{n} m^i_j\Big[n^j_k r^k_l\Big]\Big) = \Big(\sum_{k=1}^{r}\sum_{j=1}^{n} m^i_j n^j_k r^k_l\Big)\,.$$

This is the same as above so we are done. *As a fun remark, note that Einstein would simply have written $(MN)R = (m^i_j n^j_k) r^k_l = m^i_j n^j_k r^k_l = m^i_j (n^j_k r^k_l) = M(NR)$.*

References

Hefferon, Chapter Three, Section IV, parts 1-3.
Beezer, Chapter M, Section MM.
Wikipedia:

- Matrix Multiplication

Review Problems

1. Compute the following matrix products

$$\begin{pmatrix}1&2&1\\4&5&2\\7&8&2\end{pmatrix}\begin{pmatrix}-2&\frac{4}{3}&-\frac{1}{3}\\2&-\frac{5}{3}&\frac{2}{3}\\-1&2&-1\end{pmatrix}, \qquad \begin{pmatrix}1&2&3&4&5\end{pmatrix}\begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix},$$

$$\begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix}\begin{pmatrix}1&2&3&4&5\end{pmatrix}, \qquad \begin{pmatrix}1&2&1\\4&5&2\\7&8&2\end{pmatrix}\begin{pmatrix}-2&\frac{4}{3}&-\frac{1}{3}\\2&-\frac{5}{3}&\frac{2}{3}\\-1&2&-1\end{pmatrix}\begin{pmatrix}1&2&1\\4&5&2\\7&8&2\end{pmatrix},$$

$$\begin{pmatrix}x&y&z\end{pmatrix}\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}, \qquad \begin{pmatrix}2&1&2&1&2\\0&2&1&2&1\\0&1&2&1&2\\0&2&1&2&1\\0&0&0&0&2\end{pmatrix}\begin{pmatrix}1&2&1&2&1\\0&1&2&1&2\\0&2&1&2&1\\0&1&2&1&2\\0&0&0&0&1\end{pmatrix},$$

$$\begin{pmatrix}-2&\frac{4}{3}&-\frac{1}{3}\\2&-\frac{5}{3}&\frac{2}{3}\\-1&2&-1\end{pmatrix}\begin{pmatrix}4&\frac{2}{3}&-\frac{2}{3}\\6&\frac{5}{3}&-\frac{2}{3}\\12&-\frac{16}{3}&\frac{10}{3}\end{pmatrix}\begin{pmatrix}1&2&1\\4&5&2\\7&8&2\end{pmatrix}.$$

2. Let's prove the theorem $(MN)^T = N^T M^T$.

   Note: the following is a common technique for proving matrix identities.

(a) Let $M = (m^i_j)$ and let $N = (n^i_j)$. Write out a few of the entries of each matrix in the form given at the beginning of this chapter.

(b) Multiply out MN and write out a few of its entries in the same form as in part a. In terms of the entries of M and the entries of N, what is the entry in row i and column j of MN?

(c) Take the transpose $(MN)^T$ and write out a few of its entries in the same form as in part a. In terms of the entries of M and the entries of N, what is the entry in row i and column j of $(MN)^T$?

(d) Take the transposes N^T and M^T and write out a few of their entries in the same form as in part a.

(e) Multiply out $N^T M^T$ and write out a few of its entries in the same form as in part a. In terms of the entries of M and the entries of N, what is the entry in row i and column j of $N^T M^T$?

(f) Show that the answers you got in parts c and e are the same.

3. Let M be any $m \times n$ matrix. Show that $M^T M$ and $M M^T$ are symmetric. (Hint: use the result of the previous problem.) What are their sizes?

4. Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ be column vectors. Show that the dot product $x \cdot y = x^T \, \mathbb{1} \, y$.

Hint

5. Above, we showed that *left* multiplication by an $r \times s$ matrix N was a linear transformation $M^s_k \xrightarrow{N} M^r_k$. Show that *right* multiplication by a $k \times m$ matrix R is a linear transformation $M^s_k \xrightarrow{R} M^s_m$. In other words, show that right matrix multiplication obeys linearity.

Problem hint

6. Explain what happens to a matrix when:

   (a) You multiply it on the left by a diagonal matrix.

   (b) You multiply it on the right by a diagonal matrix.

   Give a few simple examples before you start explaining.

9 Properties of Matrices

9.1 Block Matrices

It is often convenient to partition a matrix M into smaller matrices called *blocks*, like so:

$$M = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 1 \\ \hline 0 & 1 & 2 & 0 \end{array}\right) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

Here $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 & 2 \end{pmatrix}$, $D = (0)$.

- The blocks of a block matrix must fit together to form a rectangle. So $\left(\begin{array}{c|c} B & A \\ \hline D & C \end{array}\right)$ makes sense, but $\left(\begin{array}{c|c} C & B \\ \hline D & A \end{array}\right)$ does not.

Reading homework: problem 9.1

- There are many ways to cut up an $n \times n$ matrix into blocks. Often context or the entries of the matrix will suggest a useful way to divide the matrix into blocks. For example, if there are large blocks of zeros in a matrix, or blocks that look like an identity matrix, it can be useful to partition the matrix accordingly.

- Matrix operations on block matrices can be carried out by treating the blocks as matrix entries. In the example above,

$$\begin{aligned} M^2 &= \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \\ &= \left(\begin{array}{c|c} A^2 + BC & AB + BD \\ \hline CA + DC & CB + D^2 \end{array}\right) \end{aligned}$$

Computing the individual blocks, we get:

$$\begin{aligned}
A^2 + BC &= \begin{pmatrix} 30 & 37 & 44 \\ 66 & 81 & 96 \\ 102 & 127 & 152 \end{pmatrix} \\
AB + BD &= \begin{pmatrix} 4 \\ 10 \\ 16 \end{pmatrix} \\
CA + DC &= \begin{pmatrix} 18 \\ 21 \\ 24 \end{pmatrix} \\
CB + D^2 &= (2)
\end{aligned}$$

Assembling these pieces into a block matrix gives:

$$\left(\begin{array}{ccc|c} 30 & 37 & 44 & 4 \\ 66 & 81 & 96 & 10 \\ 102 & 127 & 152 & 16 \\ \hline 4 & 10 & 16 & 2 \end{array}\right)$$

This is exactly M^2.

9.2 The Algebra of Square Matrices

Not every pair of matrices can be multiplied. When multiplying two matrices, the number of rows in the left matrix must equal the number of columns in the right. For an $r \times k$ matrix M and an $s \times l$ matrix N, then we must have $k = s$.

This is not a problem for square matrices of the same size, though. Two $n \times n$ matrices can be multiplied in either order. For a single matrix $M \in M_n^n$, we can form $M^2 = MM$, $M^3 = MMM$, and so on, and define $M^0 = I_n$, the identity matrix.

As a result, any polynomial equation can be evaluated on a matrix.

Example Let $f(x) = x - 2x^2 + 3x^3$.

Let $M = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Then:

$$M^2 = \begin{pmatrix} 1 & 2t \\ 0 & 1 \end{pmatrix}, M^3 = \begin{pmatrix} 1 & 3t \\ 0 & 1 \end{pmatrix}, \ldots$$

Hence:

$$
\begin{aligned}
f(M) &= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} - 2 \begin{pmatrix} 1 & 2t \\ 0 & 1 \end{pmatrix} + 3 \begin{pmatrix} 1 & 3t \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 6t \\ 0 & 2 \end{pmatrix}
\end{aligned}
$$

Suppose $f(x)$ is any function defined by a convergent Taylor Series:

$$f(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \cdots$$

Then we can define the matrix function by just plugging in M:

$$f(M) = f(0) + f'(0)M + \frac{1}{2!}f''(0)M^2 + \cdots$$

There are additional techniques to determine the convergence of Taylor Series of matrices, based on the fact that the convergence problem is simple for diagonal matrices. It also turns out that $\exp(M) = 1 + M + \frac{1}{2}M^2 + \frac{1}{3!}M^3 + \cdots$ always converges.

Matrix Exponential Example

Matrix multiplication does not commute. For *generic* $n \times n$ square matrices M and N, then $MN \neq NM$. For example:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

On the other hand:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Since $n \times n$ matrices are linear transformations $\mathbb{R}^n \to \mathbb{R}^n$, we can see that the order of successive linear transformations matters. For two linear transformations K and L taking $\mathbb{R}^n \to \mathbb{R}^n$, and $v \in \mathbb{R}^n$, then in general

$$K(L(v)) \neq L(K(v))\,.$$

Finding matrices such that $MN = NM$ is an important problem in mathematics.

Here is an example of matrices acting on objects in three dimensions that also shows matrices not commuting.

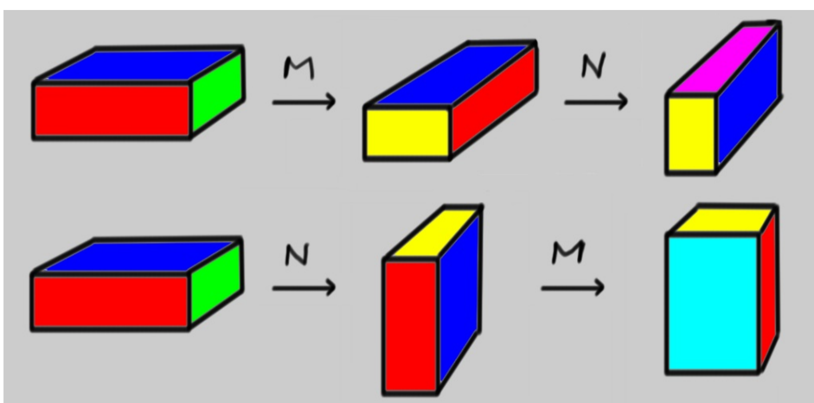
Example You learned in a Review Problem that the matrix

$$M = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

rotates vectors in the plane by an angle θ. We can generalize this, using block matrices, to three dimensions. In fact the following matrices built from a 2×2 rotation matrix, a 1×1 identity matrix and zeroes everywhere else

$$M = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix},$$

perform rotations by an angle θ in the xy and yz planes, respectively. Because, they rotate single vectors, you can also use them to rotate objects built from a collection of vectors like pretty colored blocks! Here is a picture of M and then N acting on such a block, compared with the case of N followed by M. The special case of $\theta = 90°$ is shown.



Notice how the end product of MN and NM are different, so $MN \neq NM$ here.

Trace

Matrices contain a great deal of information, so finding ways to extract essential information is useful. Here we need to assume that $n < \infty$ otherwise there are subtleties with convergence that we'd have to address.

Definition The *trace* of a square matrix $M = (m^i_j)$ is the sum of its diagonal entries.

$$\operatorname{tr} M = \sum_{i=1}^{n} m^i_i \,.$$

Example

$$\operatorname{tr}\begin{pmatrix} 2 & 7 & 6 \\ 9 & 5 & 1 \\ 4 & 3 & 8 \end{pmatrix} = 2 + 5 + 8 = 15$$

While matrix multiplication does not commute, the trace of a product of matrices does not depend on the order of multiplication:

$$\begin{aligned}
\operatorname{tr}(MN) &= \operatorname{tr}\Big(\sum_l M^i_l N^l_j\Big) \\
&= \sum_i \sum_l M^i_l N^l_i \\
&= \sum_l \sum_i N^l_i M^i_l \\
&= \operatorname{tr}\Big(\sum_i N^l_i M^i_l\Big) \\
&= \operatorname{tr}(NM).
\end{aligned}$$

Explanation of this Proof

Thus we have a Theorem:

Theorem 9.1.

$$\operatorname{tr}(MN) = \operatorname{tr}(NM)$$

for any square matrices M and N.

Example Continuing from the previous example,

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, N = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

so

$$MN = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \neq NM = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

However, $\operatorname{tr}(MN) = 2 + 1 = 3 = 1 + 2 = \operatorname{tr}(NM)$.

Another useful property of the trace is that:

$$\operatorname{tr} M = \operatorname{tr} M^T$$

This is true because the trace only uses the diagonal entries, which are fixed by the transpose. For example: $\operatorname{tr}\begin{pmatrix}1 & 1\\ 2 & 3\end{pmatrix} = 4 = \operatorname{tr}\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix} = \operatorname{tr}\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}^T$

Finally, trace is a linear transformation from matrices to the real numbers. This is easy to check.

More on the trace function

Linear Systems Redux Recall that we can view a linear system as a matrix equation

$$MX = V,$$

with M an $r \times k$ matrix of coefficients, X a $k \times 1$ matrix of unknowns, and V an $r \times 1$ matrix of constants. If M is a square matrix, then the number of equations r is the same as the number of unknowns k, so we have hope of finding a single solution.

Above we discussed functions of matrices. An extremely useful function would be $f(M) = \frac{1}{M}$, where $M\frac{1}{M} = I$. If we could compute $\frac{1}{M}$, then we would multiply both sides of the equation $MX = V$ by $\frac{1}{M}$ to obtain the solution immediately: $X = \frac{1}{M}V$.

Clearly, if the linear system has no solution, then there can be no hope of finding $\frac{1}{M}$, since if it existed we could find a solution. On the other hand, if the system has more than one solution, it also seems unlikely that $\frac{1}{M}$ would exist, since $X = \frac{1}{M}V$ yields only a single solution.

Therefore $\frac{1}{M}$ only sometimes exists. It is called the *inverse* of M, and is usually written M^{-1}.

References

Beezer: Part T, Section T
Wikipedia:

- Trace (Linear Algebra)
- Block Matrix

Review Problems

1. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{pmatrix}$. Find AA^T and A^TA. What can you say about matrices MM^T and M^TM in general? Explain.

2. Compute $\exp(A)$ for the following matrices:

   - $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$
   - $A = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$
   - $A = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$

Hint

3. Suppose $ad - bc \neq 0$, and let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

   (a) Find a matrix M^{-1} such that $MM^{-1} = I$.

   (b) Explain why your result explains what you found in a previous homework exercise.

   (c) Compute $M^{-1}M$.

4. Let $M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$. Divide M into named blocks, and then multiply blocks to compute M^2.

Definition A square matrix M is

INVERTIBLE / NON-SINGULAR

if there exists M^{-1} such that

$$\boxed{M^{-1}M = I = MM^{-1}}$$

5. A matrix A is called *anti-symmetric* (or skew-symmetric) if $A^T = -A$. Show that for every $n \times n$ matrix M, we can write $M = A + S$ where A is an anti-symmetric matrix and S is a symmetric matrix.

   Hint: What kind of matrix is $M + M^T$? How about $M - M^T$?

10 Inverse Matrix

Definition A square matrix M is *invertible* (or *nonsingular*) if there exists a matrix M^{-1} such that

$$M^{-1}M = I = M^{-1}M.$$

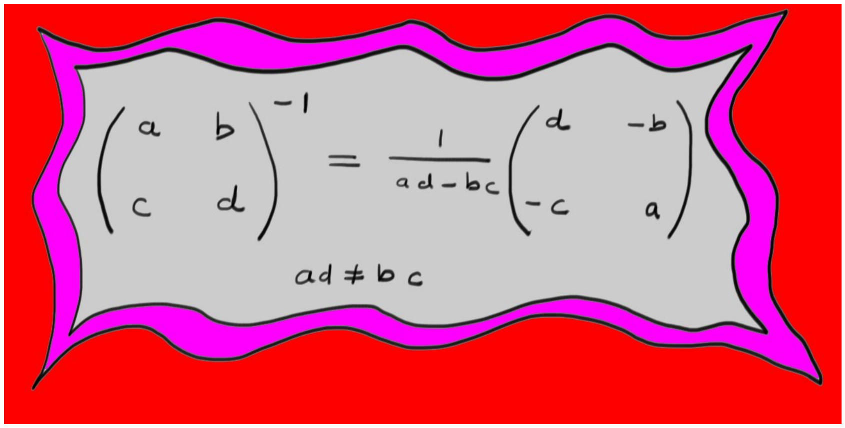
Inverse of a 2×2 Matrix Let M and N be the matrices:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad N = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Multiplying these matrices gives:

$$MN = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = (ad - bc)I$$

Then $M^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, so long as $ad - bc \neq 0$.



10.1 Three Properties of the Inverse

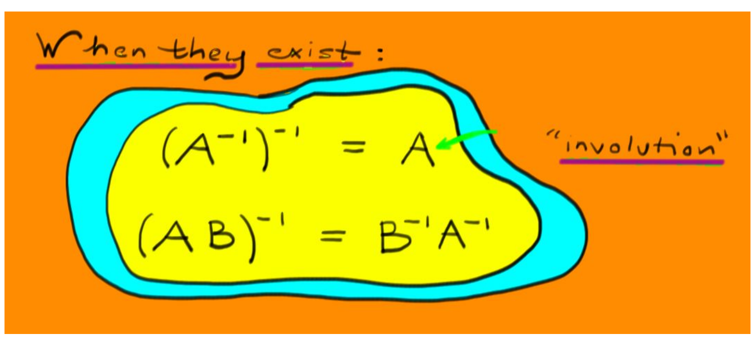
1. If A is a square matrix and B is the inverse of A, then A is the inverse of B, since $AB = I = BA$. Then we have the identity:

$$(A^{-1})^{-1} = A$$

2. Notice that $B^{-1}A^{-1}AB = B^{-1}IB = I = ABB^{-1}A^{-1}$. Then:

$$(AB)^{-1} = B^{-1}A^{-1}$$

Then much like the transpose, taking the inverse of a product *reverses* the order of the product.



3. Finally, recall that $(AB)^T = B^T A^T$. Since $I^T = I$, then $(A^{-1}A)^T = A^T(A^{-1})^T = I$. Similarly, $(AA^{-1})^T = (A^{-1})^T A^T = I$. Then:

$$(A^{-1})^T = (A^T)^{-1}$$

As such, we could even write A^{-T} for the inverse of the transpose of A (or equivalently the transpose of the inverse).

$$\boxed{(A^T)^{-1} = (A^{-1})^T}$$

Example

10.2 Finding Inverses

Suppose M is a square matrix and $MX = V$ is a linear system with unique solution X_0. Since there is a unique solution, $M^{-1}V$, then the reduced row echelon form of the linear system has an identity matrix on the left:

$$\big(M \mid V\big) \sim \big(I \mid M^{-1}V\big)$$

Solving the linear system $MX = V$ then tells us what $M^{-1}V$ is.

To solve many linear systems at once, we can consider augmented matrices with a matrix on the right side instead of a column vector, and then apply Gaussian row reduction to the left side of the matrix. Once the identity matrix is on the left side of the augmented matrix, then the solution of each of the individual linear systems is on the right.

To compute M^{-1}, we would like M^{-1}, rather than $M^{-1}V$ to appear on the right side of our augmented matrix. This is achieved by solving the collection of systems $MX = e_k$, where e_k is the column vector of zeroes with a 1 in the kth entry. *I.e.*, the $n \times n$ identity matrix can be viewed as a bunch of column vectors $I_n = (e_1\ e_2\ \cdots e_n)$. So, putting the e_k's together into an identity matrix, we get:

$$\big(M \mid I\big) \sim \big(I \mid M^{-1}I\big) = \big(I \mid M^{-1}\big)$$

Example Find $\begin{pmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{pmatrix}^{-1}$. Start by writing the augmented matrix, then apply row reduction to the left side.

$$
\left(\begin{array}{ccc|ccc}
-1 & 2 & -3 & 1 & 0 & 0 \\
2 & 1 & 0 & 0 & 1 & 0 \\
4 & -2 & 5 & 0 & 0 & 1
\end{array}\right)
\sim
\left(\begin{array}{ccc|ccc}
1 & -2 & 3 & 1 & 0 & 0 \\
0 & 5 & -6 & 2 & 1 & 0 \\
0 & 6 & -7 & 4 & 0 & 1
\end{array}\right)
$$

$$
\sim
\left(\begin{array}{ccc|ccc}
1 & 0 & \frac{3}{5} & \frac{-1}{4} & \frac{2}{5} & 0 \\
0 & 1 & \frac{-6}{5} & \frac{2}{5} & \frac{1}{5} & 0 \\
0 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{-6}{5} & 1
\end{array}\right)
$$

$$
\sim
\left(\begin{array}{ccc|ccc}
1 & 0 & 0 & -5 & 4 & -3 \\
0 & 1 & 0 & 10 & -7 & 6 \\
0 & 0 & 1 & 8 & -6 & 5
\end{array}\right)
$$

At this point, we know M^{-1} assuming we didn't goof up. However, row reduction is a lengthy and arithmetically involved process, so we should *check our answer,* by confirming that $MM^{-1} = I$ (or if you prefer $M^{-1}M = I$):

$$
MM^{-1} = \begin{pmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{pmatrix} \begin{pmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

The product of the two matrices is indeed the identity matrix, so we're done.

Reading homework: problem 10.1

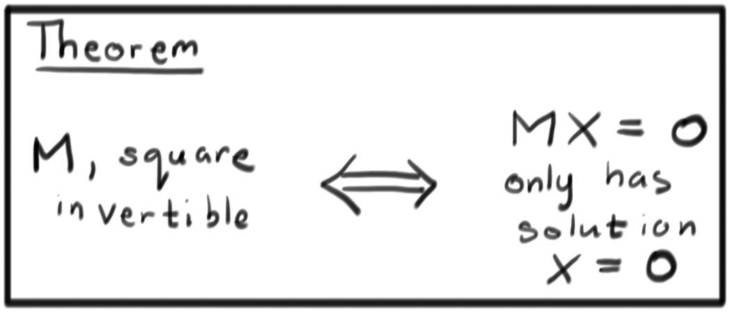
10.3 Linear Systems and Inverses

If M^{-1} exists and is known, then we can immediately solve linear systems associated to M.

Example Consider the linear system:

$$
\begin{aligned}
-x + 2y - 3z &= 1 \\
2x + y \phantom{{}-3z} &= 2 \\
4x - 2y + 5z &= 0
\end{aligned}
$$

The associated matrix equation is $MX = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, where M is the same as in the previous section. Then:



$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix}$$

Then $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix}$. In summary, when M^{-1} exists, then

$$MX = V \Rightarrow X = M^{-1}V\,.$$

Reading homework: problem 10.2

10.4 Homogeneous Systems

Theorem 10.1. *A square matrix M is invertible if and only if the homogeneous system*

$$MX = 0$$

has no non-zero solutions.

Proof. First, suppose that M^{-1} exists. Then $MX = 0 \Rightarrow X = M^{-1}0 = 0$. Thus, if M is invertible, then $MX = 0$ has no non-zero solutions.

On the other hand, $MX = 0$ always has the solution $X = 0$. If no other solutions exist, then M can be put into reduced row echelon form with every variable a pivot. In this case, M^{-1} can be computed using the process in the previous section. □

A great test of your linear algebra knowledge is to make a list of conditions for a matrix to be singular. You will learn more of these as the course goes by, but can also skip straight to the list in Section 24.1.

10.5 Bit Matrices

In computer science, information is recorded using binary strings of data. For example, the following string contains an English word:

$$011011000110100101101110011001010110000101110010$$

A *bit* is the basic unit of information, keeping track of a single one or zero. Computers can add and multiply individual bits very quickly.

Consider the set $\mathbb{Z}_2 = \{0,1\}$ with addition and multiplication given by the following tables:

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \qquad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

Notice that $-1 = 1$, since $1+1=0$.

It turns out that $\mathbb{Z}_2$ is almost as good as the real or complex numbers (they are all fields), so we can apply all of the linear algebra we have learned thus far to matrices with $\mathbb{Z}_2$ entries. A matrix with entries in $\mathbb{Z}_2$ is sometimes called a *bit matrix*[3].

Example $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ is an invertible matrix over $\mathbb{Z}_2$:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

This can be easily verified by multiplying:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Application: Cryptography A very simple way to hide information is to use a substitution cipher, in which the alphabet is permuted and each letter in a message is systematically exchanged for another. For example, the ROT-13 cypher just exchanges a letter with the letter thirteen places before or after it in the alphabet. For example, HELLO becomes URYYB. Applying the algorithm again decodes the message, turning

[3]Note that bits in a bit arithmetic shorthand do not "add" and "multiply" as elements in $\mathbb{Z}_2$ does since these operators corresponding to "bitwise or" and "bitwise and" respectively.

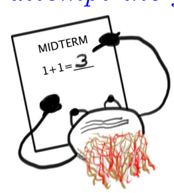
URYYB back into HELLO. Substitution ciphers are easy to break, but the basic idea can be extended to create cryptographic systems that are practically uncrackable. For example, a *one-time pad* is a system that uses a different substitution for each letter in the message. So long as a particular set of substitutions is not used on more than one message, the one-time pad is unbreakable.

English characters are often stored in computers in the ASCII format. In ASCII, a single character is represented by a string of eight bits, which we can consider as a vector in $\mathbb{Z}_2^8$ (which is like vectors in $\mathbb{R}^8$, where the entries are zeros and ones). One way to create a substitution cipher, then, is to choose an 8×8 invertible bit matrix M, and multiply each letter of the message by M. Then to decode the message, each string of eight characters would be multiplied by M^{-1}.

To make the message a bit tougher to decode, one could consider pairs (or longer sequences) of letters as a single vector in $\mathbb{Z}_2^{16}$ (or a higher-dimensional space), and then use an appropriately-sized invertible matrix.

For more on cryptography, see "The Code Book," by Simon Singh (1999, Doubleday).

You are now ready to attempt the first sample midterm.



References

Hefferon: Chapter Three, Section IV.2
Beezer: Chapter M, Section MISLE
Wikipedia: Invertible Matrix

Review Problems

1. Find formulas for the inverses of the following matrices, when they are not singular:

(a) $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$

When are these matrices singular?

2. Write down all 2×2 bit matrices and decide which of them are singular. For those which are not singular, pair them with their inverse.

3. Let M be a square matrix. Explain why the following statements are equivalent:

   (a) $MX = V$ has a *unique* solution for every column vector V.

   (b) M is non-singular.

   (In general for problems like this, think about the key words:

   First, suppose that there is some column vector V such that the equation $MX = V$ has two distinct solutions. Show that M must be singular; that is, show that M can have no inverse.

   Next, suppose that there is some column vector V such that the equation $MX = V$ has no solutions. Show that M must be singular.

   Finally, suppose that M is non-singular. Show that no matter what the column vector V is, there is a unique solution to $MX = V$.)

Hints for Problem 3

4. *Left and Right Inverses:* So far we have only talked about inverses of square matrices. This problem will explore the notion of a left and right inverse for a matrix that is not square. Let

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

   (a) Compute:

i. AA^T,

ii. $\left(AA^T\right)^{-1}$,

iii. $B := A^T\left(AA^T\right)^{-1}$

(b) Show that the matrix B above is a *right inverse* for A, *i.e.*, verify that
$$AB = I\,.$$

(c) Does BA make sense? (Why not?)

(d) Let A be an $n \times m$ matrix with $n > m$. Suggest a formula for a left inverse C such that
$$CA = I$$

Hint: you may assume that A^TA has an inverse.

(e) Test your proposal for a left inverse for the simple example
$$A = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\,,$$

(f) True or false: Left and right inverses are unique. If false give a counterexample.

`Left and Right Inverses`

11 LU Decomposition

Certain matrices are easier to work with than others. In this section, we will see how to write any square[4] matrix M as the product of two simpler matrices. We will write

$$M = LU\,,$$

where:

- L is *lower triangular*. This means that all entries above the main diagonal are zero. In notation, $L = (l^i_j)$ with $l^i_j = 0$ for all $j > i$.

$$L = \begin{pmatrix} l^1_1 & 0 & 0 & \cdots \\ l^2_1 & l^2_2 & 0 & \cdots \\ l^3_1 & l^3_2 & l^3_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- U is *upper triangular*. This means that all entries below the main diagonal are zero. In notation, $U = (u^i_j)$ with $u^i_j = 0$ for all $j < i$.

$$U = \begin{pmatrix} u^1_1 & u^1_2 & u^1_3 & \cdots \\ 0 & u^2_2 & u^2_3 & \cdots \\ 0 & 0 & u^3_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$M = LU$ is called an LU *decomposition* of M.

This is a useful trick for many computational reasons. It is much easier to compute the inverse of an upper or lower triangular matrix. Since inverses are useful for solving linear systems, this makes solving any linear system associated to the matrix much faster as well. The determinant—a very important quantity associated with any square matrix—is very easy to compute for triangular matrices.

Example Linear systems associated to upper triangular matrices are very easy to solve by back substitution.

$$\left(\begin{array}{cc|c} a & b & 1 \\ 0 & c & e \end{array}\right) \Rightarrow y = \frac{e}{c}\,, \quad x = \frac{1}{a}\left(1 - \frac{be}{c}\right)$$

[4] The case where M is not square is dealt with at the end of the lecture.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & d \\ a & 1 & 0 & e \\ b & c & 1 & f \end{array}\right) \Rightarrow x = d\,, \qquad y = e - ad\,, \qquad z = f - bd - c(e - ad)$$

For lower triangular matrices, *back* substitution gives a quick solution; for upper triangular matrices, *forward* substitution gives the solution.

11.1 Using LU Decomposition to Solve Linear Systems

Suppose we have $M = LU$ and want to solve the system

$$MX = LUX = V.$$

- Step 1: Set $W = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = UX$.
- Step 2: Solve the system $LW = V$. This should be simple by forward substitution since L is lower triangular. Suppose the solution to $LW = V$ is W_0.
- Step 3: Now solve the system $UX = W_0$. This should be easy by backward substitution, since U is upper triangular. The solution to this system is the solution to the original system.

We can think of this as using the matrix L to perform row operations on the matrix U in order to solve the system; this idea also appears in the study of determinants.

Reading homework: problem 11.1

Example Consider the linear system:

$$\begin{aligned} 6x + 18y + 3z &= 3 \\ 2x + 12y + z &= 19 \\ 4x + 15y + 3z &= 0 \end{aligned}$$

An LU decomposition for the associated matrix M is:

$$\begin{pmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 6 & 0 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 6 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- Step 1: Set $W = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = UX$.

- Step 2: Solve the system $LW = V$:

$$\begin{pmatrix} 3 & 0 & 0 \\ 1 & 6 & 0 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 3 \\ 19 \\ 0 \end{pmatrix}$$

By substitution, we get $u = 1$, $v = 3$, and $w = -11$. Then

$$W_0 = \begin{pmatrix} 1 \\ 3 \\ -11 \end{pmatrix}$$

- Step 3: Solve the system $UX = W_0$.

$$\begin{pmatrix} 2 & 6 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -11 \end{pmatrix}$$

Back substitution gives $z = -11, y = 3$, and $x = -3$.

Then $X = \begin{pmatrix} -3 \\ 3 \\ -11 \end{pmatrix}$, and we're done.

`Using a` LU `decomposition`

11.2 Finding an LU Decomposition.

For any given matrix, there are actually many different LU decompositions. However, there is a unique LU decomposition in which the L matrix has ones on the diagonal; then L is called a *lower unit triangular matrix*.

To find the LU decomposition, we'll create two sequences of matrices $L_0, L_1, \ldots$ and $U_0, U_1, \ldots$ such that at each step, $L_iU_i = M$. Each of the L_i will be lower triangular, but only the last U_i will be upper triangular.

Start by setting $L_0 = I$ and $U_0 = M$, because $L_0U_0 = M$. A main concept of this calculation is captured by the following example:

LU method for $MX = V$

(i) Write $M = LU \Rightarrow L\underbrace{UX}_{W} = V$

(ii) Solve $LW = V$

(iii) Solve $UX = W$

Example Consider

$$E = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}, \qquad M = \begin{pmatrix} a & b & c & \cdots \\ d & e & f & \cdots \end{pmatrix}.$$

Lets compute EM

$$EM = \begin{pmatrix} a & b & c & \cdots \\ d + \lambda a & e + \lambda b & f + \lambda c & \cdots \end{pmatrix},.$$

Something neat happened here: multiplying M by E performed the row operation $R_2 \to R_2 + \lambda R - 1$ on M. Another interesting fact:

$$E^{-1} := \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}$$

obeys (check this yourself...)

$$E^{-1}E = 1\,.$$

Hence $M = E^{-1}EM$ or, writing this out

$$\begin{pmatrix} a & b & c & \cdots \\ d & e & f & \cdots \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix} \begin{pmatrix} a & b & c & \cdots \\ d + \lambda a & e + \lambda b & f + \lambda c & \cdots \end{pmatrix}.$$

Here the matrix on the left is lower triangular, while the matrix on the right has had a row operation performed on it.

We would like to use the first row of U_0 to zero out the first entry of every row below it. For our running example,

$$U_0 = M = \begin{pmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{pmatrix},$$

so we would like to perform the row operations $R_2 \to R_2 - \frac{1}{3}R_1$ and $R_3 \to R_3 - \frac{2}{3}R_1$. If we perform these row operations on U_0 to produce

$$U_1 = \begin{pmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 3 & 1 \end{pmatrix},$$

we need to multiply this on the left by a lower triangular matrix L_1 so that the product $L_1U_1 = M$ still. The above example shows how to do this: Set L_1 to be the lower triangular matrix whose first column is filled with the minus constants used to zero out the first column of M. Then

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & 0 & 1 \end{pmatrix}.$$

By construction $L_1U_1 = M$, but you should compute this yourself as a double check.

Now repeat the process by zeroing the second column of U_1 below the diagonal using the second row of U_1 using the row operation $R_3 \to R_3 - \frac{1}{2}R_2$ to produce

$$U_2 = \begin{pmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrix that undoes this row operation is obtained in the same way we found L_1 above and is:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

Thus our answer for L_2 is the product of this matrix with L_1, namely

$$L_2 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & \frac{1}{2} & 1 \end{pmatrix}.$$

Notice that it is lower triangular because

THE PRODUCT OF LOWER TRIANGULAR MATRICES IS ALWAYS LOWER TRIANGULAR!

Moreover it is obtained by recording minus the constants used for all our row operations in the appropriate columns (this always works this way). Moreover, U_2 is upper triangular and $M = L_2 U_2$, we are done! Putting this all together we have

$$M = \begin{pmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

If the matrix you're working with has more than three rows, just continue this process by zeroing out the next column below the diagonal, and repeat until there's nothing left to do.

Another LU decomposition example

The fractions in the L matrix are admittedly ugly. For two matrices LU, we can multiply one entire column of L by a constant λ and divide the corresponding row of U by the same constant without changing the product of the two matrices. Then:

$$\begin{aligned} LU &= \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & \frac{1}{2} & 1 \end{pmatrix} I \begin{pmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 & 0 \\ 1 & 6 & 0 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 6 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \end{aligned}$$

The resulting matrix looks nicer, but isn't in standard form.

Reading homework: problem 11.2

For matrices that are not square, LU decomposition still makes sense. Given an $m \times n$ matrix M, for example we could write $M = LU$ with L a square lower unit triangular matrix, and U a rectangular matrix. Then L will be an $m \times m$ matrix, and U will be an $m \times n$ matrix (of the same shape as M). From here, the process is exactly the same as for a square matrix. We create a sequence of matrices L_i and U_i that is eventually the LU decomposition. Again, we start with $L_0 = I$ and $U_0 = M$.

Example Let's find the LU decomposition of $M = U_0 = \begin{pmatrix} -2 & 1 & 3 \\ -4 & 4 & 1 \end{pmatrix}$. Since M is a 2×3 matrix, our decomposition will consist of a 2×2 matrix and a 2×3 matrix. Then we start with $L_0 = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The next step is to zero-out the first column of M below the diagonal. There is only one row to cancel, then, and it can be removed by subtracting 2 times the first row of M to the second row of M. Then:

$$L_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \qquad U_1 = \begin{pmatrix} -2 & 1 & 3 \\ 0 & 2 & -5 \end{pmatrix}$$

Since U_1 is upper triangular, we're done. With a larger matrix, we would just continue the process.

11.3 Block LDU Decomposition

Let M be a square block matrix with square blocks X, Y, Z, W such that X^{-1} exists. Then M can be decomposed as a block LDU decomposition, where D is block diagonal, as follows:

$$M = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$$

Then:

$$M = \begin{pmatrix} I & 0 \\ ZX^{-1} & I \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & W - ZX^{-1}Y \end{pmatrix} \begin{pmatrix} I & X^{-1}Y \\ 0 & I \end{pmatrix}.$$

This can be checked explicitly simply by block-multiplying these three matrices.

`Block` LDU `Explanation`

Example For a 2×2 matrix, we can regard each entry as a block.

$$\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix} = \begin{pmatrix}1 & 0\\3 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -2\end{pmatrix}\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$$

By multiplying the diagonal matrix by the upper triangular matrix, we get the standard LU decomposition of the matrix.

References

Wikipedia:

- LU Decomposition
- Block LU Decomposition

Review Problems

1. Consider the linear system:

$$\begin{array}{lcr} x^1 & & = v^1 \\ l^2_1 x^1 & +x^2 & = v^2 \\ \vdots & & \vdots \\ l^n_1 x^1 & +l^n_2 x^2 + \cdots + x^n & = v^n \end{array}$$

   i. Find x^1.

   ii. Find x^2.

   iii. Find x^3.

   k. Try to find a formula for x^k. Don't worry about simplifying your answer.

2. Let $M = \begin{pmatrix}X & Y\\Z & W\end{pmatrix}$ be a square $n \times n$ block matrix with W invertible.

   i. If W has r rows, what size are X, Y, and Z?

   ii. Find a UDL decomposition for M. In other words, fill in the stars in the following equation:

$$\begin{pmatrix}X & Y\\Z & W\end{pmatrix} = \begin{pmatrix}I & *\\0 & I\end{pmatrix}\begin{pmatrix}* & 0\\0 & *\end{pmatrix}\begin{pmatrix}I & 0\\ * & I\end{pmatrix}$$

12 Elementary Matrices and Determinants

Given a square matrix, is there an easy way to know when it is invertible? Answering this fundamental question is our next goal.

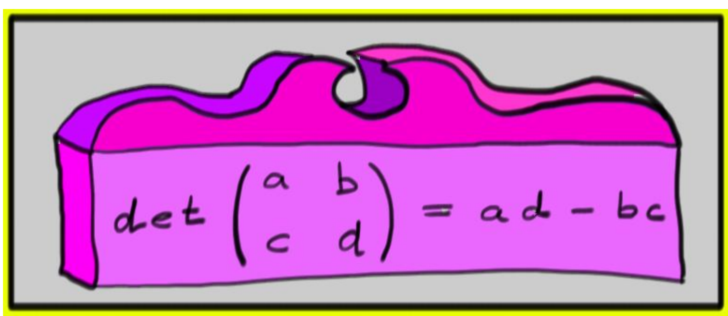
For small cases, we already know the answer. If M is a 1×1 matrix, then $M = (m) \Rightarrow M^{-1} = (1/m)$. Then M is invertible if and only if $m \neq 0$.

For M a 2×2 matrix, we showed in Section 10 that if $M = \begin{pmatrix} m^1_1 & m^1_2 \\ m^2_1 & m^2_2 \end{pmatrix}$, then $M^{-1} = \frac{1}{m^1_1 m^2_2 - m^1_2 m^2_1} \begin{pmatrix} m^2_2 & -m^1_2 \\ -m^2_1 & m^1_1 \end{pmatrix}$. Thus M is invertible if and only if

$$m^1_1 m^2_2 - m^1_2 m^2_1 \neq 0\,.$$

For 2×2 matrices, this quantity is called the *determinant of* M.

$$\det M = \det \begin{pmatrix} m^1_1 & m^1_2 \\ m^2_1 & m^2_2 \end{pmatrix} = m^1_1 m^2_2 - m^1_2 m^2_1$$



Example For a 3×3 matrix, $M = \begin{pmatrix} m^1_1 & m^1_2 & m^1_3 \\ m^2_1 & m^2_2 & m^2_3 \\ m^3_1 & m^3_2 & m^3_3 \end{pmatrix}$, then (by the first review question) M is non-singular if and only if:

$$\det M = m^1_1 m^2_2 m^3_3 - m^1_1 m^2_3 m^3_2 + m^1_2 m^2_3 m^3_1 - m^1_2 m^2_1 m^3_3 + m^1_3 m^2_1 m^3_2 - m^1_3 m^2_2 m^3_1 \neq 0.$$

Notice that in the subscripts, each ordering of the numbers 1, 2, and 3 occurs exactly once. Each of these is a *permutation* of the set $\{1, 2, 3\}$.

12.1 Permutations

Consider n objects labeled 1 through n and shuffle them. Each possible shuffle is called a *permutation* σ. For example, here is an example of a permutation of 5:

$$\rho = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{bmatrix}$$

We can consider a permutation σ as a function from the set of numbers $[n] := \{1, 2, \ldots, n\}$ to $[n]$, and write $\rho(3) = 5$ from the above example. In general we can write

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) & \sigma(5), \end{bmatrix}$$

but since the top line of any permutation is always the same, we can omit the top line and just write:

$$\sigma = \begin{bmatrix} \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) & \sigma(5) \end{bmatrix}$$

so we can just write $\rho = [4, 2, 5, 1, 3]$. There is also one more notation called cycle notation, but we do not discuss it here.

The mathematics of permutations is extensive and interesting; there are a few properties of permutations that we'll need.

- There are $n!$ permutations of n distinct objects, since there are n choices for the first object, $n - 1$ choices for the second once the first has been chosen, and so on.

- Every permutation can be built up by successively swapping pairs of objects. For example, to build up the permutation $\begin{bmatrix} 3 & 1 & 2 \end{bmatrix}$ from the trivial permutation $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, you can first swap 2 and 3, and then swap 1 and 3.

- For any given permutation σ, there is some number of swaps it takes to build up the permutation. (It's simplest to use the minimum number of swaps, but you don't have to: it turns out that *any* way of building up the permutation from swaps will have have the same parity of swaps, either even or odd.) If this number happens to be even, then σ is called an *even permutation*; if this number is odd, then σ is an *odd permutation*. In fact, $n!$ is even for all $n \geq 2$, and exactly half of the permutations are even and the other half are odd. It's worth noting that the trivial permutation (which sends $i \to i$ for every i) is an even permutation, since it uses zero swaps.

Definition The *sign function* is a function $\operatorname{sgn}(\sigma)$ that sends permutations to the set $\{-1, 1\}$, defined by:

$$\operatorname{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even;} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

For more on the swaps (also known as inversions) and the sign function, see Problem 4.

Permutation Example

Reading homework: problem 12.1

We can use permutations to give a definition of the determinant.

Definition For an $n \times n$ matrix M, the *determinant* of M (sometimes written $|M|$) is given by:

$$\det M = \sum_{\sigma} \operatorname{sgn}(\sigma)\, m^1_{\sigma(1)} m^2_{\sigma(2)} \cdots m^n_{\sigma(n)}.$$

The sum is over all permutations of n. Each summand is a product of a single entry from each row, but with the column numbers shuffled by the permutation σ.

The last statement about the summands yields a nice property of the determinant:

Theorem 12.1. *If M has a row consisting entirely of zeros, then $m^i_{\sigma(i)} = 0$ for every σ. Then* $\det M = 0$.

Example Because there are many permutations of n, writing the determinant this way for a general matrix gives a very long sum. For $n = 4$, there are $24 = 4!$ permutations, and for $n = 5$, there are already $120 = 5!$ permutations.

For a 4×4 matrix, $M = \begin{pmatrix} m^1_1 & m^1_2 & m^1_3 & m^1_4 \\ m^2_1 & m^2_2 & m^2_3 & m^2_4 \\ m^3_1 & m^3_2 & m^3_3 & m^3_4 \\ m^4_1 & m^4_2 & m^4_3 & m^4_4 \end{pmatrix}$, then $\det M$ is:

$$\begin{array}{rcl} \det M & = & m^1_1 m^2_2 m^3_3 m^4_4 - m^1_1 m^2_3 m^3_2 m^4_4 - m^1_1 m^2_2 m^3_4 m^4_3 \\ & - & m^1_2 m^2_1 m^3_3 m^4_4 + m^1_1 m^2_3 m^3_4 m^4_2 + m^1_1 m^2_4 m^3_2 m^4_3 \\ & + & m^1_2 m^2_3 m^3_1 m^4_4 + m^1_2 m^2_1 m^3_4 m^4_3 \pm 16 \text{ more terms.} \end{array}$$

This is very cumbersome.

Luckily, it is very easy to compute the determinants of certain matrices. For example, if M is diagonal, then $M^i_j = 0$ whenever $i \neq j$. Then all summands of the determinant involving off-diagonal entries vanish, so:

$$\det M = \sum_\sigma \operatorname{sgn}(\sigma) m^1_{\sigma(1)} m^2_{\sigma(2)} \cdots m^n_{\sigma(n)} = m^1_1 m^2_2 \cdots m^n_n.$$

Thus, the determinant of a diagonal matrix is just the product of its diagonal entries.

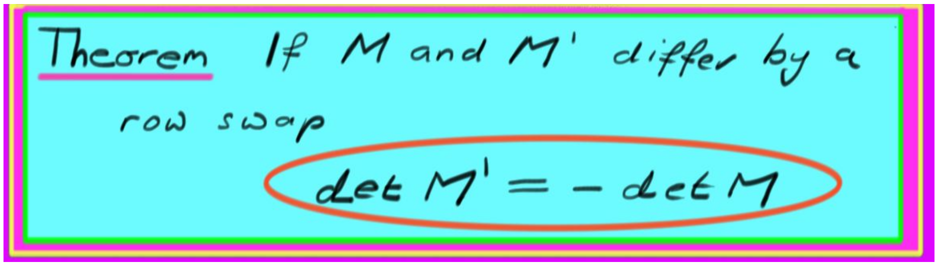
Since the identity matrix is diagonal with all diagonal entries equal to one, we have:

$$\det I = 1.$$

We would like to use the determinant to decide whether a matrix is invertible or not. Previously, we computed the inverse of a matrix by applying row operations. As such, it makes sense to ask what happens to the determinant when row operations are applied to a matrix.

Swapping rows Swapping rows i and j (with $i < j$) in a matrix changes the determinant. For a permutation σ, let $\hat{\sigma}$ be the permutation obtained by swapping positions i and j. The sign of $\hat{\sigma}$ is the opposite of the sign of σ. Let M be a matrix, and M' be the same matrix, but with rows i and j swapped. Then the determinant of M' is:

$$\begin{array}{rcl} \det M' & = & \sum_\sigma \operatorname{sgn}(\sigma)\, m^1_{\sigma(1)} \cdots m^j_{\sigma(i)} \cdots m^i_{\sigma(j)} \cdots m^n_{\sigma(n)} \\ & = & \sum_\sigma \operatorname{sgn}(\sigma)\, m^1_{\sigma(1)} \cdots m^i_{\sigma(j)} \cdots m^j_{\sigma(i)} \cdots m^n_{\sigma(n)} \\ & = & \sum_\sigma (-\operatorname{sgn}(\hat{\sigma}))\, m^1_{\hat{\sigma}(1)} \cdots m^i_{\hat{\sigma}(j)} \cdots m^j_{\hat{\sigma}(i)} \cdots m^n_{\hat{\sigma}(n)} \\ & = & -\sum_{\hat{\sigma}} \operatorname{sgn}(\hat{\sigma})\, m^1_{\hat{\sigma}(1)} \cdots m^i_{\hat{\sigma}(j)} \cdots m^j_{\hat{\sigma}(i)} \cdots m^n_{\hat{\sigma}(n)} \\ & = & -\det M. \end{array}$$



Thus we see that swapping rows changes the sign of the determinant. *I.e.*

$$M' = -\det M\,.$$

Reading homework: problem 12.2

Applying this result to $M = I$ (the identity matrix) yields

$$\det E^i_j = -1\,,$$

where the matrix E^i_j is the identity matrix with rows i and j swapped. It is our first example of an elementary matrix and we will meet it again soon.

This implies another nice property of the determinant. If two rows of the matrix are identical, then swapping the rows changes the sign of the matrix, but leaves the matrix unchanged. Then we see the following:

Theorem 12.2. *If M has two identical rows, then* $\det M = 0$.

12.2 Elementary Matrices

Our next goal is to find matrices that emulate the Gaussian row operations on a matrix. In other words, for any matrix M, and a matrix M' equal to M after a row operation, we wish to find a matrix R such that $M' = RM$.

See Some Ideas Explained

We will first find a matrix that, when it multiplies a matrix M, rows i and j of M are swapped.

Let R^1 through R^n denote the rows of M, and let M' be the matrix M with rows i and j swapped. Then M and M' can be regarded as a block matrices:

$$M = \begin{pmatrix} \vdots \\ R^i \\ \vdots \\ R^j \\ \vdots \end{pmatrix}, \text{ and } M' = \begin{pmatrix} \vdots \\ R^j \\ \vdots \\ R^i \\ \vdots \end{pmatrix}.$$

Then notice that:

$$M' = \begin{pmatrix} \vdots \\ R^j \\ \vdots \\ R^i \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & & 1 & \\ & & & \ddots & & \\ & & 1 & & 0 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \begin{pmatrix} \vdots \\ R^i \\ \vdots \\ R^j \\ \vdots \end{pmatrix}$$

The matrix

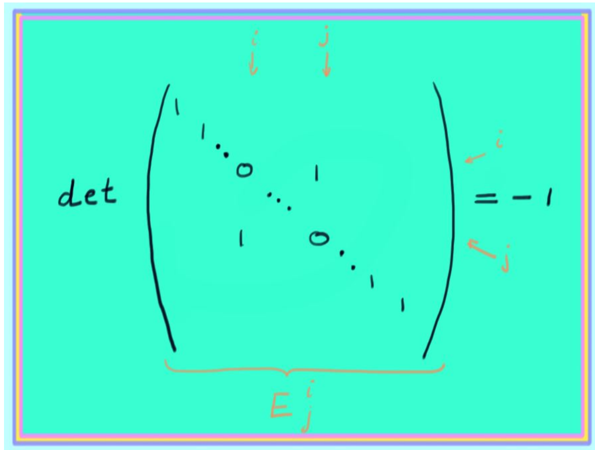
$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & & 1 & \\ & & & \ddots & & \\ & & 1 & & 0 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} =: E^i_j$$

is just the identity matrix with rows i and j swapped. This matrix E^i_j is called an *elementary matrix*. Then, symbolically,

$$M' = E^i_j M \,.$$

Because $\det I = 1$ and swapping a pair of rows changes the sign of the determinant, we have found that

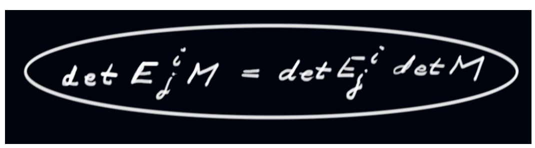
$$\det E^i_j = -1 \,.$$



Moreover, since swapping a pair of rows flips the sign of the determinant, $\det E^i_j = -1$ and $\det E^i_j M$ is the matrix M with rows i and j swapped we have that

$$\det E^i_j M = \det E^i_j \, \det M\,.$$

This result hints at an even better one for determinants of products of determinants. Stare at it again before reading the next Lecture:



References

Hefferon, Chapter Four, Section I.1 and I.3
Beezer, Chapter D, Section DM, Subsection EM
Beezer, Chapter D, Section PDM
Wikipedia:

- Determinant
- Permutation
- Elementary Matrix

Review Problems

1. Let $M = \begin{pmatrix} m_1^1 & m_2^1 & m_3^1 \\ m_1^2 & m_2^2 & m_3^2 \\ m_1^3 & m_2^3 & m_3^3 \end{pmatrix}$. Use row operations to put M into *row echelon form*. For simplicity, assume that $m_1^1 \neq 0 \neq m_1^1 m_2^2 - m_1^2 m_2^1$.

   Prove that M is non-singular if and only if:

   $$m_1^1 m_2^2 m_3^3 - m_1^1 m_3^2 m_2^3 + m_2^1 m_3^2 m_1^3 - m_2^1 m_1^2 m_3^3 + m_3^1 m_1^2 m_2^3 - m_3^1 m_2^2 m_1^3 \neq 0$$

2. (a) What does the matrix $E_2^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ do to $M = \begin{pmatrix} a & b \\ d & c \end{pmatrix}$ under left multiplication? What about right multiplication?

   (b) Find elementary matrices $R^1(\lambda)$ and $R^2(\lambda)$ that respectively multiply rows 1 and 2 of M by λ but otherwise leave M the same under left multiplication.

   (c) Find a matrix $S_2^1(\lambda)$ that adds a multiple λ of row 2 to row 1 under left multiplication.

3. Let M be a matrix and $S_j^i M$ the same matrix with rows i and j switched. Explain every line of the series of equations proving that $\det M = -\det(S_j^i M)$.

4. This problem is a "hands-on" look at why the property describing the parity of permutations is true.

   The *inversion number* of a permutation σ is the number of pairs $i < j$ such that $\sigma(i) > \sigma(j)$; it's the number of "numbers that appear left of smaller numbers" in the permutation. For example, for the permutation $\rho = [4, 2, 3, 1]$, the inversion number is 5. The number 4 comes before $2, 3$, and 1, and 2 and 3 both come before 1.

   Given a permutation σ, we can make a new permutation $\tau_{i,j}\sigma$ by exchanging the ith and jth entries of σ.

   (a) What is the inversion number of the permutation $\mu = [1, 2, 4, 3]$ that exchanges 4 and 3 and leaves everything else alone? Is it an even or an odd permutation?

(b) What is the inversion number of the permutation $\rho = [4, 2, 3, 1]$ that exchanges 1 and 4 and leaves everything else alone? Is it an even or an odd permutation?

(c) What is the inversion number of the permutation $\tau_{1,3}\mu$? Compare the parity[5] of μ to the parity of $\tau_{1,3}\mu$.

(d) What is the inversion number of the permutation $\tau_{2,4}\rho$? Compare the parity of ρ to the parity of $\tau_{2,4}\rho$.

(e) What is the inversion number of the permutation $\tau_{3,4}\rho$? Compare the parity of ρ to the parity of $\tau_{3,4}\rho$.

Problem 4 hints

5. (Extra credit) Here we will examine a (very) small set of the general properties about permutations and their applications. In particular, we will show that one way to compute the sign of a permutation is by finding the inversion number N of σ and we have

$$\operatorname{sgn}(\sigma) = (-1)^N.$$

For this problem, let $\mu = [1, 2, 4, 3]$.

(a) Show that every permutation σ can be sorted by only taking simple (adjacent) transpositions s_i where s_i interchanges the numbers in position i and $i + 1$ of a permutation σ (in our other notation $s_i = \tau_{i,i+1}$). For example $s_2\mu = [1, 4, 2, 3]$, and to sort μ we have $s_3\mu = [1, 2, 3, 4]$.

(b) We can compose simple transpositions together to represent a permutation (note that the sequence of compositions is not unique), and these are associative, we have an identity (the trivial permutation where the list is in order or we do nothing on our list), and we have an inverse since it is clear that $s_i s_i \sigma = \sigma$. Thus permutations of $[n]$ under composition are an example of a group. However

[5] The *parity* of an integer refers to whether the integer is even or odd. Here the parity of a permutation μ refers to the parity of its inversion number.

note that not all simple transpositions commute with each other since

$$s_1 s_2[1,2,3] = s_1[1,3,2] = [3,1,2]$$
$$s_2 s_1[1,2,3] = s_2[2,1,3] = [2,3,1]$$

(you will prove here when simple transpositions commute). When we consider our initial permutation to be the trivial permutation $e = [1, 2, \ldots, n]$, we do not write it; for example $s_i \equiv s_i e$ and $\mu = s_3 \equiv s_3 e$. This is analogous to not writing 1 when multiplying. Show that $s_i s_i = e$ (in shorthand $s_i^2 = e$), $s_{i+1} s_i s_{i+1} = s_i s_{i+1} s_i$ for all i, and s_i and s_j commute for all $|i - j| \geq 2$.

(c) Show that every way of expressing σ can be obtained from using the relations proved in part 5b. In other words, show that for any expression w of simple transpositions representing the trivial permutation e, using the proved relations.

Hint: Use induction on n. For the induction step, follow the path of the $(n+1)$-th strand by looking at $s_n s_{n-1} \cdots s_k s_{k\pm1} \cdots s_n$ and argue why you can write this as a subexpression for any expression of e. Consider using diagrams of these paths to help.

(d) The simple transpositions acts on an n-dimensional vector space V by $s_i v = E^i_{i+1} v$ (where E^i_j is an elementary matrix) for all vectors $v \in V$. Therefore we can just represent a permutation σ as the matrix M_σ[6], and we have $\det(M_{s_i}) = \det(E^i_{i+1}) = -1$. Thus prove that $\det(M_\sigma) = (-1)^N$ where N is a number of simple transpositions needed to represent σ as a permutation. You can assume that $M_{s_i s_j} = M_{s_i} M_{s_j}$ (it is not hard to prove) and that $\det(AB) = \det(A)\det(B)$ from Chapter 13.

Hint: You to make sure $\det(M_\sigma)$ is well-defined since there are infinite ways to represent σ as simple transpositions.

(e) Show that $s_{i+1} s_i s_{i+1} = \tau_{i,i+2}$, and so give one way of writing $\tau_{i,j}$ in terms of simple transpositions? Is $\tau_{i,j}$ an even or an odd permutation? What is $\det(M_{\tau_{i,j}})$? What is the inversion number of $\tau_{i,j}$?

(f) The minimal number of simple transpositions needed to express σ is called the *length* of σ; for example the length of μ is 1 since

[6] Often people will just use σ for the matrix when the context is clear.

$\mu = s_3$. Show that the length of σ is equal to the inversion number of σ.

Hint: Find an procedure which gives you a new permutation σ' where $\sigma = s_i\sigma'$ for some i and the inversion number for σ' is 1 less than the inversion number for σ.

(g) Show that $(-1)^N = \operatorname{sgn}(\sigma) = \det(M_\sigma)$, where σ is a permutation with N inversions. Note that this immediately implies that $\operatorname{sgn}(\sigma\rho) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\rho)$ for any permutations σ and ρ.

13 Elementary Matrices and Determinants II

In Lecture 12, we saw the definition of the determinant and derived an elementary matrix that exchanges two rows of a matrix. Next, we need to find elementary matrices corresponding to the other two row operations; multiplying a row by a scalar, and adding a multiple of one row to another. As a consequence, we will derive some important properties of the determinant.

Consider $M = \begin{pmatrix} R^1 \\ \vdots \\ R^n \end{pmatrix}$, where R^i are row vectors. Let $R^i(\lambda)$ be the identity matrix, with the ith diagonal entry replaced by λ, not to be confused with the row vectors. *I.e.*

$$R^i(\lambda) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Then:

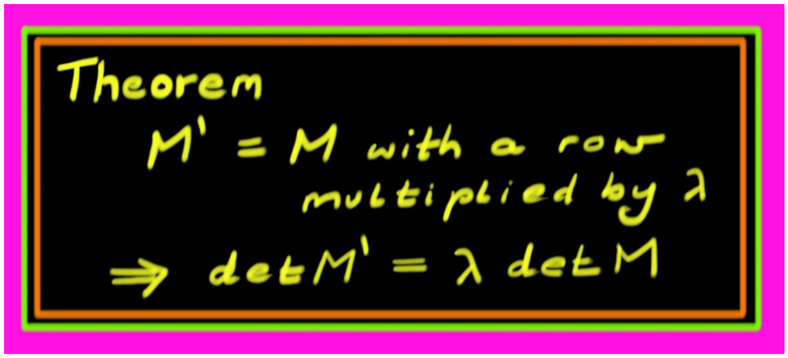
$$M' = R^i(\lambda)M = \begin{pmatrix} R^1 \\ \vdots \\ \lambda R^i \\ \vdots \\ R^n \end{pmatrix}$$

What effect does multiplication by $R^i(\lambda)$ have on the determinant?

$$\begin{aligned} \det M' &= \sum_\sigma \operatorname{sgn}(\sigma) m^1_{\sigma(1)} \cdots \lambda m^i_{\sigma(i)} \cdots m^n_{\sigma(n)} \\ &= \lambda \sum_\sigma \operatorname{sgn}(\sigma) m^1_{\sigma(1)} \cdots m^i_{\sigma(i)} \cdots m^n_{\sigma(n)} \\ &= \lambda \det M \end{aligned}$$

Thus, multiplying a row by λ multiplies the determinant by λ. *I.e.,*

$$\det R^i(\lambda)M = \lambda \det M\,.$$



Since $R^i(\lambda)$ is just the identity matrix with a single row multiplied by λ, then by the above rule, the determinant of $R^i(\lambda)$ is λ. Thus:

$$\det R^i(\lambda) = \det \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} = \lambda$$

The final row operation is adding λR^j to R^i. This is done with the matrix $S^i_j(\lambda)$, which is an identity matrix but with a λ in the i, j position.

$$S^i_j(\lambda) = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & \lambda & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

Then multiplying $S^i_j(\lambda)$ by M gives the following:

$$\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & \lambda & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \begin{pmatrix} \vdots \\ R^i \\ \vdots \\ R^j \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ R^i + \lambda R^j \\ \vdots \\ R^j \\ \vdots \end{pmatrix}$$

What is the effect of multiplying by $S^i_j(\lambda)$ on the determinant? Let $M' = S^i_j(\lambda)M$, and let M'' be the matrix M but with R^i replaced by R^j.

$$\begin{aligned} \det M' &= \sum_\sigma \operatorname{sgn}(\sigma) m^1_{\sigma(1)} \cdots (m^i_{\sigma(i)} + \lambda m^j_{\sigma(j)}) \cdots m^n_{\sigma(n)} \\ &= \sum_\sigma \operatorname{sgn}(\sigma) m^1_{\sigma(1)} \cdots m^i_{\sigma(i)} \cdots m^n_{\sigma(n)} \\ &\qquad + \sum_\sigma \operatorname{sgn}(\sigma) m^1_{\sigma(1)} \cdots \lambda m^j_{\sigma(j)} \cdots m^j_{\sigma(j)} \cdots m^n_{\sigma(n)} \\ &= \det M + \lambda \det M'' \end{aligned}$$

Since M'' has two identical rows, its determinant is 0. Then

$$\det S^i_j(\lambda) M = \det M\,.$$

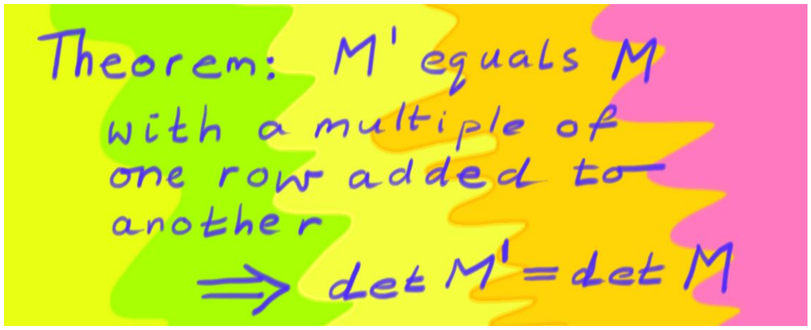
Notice that if M is the identity matrix, then we have

$$\det S^i_j(\lambda) = \det(S^i_j(\lambda) I) = \det I = 1\,.$$

We now have elementary matrices associated to each of the row operations.

$$\begin{aligned} E^i_j &= I \text{ with rows } i,j \text{ swapped;} \quad \det E^i_j = -1 \\ R^i(\lambda) &= I \text{ with } \lambda \text{ in position } i,i; \quad \det R^i(\lambda) = \lambda \\ S^i_j(\lambda) &= I \text{ with } \lambda \text{ in position } i,j; \quad \det S^i_j(\lambda) = 1 \end{aligned}$$

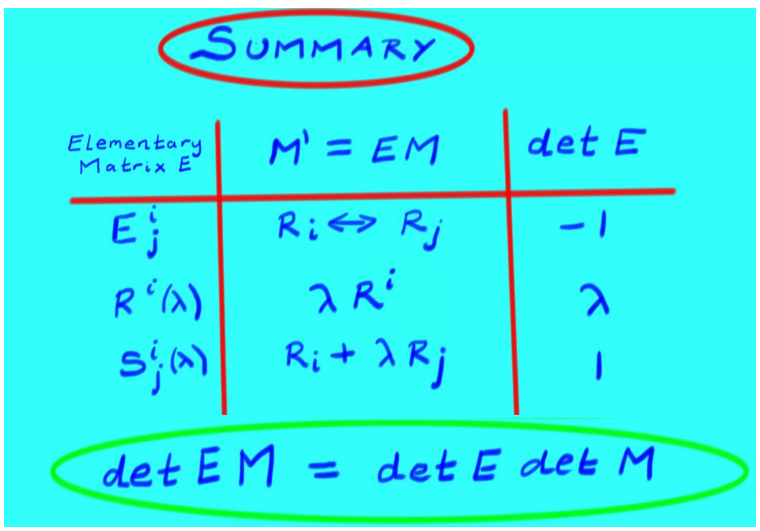
`Elementary Determinants`



We have also proved the following theorem along the way:

Theorem 13.1. *If* E *is* any *of the elementary matrices* $E^i_j, R^i(\lambda), S^i_j(\lambda)$, *then* $\det(EM) = \det E \det M$.

Reading homework: problem 13.1



We have seen that any matrix M can be put into reduced row echelon form via a sequence of row operations, and we have seen that any row operation can be emulated with left matrix multiplication by an elementary matrix. Suppose that $\mathrm{RREF}(M)$ is the reduced row echelon form of M. Then $\mathrm{RREF}(M) = E_1 E_2 \cdots E_k M$ where each E_i is an elementary matrix.

What is the determinant of a square matrix in reduced row echelon form?

- If M is not invertible, then some row of $\text{RREF}(M)$ contains only zeros. Then we can multiply the zero row by any constant λ without changing M; by our previous observation, this scales the determinant of M by λ. Thus, if M is not invertible, $\det \text{RREF}(M) = \lambda \det \text{RREF}(M)$, and so $\det \text{RREF}(M) = 0$.

- Otherwise, every row of $\text{RREF}(M)$ has a pivot on the diagonal; since M is square, this means that $\text{RREF}(M)$ is the identity matrix. Then if M is invertible, $\det \text{RREF}(M) = 1$.

- Additionally, notice that $\det \text{RREF}(M) = \det(E_1E_2\cdots E_kM)$. Then by the theorem above, $\det \text{RREF}(M) = \det(E_1)\cdots\det(E_k)\det M$. Since each E_i has non-zero determinant, then $\det \text{RREF}(M) = 0$ if and only if $\det M = 0$.

Then we have shown:

Theorem 13.2. *For any square matrix M, $\det M \neq 0$ if and only if M is invertible.*

Since we know the determinants of the elementary matrices, we can immediately obtain the following:

`Determinants and Inverses`

Corollary 13.3. *Any elementary matrix $E^i_j, R^i(\lambda), S^i_j(\lambda)$ is invertible, except for $R^i(0)$. In fact, the inverse of an elementary matrix is another elementary matrix.*

To obtain one last important result, suppose that M and N are square $n \times n$ matrices, with reduced row echelon forms such that, for elementary matrices E_i and F_i,

$$M = E_1E_2\cdots E_k \ \text{RREF}(M)\,,$$

and

$$N = F_1 F_2 \cdots F_l \ \mathrm{RREF}(N)\,.$$

If $\mathrm{RREF}(M)$ is the identity matrix (*i.e.*, M is invertible), then:

$$\begin{aligned}
\det(MN) &= \det(E_1 E_2 \cdots E_k \ \mathrm{RREF}(M) F_1 F_2 \cdots F_l \ \mathrm{RREF}(N)) \\
&= \det(E_1 E_2 \cdots E_k I F_1 F_2 \cdots F_l \ \mathrm{RREF}(N)) \\
&= \det(E_1) \cdots \det(E_k) \det(I) \det(F_1) \cdots \det(F_l) \det(\mathrm{RREF}(N) \\
&= \det(M) \det(N)
\end{aligned}$$

Otherwise, M is not invertible, and $\det M = 0 = \det \mathrm{RREF}(M)$. Then there exists a row of zeros in $\mathrm{RREF}(M)$, so $R^n(\lambda)\,\mathrm{RREF}(M) = \mathrm{RREF}(M)$. Then:

$$\begin{aligned}
\det(MN) &= \det(E_1 E_2 \cdots E_k \ \mathrm{RREF}(M) N) \\
&= \det(E_1 E_2 \cdots E_k \ \mathrm{RREF}(M) N) \\
&= \det(E_1) \cdots \det(E_k) \det(\mathrm{RREF}(M) N) \\
&= \det(E_1) \cdots \det(E_k) \det(R^n(\lambda) \ \mathrm{RREF}(M) N) \\
&= \det(E_1) \cdots \det(E_k) \lambda \det(\mathrm{RREF}(M) N) \\
&= \lambda \det(MN)
\end{aligned}$$

Which implies that $\det(MN) = 0 = \det M \det N$.

Thus we have shown that for *any* matrices M and N,

$$\det(MN) = \det M \det N$$

This result is *extremely important*; do not forget it!

`Alternative proof`

Reading homework: problem 13.2

References

Hefferon, Chapter Four, Section I.1 and I.3
Beezer, Chapter D, Section DM, Subsection EM
Beezer, Chapter D, Section PDM
Wikipedia:

- Determinant
- Elementary Matrix

Review Problems

1. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $N = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Compute the following:

   (a) $\det M$.

   (b) $\det N$.

   (c) $\det(MN)$.

   (d) $\det M \det N$.

   (e) $\det(M^{-1})$ assuming $ad - bc \neq 0$.

   (f) $\det(M^T)$

   (g) $\det(M+N) - (\det M + \det N)$. Is the determinant a linear transformation from square matrices to real numbers? Explain.

2. Suppose $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible. Write M as a product of elementary row matrices times $\mathrm{RREF}(M)$.

3. Find the inverses of each of the elementary matrices, $E^i_j, R^i(\lambda), S^i_j(\lambda)$. Make sure to show that the elementary matrix times its inverse is actually the identity.

4. (Extra Credit) Let e^i_j denote the matrix with a 1 in the i-th row and j-th column and 0's everywhere else, and let A be an arbitrary 2×2 matrix. Compute $\det(A + tI_2)$, and what is first order term (the coefficient of t)? Can you express your results in terms of $\mathrm{tr}(A)$? What about the first order term in $\det(A + tI_n)$ for any arbitrary $n \times n$ matrix A in terms of $\mathrm{tr}(A)$?

   We note that the result of $\det(A+tI_2)$ is a polynomial in the variable t and by taking $t = -\lambda$ is what is known as the *characteristic polynomial* from Chapter 18.

5. (Extra Credit: (Directional) Derivative of the Determinant) Notice that $\det\colon \mathbb{M}_n \to \mathbb{R}$ where $\mathbb{M}_n$ is the vector space of all $n \times n$ matrices, and so we can take directional derivatives of det. Let A be an arbitrary $n \times n$ matrix, and for all i and j compute the following:

   (a)
$$\lim_{t\to 0} \frac{\det(I_2 + te^i_j) - \det(I_2)}{t}$$

   (b)
$$\lim_{t\to 0} \frac{\det(I_3 + te^i_j) - \det(I_3)}{t}$$

   (c)
$$\lim_{t\to 0} \frac{\det(I_n + te^i_j) - \det(I_n)}{t}$$

   (d)
$$\lim_{t\to 0} \frac{\det(I_n + At) - \det(I_n)}{t}$$

(Recall that what you are calculating is the directional derivative in the e^i_j and A directions.) Can you express your results in terms of the trace function?

Hint: Use the results from Problem 4 and what you know about the derivatives of polynomials evaluated at 0 (i.e. what is $p'(0)$*?).*

14 Properties of the Determinant

In Lecture 13 we showed that the determinant of a matrix is non-zero if and only if that matrix is invertible. We also showed that the determinant is a *multiplicative* function, in the sense that $\det(MN) = \det M \det N$. Now we will devise some methods for calculating the determinant.

Recall that:

$$\det M = \sum_{\sigma} \operatorname{sgn}(\sigma) m^1_{\sigma(1)} m^2_{\sigma(2)} \cdots m^n_{\sigma(n)}.$$

A *minor* of an $n \times n$ matrix M is the determinant of any square matrix obtained from M by deleting rows and columns. In particular, any entry m^i_j of a square matrix M is associated to a minor obtained by deleting the ith row and jth column of M.

It is possible to write the determinant of a matrix in terms of its minors as follows:

$$\begin{aligned}
\det M &= \sum_{\sigma} \operatorname{sgn}(\sigma)\, m^1_{\sigma(1)} m^2_{\sigma(2)} \cdots m^n_{\sigma(n)} \\
&= m^1_1 \sum_{\hat\sigma} \operatorname{sgn}(\hat\sigma)\, m^2_{\hat\sigma(2)} \cdots m^n_{\hat\sigma(n)} \\
&- m^1_2 \sum_{\hat\sigma} \operatorname{sgn}(\hat\sigma)\, m^2_{\hat\sigma(1)} m^3_{\hat\sigma(3)} \cdots m^n_{\hat\sigma(n)} \\
&+ m^1_3 \sum_{\hat\sigma} \operatorname{sgn}(\hat\sigma)\, m^2_{\hat\sigma(1)} m^3_{\hat\sigma(2)} m^4_{\hat\sigma(4)} \cdots m^n_{\hat\sigma(n)} \pm \cdots
\end{aligned}$$

Here the symbols $\hat\sigma$ refer to permutations of $n-1$ objects. What we're doing here is collecting up all of the terms of the original sum that contain the first row entry m^1_j for each column number j. Each term in that collection is associated to a permutation sending $1 \to j$. The remainder of any such permutation maps the set $\{2, \ldots, n\} \to \{1, \ldots, j-1, j+1, \ldots, n\}$. We call this partial permutation $\hat\sigma = \begin{bmatrix} \sigma(2) & \cdots & \sigma(n) \end{bmatrix}$.

The last issue is that the permutation $\hat\sigma$ may not have the same sign as σ. From previous homework, we know that a permutation has the same parity as its inversion number. Removing $1 \to j$ from a permutation reduces the inversion number by the number of elements right of j that are less than j. Since j comes first in the permutation $\begin{bmatrix} j & \sigma(2) & \cdots & \sigma(n) \end{bmatrix}$, the inversion number of $\hat{\sigma}$ is reduced by $j-1$. Then the sign of σ differs from the sign of $\hat{\sigma}$ if σ sends 1 to an even number.

In other words, to expand by minors we pick an entry m^1_j of the first row, then add $(-1)^{j-1}$ times the determinant of the matrix with row i and column j deleted.

Example Let's compute the determinant of $M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ using expansion by minors.

$$\begin{aligned} \det M &= 1 \det \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} - 2 \det \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} + 3 \det \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix} \\ &= 1(5 \cdot 9 - 8 \cdot 6) - 2(4 \cdot 9 - 7 \cdot 6) + 3(4 \cdot 8 - 7 \cdot 5) \\ &= 0 \end{aligned}$$

Here, M^{-1} does not exist because[7] $\det M = 0$.

Example Sometimes the entries of a matrix allow us to simplify the calculation of the determinant. Take $N = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix}$. Notice that the second row has many zeros; then we can switch the first and second rows of N to get:

$$\begin{aligned} \det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix} &= -\det \begin{pmatrix} 4 & 0 & 0 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix} \\ &= -4 \det \begin{pmatrix} 2 & 3 \\ 8 & 9 \end{pmatrix} \\ &= 24 \end{aligned}$$

Example

[7] A fun exercise is to compute the determinant of a 4×4 matrix filled in order, from left to right, with the numbers $1, 2, 3, \ldots 16$. What do you observe? Try the same for a 5×5 matrix with $1, 2, 3 \ldots 25$. Is there a pattern? Can you explain it?

Theorem 14.1. *For any square matrix M, we have:*

$$\det M^T = \det M$$

Proof. By definition,

$$\det M = \sum_{\sigma} \operatorname{sgn}(\sigma) m^1_{\sigma(1)} m^2_{\sigma(2)} \cdots m^n_{\sigma(n)}.$$

For any permutation σ, there is a unique inverse permutation σ^{-1} that undoes σ. If σ sends $i \to j$, then σ^{-1} sends $j \to i$. In the two-line notation for a permutation, this corresponds to just flipping the permutation over. For example, if $\sigma = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$, then we can find σ^{-1} by flipping the permutation and then putting the columns in order:

$$\sigma^{-1} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

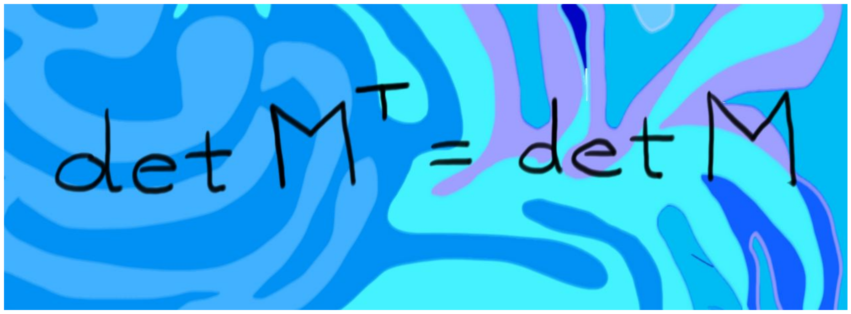
Since any permutation can be built up by transpositions, one can also find the inverse of a permutation σ by undoing each of the transpositions used to build up σ; this shows that one can use the same number of transpositions to build σ and σ^{-1}. In particular, $\operatorname{sgn} \sigma = \operatorname{sgn} \sigma^{-1}$.

Reading homework: problem 14.1

Then we can write out the above in formulas as follows:

$$\begin{aligned}
\det M &= \sum_{\sigma} \operatorname{sgn}(\sigma) m^1_{\sigma(1)} m^2_{\sigma(2)} \cdots m^n_{\sigma(n)} \\
&= \sum_{\sigma} \operatorname{sgn}(\sigma) m_1^{\sigma^{-1}(1)} m_2^{\sigma^{-1}(2)} \cdots m_n^{\sigma^{-1}(n)} \\
&= \sum_{\sigma} \operatorname{sgn}(\sigma^{-1}) m_1^{\sigma^{-1}(1)} m_2^{\sigma^{-1}(2)} \cdots m_n^{\sigma^{-1}(n)} \\
&= \sum_{\sigma} \operatorname{sgn}(\sigma) m_1^{\sigma(1)} m_2^{\sigma(2)} \cdots m_n^{\sigma(n)} \\
&= \det M^T.
\end{aligned}$$

The second-to-last equality is due to the existence of a unique inverse permutation: summing over permutations is the same as summing over all inverses of permutations. The final equality is by the definition of the transpose. $\square$



Example Because of this theorem, we see that expansion by minors also works over columns. Let $M = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 8 & 9 \end{pmatrix}$. Then

$$\det M = \det M^T = 1 \det \begin{pmatrix} 5 & 8 \\ 6 & 9 \end{pmatrix} = -3\,.$$

14.1 Determinant of the Inverse

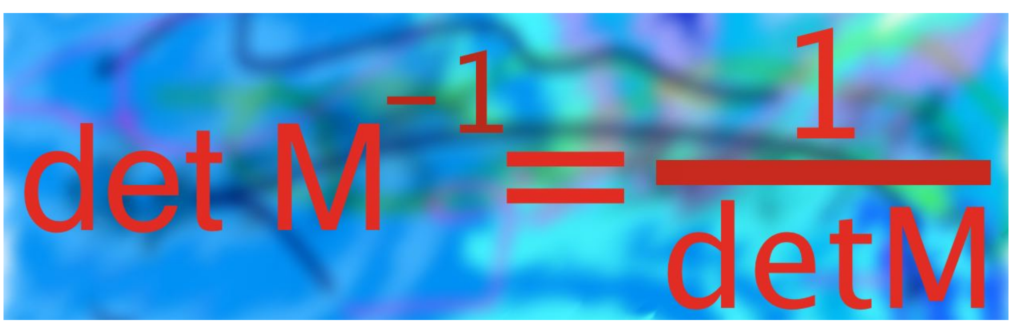
Let M and N be $n \times n$ matrices. We previously showed that

$$\det(MN) = \det M \det N, \text{ and } \det I = 1.$$

Then $1 = \det I = \det(MM^{-1}) = \det M \det M^{-1}$. As such we have:

Theorem 14.2.
$$\det M^{-1} = \frac{1}{\det M}$$

Just so you don't forget this:



14.2 Adjoint of a Matrix

Recall that for the 2×2 matrix

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot I$$

Or in a more careful notation: if $M = \begin{pmatrix} m^1_1 & m^1_2 \\ m^2_1 & m^2_2 \end{pmatrix}$, then

$$M^{-1} = \frac{1}{m^1_1 m^2_2 - m^1_2 m^2_1} \begin{pmatrix} m^2_2 & -m^1_2 \\ -m^2_1 & m^1_1 \end{pmatrix} ,$$

so long as $\det M = m^1_1 m^2_2 - m^1_2 m^2_1 \neq 0$. The matrix $\begin{pmatrix} m^2_2 & -m^1_2 \\ -m^2_1 & m^1_1 \end{pmatrix}$ that appears above is a special matrix, called the *adjoint* of M. Let's define the adjoint for an $n \times n$ matrix.

A *cofactor* of M is obtained choosing any entry m^i_j of M and then deleting the ith row and jth column of M, taking the determinant of the resulting matrix, and multiplying by$(-1)^{i+j}$. This is written cofactor(m^i_j).

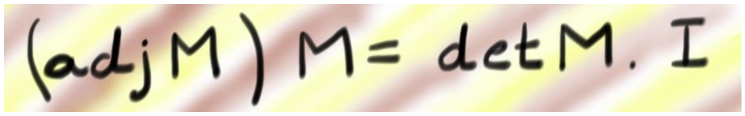
Definition For $M = (m^i_j)$ a square matrix, The *adjoint matrix* $\operatorname{adj} M$ is given by:

$$\operatorname{adj} M = (\text{cofactor}(m^i_j))^T$$

Example

$$\operatorname{adj}\begin{pmatrix} 3 & -1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \det\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} & -\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \det\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ -\det\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} & \det\begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix} & -\det\begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix} \\ \det\begin{pmatrix} -1 & -1 \\ 2 & 0 \end{pmatrix} & -\det\begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} & \det\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} \end{pmatrix}^T$$

Reading homework: problem 14.2



Let's multiply $M \operatorname{adj} M$. For any matrix N, the i, j entry of MN is given by taking the dot product of the ith row of M and the jth column of N. Notice that the dot product of the ith row of M and the ith column of $\operatorname{adj} M$ is just the expansion by minors of $\det M$ in the ith row. Further, notice that the dot product of the ith row of M and the jth column of $\operatorname{adj} M$ with $j \neq i$ is the same as expanding M by minors, but with the jth row replaced by the ith row. Since the determinant of any matrix with a row repeated is zero, then these dot products are zero as well.

We know that the i, j entry of the product of two matrices is the dot product of the ith row of the first by the jth column of the second. Then:

$$M \operatorname{adj} M = (\det M) I$$

Thus, when $\det M \neq 0$, the adjoint gives an explicit formula for M^{-1}.

Theorem 14.3. *For M a square matrix with $\det M \neq 0$ (equivalently, if M is invertible), then*

$$M^{-1} = \frac{1}{\det M} \operatorname{adj} M$$

`The Adjoint Matrix`

Example Continuing with the previous example,

$$\operatorname{adj} \begin{pmatrix} 3 & -1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ -1 & 3 & -1 \\ 1 & -3 & 7 \end{pmatrix}.$$

Now, multiply:

$$\begin{pmatrix} 3 & -1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 \\ -1 & 3 & -1 \\ 1 & -3 & 7 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3 & -1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \frac{1}{6} \begin{pmatrix} 2 & 0 & 2 \\ -1 & 3 & -1 \\ 1 & -3 & 7 \end{pmatrix}$$

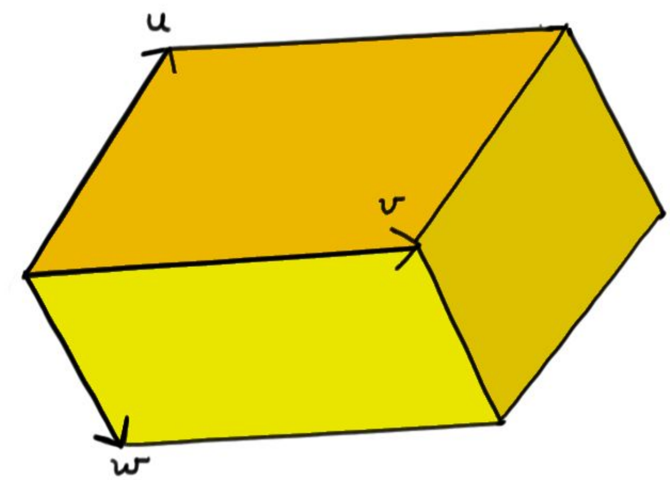


Figure 1: A parallelepiped.

This process for finding the inverse matrix is sometimes called *Cramer's Rule* .

14.3 Application: Volume of a Parallelepiped

Given three vectors u, v, w in $\mathbb{R}^3$, the parallelepiped determined by the three vectors is the "squished" box whose edges are parallel to u, v, and w as depicted in Figure 1.

From calculus, we know that the volume of this object is $|u \cdot (v \times w)|$. This is the same as expansion by minors of the matrix whose columns are u, v, w. Then:

$$\text{Volume} = \left| \det \begin{pmatrix} u & v & w \end{pmatrix} \right|$$

References

Hefferon, Chapter Four, Section I.1 and I.3
Beezer, Chapter D, Section DM, Subsection DD
Beezer, Chapter D, Section DM, Subsection CD
Wikipedia:

- Determinant
- Elementary Matrix
- Cramer's Rule

Review Problems

1. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Show:
$$\det M = \frac{1}{2}(\operatorname{tr} M)^2 - \frac{1}{2}\operatorname{tr}(M^2)$$
Suppose M is a 3×3 matrix. Find and verify a similar formula for $\det M$ in terms of $\operatorname{tr}(M^3)$, $\operatorname{tr}(M^2)$, and $\operatorname{tr} M$.

2. Suppose $M = LU$ is an LU decomposition. Explain how you would efficiently compute $\det M$ in this case.

3. In computer science, the *complexity* of an algorithm is (roughly) computed by counting the number of times a given operation is performed. Suppose adding or subtracting any two numbers takes a seconds, and multiplying two numbers takes m seconds. Then, for example, computing $2 \cdot 6 - 5$ would take $a + m$ seconds.

   (a) How many additions and multiplications does it take to compute the determinant of a general 2×2 matrix?

   (b) Write a formula for the number of additions and multiplications it takes to compute the determinant of a general $n \times n$ matrix using the definition of the determinant. Assume that finding and multiplying by the sign of a permutation is free.

   (c) How many additions and multiplications does it take to compute the determinant of a general 3×3 matrix using expansion by minors? Assuming $m = 2a$, is this faster than computing the determinant from the definition?

Problem 3 hint


15 Subspaces and Spanning Sets

It is time to study vector spaces more carefully and answer some fundamental questions.

1. *Subspaces*: When is a subset of a vector space itself a vector space? (This is the notion of a *subspace*.)

2. *Linear Independence*: Given a collection of vectors, is there a way to tell whether they are independent, or if one is a "linear combination" of the others?

3. *Dimension*: Is there a consistent definition of how "big" a vector space is?

4. *Basis*: How do we label vectors? Can we write any vector as a sum of some basic set of vectors? How do we change our point of view from vectors labeled one way to vectors labeled in another way?

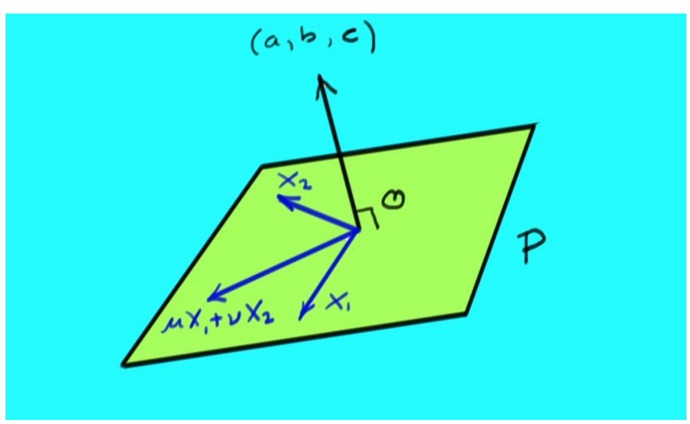
Let's start at the top!

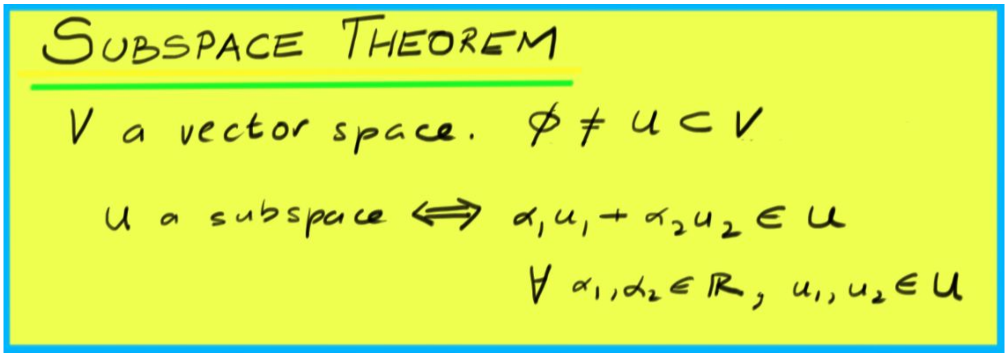
15.1 Subspaces

Definition We say that a subset U of a vector space V is a *subspace* of V if U is a vector space under the inherited addition and scalar multiplication operations of V.

Example Consider a plane P in $\mathbb{R}^3$ through the origin:

$$ax + by + cz = 0.$$





This equation can be expressed as the homogeneous system $\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$, or $MX = 0$ with M the matrix $\begin{pmatrix} a & b & c \end{pmatrix}$. If X_1 and X_2 are both solutions to $MX = 0$, then, by linearity of matrix multiplication, so is $\mu X_1 + \nu X_2$:

$$M(\mu X_1 + \nu X_2) = \mu M X_1 + \nu M X_2 = 0.$$

So P is closed under addition and scalar multiplication. Additionally, P contains the origin (which can be derived from the above by setting $\mu = \nu = 0$). All other vector space requirements hold for P because they hold for all vectors in $\mathbb{R}^3$.

Theorem 15.1 (Subspace Theorem)**.** *Let U be a non-empty subset of a vector space V. Then U is a subspace if and only if $\mu u_1 + \nu u_2 \in U$ for arbitrary u_1, u_2 in U, and arbitrary constants μ, ν.*

Proof. One direction of this proof is easy: if U is a subspace, then it is a vector space, and so by the additive closure and multiplicative closure properties of vector spaces, it has to be true that $\mu u_1 + \nu u_2 \in U$ for all u_1, u_2 in U and all constants constants μ, ν.

The other direction is almost as easy: we need to show that if $\mu u_1 + \nu u_2 \in U$ for all u_1, u_2 in U and all constants μ, ν, then U is a vector space. That is, we need to show that the ten properties of vector spaces are satisfied. We know that the additive closure and multiplicative closure properties are satisfied. Each of the other eight properties is true in U because it is true in V. The details of this are left as an exercise. □

Note that the requirements of the subspace theorem are often referred to as "closure".

From now on, we can use this theorem to check if a set is a vector space. That is, if we have some set U of vectors that come from some bigger vector space V, to check if U itself forms a smaller vector space we need check only two things: if we add any two vectors in U, do we end up with a vector in U? And, if we multiply any vector in U by any constant, do we end up with a vector in U? If the answer to both of these questions is yes, then U is a vector space. If not, U is not a vector space.

Reading homework: problem 15.1

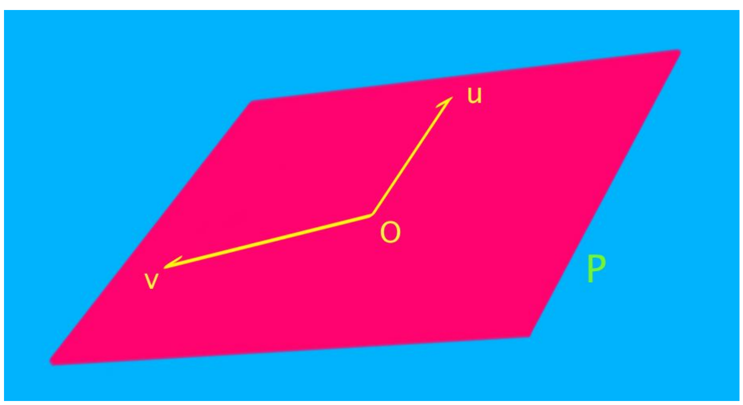
15.2 Building Subspaces

Consider the set

$$U = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \subset \mathbb{R}^3.$$

Because U consists of only two vectors, it clear that U is *not* a vector space, since any constant multiple of these vectors should also be in U. For example, the 0-vector is not in U, nor is U closed under vector addition.

But we know that any two vectors define a plane:



In this case, the vectors in U define the xy-plane in $\mathbb{R}^3$. We can consider the xy-plane as the set of all vectors that arise as a linear combination of the two vectors in U. Call this set of all linear combinations the *span* of U:

$$\operatorname{span}(U) = \left\{ x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \middle| x, y \in \mathbb{R} \right\}.$$

Notice that any vector in the xy-plane is of the form

$$\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in \operatorname{span}(U).$$

Definition Let V be a vector space and $S = \{s_1, s_2, \ldots\} \subset V$ a subset of V. Then the *span of* S is the set:

$$\operatorname{span}(S) = \{r^1 s_1 + r^2 s_2 + \cdots + r^N s_N | r^i \in \mathbb{R}, N \in \mathbb{N}\}.$$

That is, the span of S is the set of all finite linear combinations[8] of elements of S. Any *finite* sum of the form (a constant times s_1 plus a constant times s_2 plus a constant times s_3 and so on) is in the span of S.

It is important that we only allow finite linear combinations. In the definition above, N must be a finite number. It can be any finite number, but it must be finite.

Example Let $V = \mathbb{R}^3$ and $X \subset V$ be the x-axis. Let $P = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and set

$$S = X \cup P\,.$$

The elements of $\operatorname{span}(S)$ are linear combinations of vectors in the x-axis and the vector P.

The vector $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ is in $\operatorname{span}(S)$, because $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Similarly, the vector $\begin{pmatrix} -12 \\ 17.5 \\ 0 \end{pmatrix}$ is in $\operatorname{span}(S)$, because $\begin{pmatrix} -12 \\ 17.5 \\ 0 \end{pmatrix} = \begin{pmatrix} -12 \\ 0 \\ 0 \end{pmatrix} + 17.5 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Similarly, any vector of the form

$$\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

is in $\operatorname{span}(S)$. On the other hand, any vector in $\operatorname{span}(S)$ must have a zero in the z-coordinate. (Why?)

So $\operatorname{span}(S)$ is the xy-plane, which is a vector space. (Try drawing a picture to verify this!)

[8] Usually our vector spaces are defined over $\mathbb{R}$, but in general we can have vector spaces defined over different base fields such as $\mathbb{C}$ or $\mathbb{Z}_2$. The coefficients r^i should come from whatever our base field is (usually $\mathbb{R}$).

Reading homework: problem 15.2

Lemma 15.2. *For any subset* $S \subset V$*,* $\operatorname{span}(S)$ *is a subspace of* V*.*

Proof. We need to show that $\operatorname{span}(S)$ is a vector space.

It suffices to show that $\operatorname{span}(S)$ is closed under linear combinations. Let $u, v \in \operatorname{span}(S)$ and λ, μ be constants. By the definition of $\operatorname{span}(S)$, there are constants c^i and d^i (some of which could be zero) such that:

$$
\begin{aligned}
u &= c^1 s_1 + c^2 s_2 + \cdots \\
v &= d^1 s_1 + d^2 s_2 + \cdots \\
\Rightarrow \lambda u + \mu v &= \lambda(c^1 s_1 + c^2 s_2 + \cdots) + \mu(d^1 s_1 + d^2 s_2 + \cdots) \\
&= (\lambda c^1 + \mu d^1) s_1 + (\lambda c^2 + \mu d^2) s_2 + \cdots
\end{aligned}
$$

This last sum is a linear combination of elements of S, and is thus in $\operatorname{span}(S)$. Then $\operatorname{span}(S)$ is closed under linear combinations, and is thus a subspace of V. $\square$

Note that this proof, like many proofs, consisted of little more than just writing out the definitions.

Example For which values of a does

$$\operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} a \\ 1 \\ 0 \end{pmatrix} \right\} = \mathbb{R}^3 ?$$

Given an arbitrary vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $\mathbb{R}^3$, we need to find constants r^1, r^2, r^3 such that

$$r^1 \begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix} + r^2 \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + r^3 \begin{pmatrix} a \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

We can write this as a linear system in the unknowns r^1, r^2, r^3 as follows:

$$\begin{pmatrix} 1 & 1 & a \\ 0 & 2 & 1 \\ a & -3 & 0 \end{pmatrix} \begin{pmatrix} r^1 \\ r^2 \\ r^3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

If the matrix $M = \begin{pmatrix} 1 & 1 & a \\ 0 & 2 & 1 \\ a & -3 & 0 \end{pmatrix}$ is invertible, then we can find a solution

$$M^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r^1 \\ r^2 \\ r^3 \end{pmatrix}$$

for *any* vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.

Therefore we should choose a so that M is invertible:

$$i.e., \; 0 \neq \det M = -2a^2 + 3 + a = -(2a - 3)(a + 1).$$

Then the span is $\mathbb{R}^3$ if and only if $a \neq -1, \frac{3}{2}$.

`Linear systems as spanning sets`

References

Hefferon, Chapter Two, Section I.2: Subspaces and Spanning Sets
Beezer, Chapter VS, Section S
Beezer, Chapter V, Section LC
Beezer, Chapter V, Section SS
Wikipedia:

- Linear Subspace
- Linear Span

Review Problems

1. (Subspace Theorem) Suppose that V is a vector space and that $U \subset V$ is a subset of V. Show that

$$\mu u_1 + \nu u_2 \in U \text{ for all } u_1, u_2 \in U, \mu, \nu \in \mathbb{R}$$

implies that U is a subspace of V. (In other words, check all the vector space requirements for U.)

2. Let $P_3^{\mathbb{R}}$ be the vector space of polynomials of degree 3 or less in the variable x. Check whether

$$x - x^3 \in \operatorname{span}\{x^2, 2x + x^2, x + x^3\}$$

Hint for Problem 2

3. Let U and W be subspaces of V. Are:

   (a) $U \cup W$

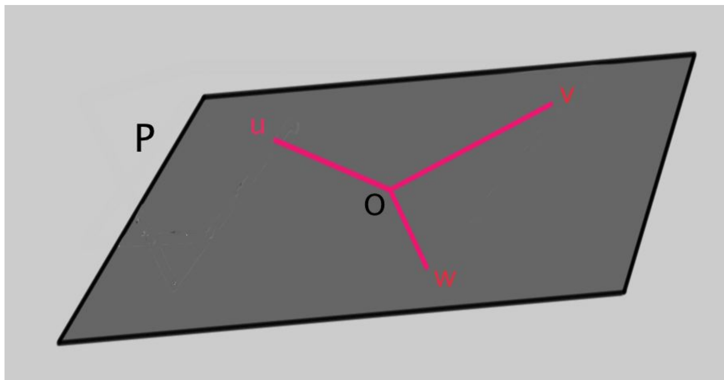
   (b) $U \cap W$

   also subspaces? Explain why or why not. Draw examples in $\mathbb{R}^3$.

Hint for Problem 3

16 Linear Independence

Consider a plane P that includes the origin in $\mathbb{R}^3$ and a collection $\{u, v, w\}$ of non-zero vectors in P:



If no two of u, v and w are parallel, then $P = \operatorname{span}\{u, v, w\}$. But any two vectors determines a plane, so we should be able to span the plane using only two of the vectors u, v, w. Then we could choose two of the vectors in $\{u, v, w\}$ whose span is P, and express the other as a linear combination of those two. Suppose u and v span P. Then there exist constants d^1, d^2 (not both zero) such that $w = d^1u + d^2v$. Since w can be expressed in terms of u and v we say that it is not independent. More generally, the relationship

$$c^1u + c^2v + c^3w = 0 \qquad c^i \in \mathbb{R},\ \text{some } c^i \neq 0$$

expresses the fact that u, v, w are not all independent.

Definition We say that the vectors $v_1, v_2, \ldots, v_n$ are *linearly dependent* if there exist constants[9] $c^1, c^2, \ldots, c^n$ not all zero such that

$$c^1v_1 + c^2v_2 + \cdots + c^nv_n = 0.$$

Otherwise, the vectors $v_1, v_2, \ldots, v_n$ are *linearly independent.*

[9]Usually our vector spaces are defined over $\mathbb{R}$, but in general we can have vector spaces defined over different base fields such as $\mathbb{C}$ or $\mathbb{Z}_2$. The coefficients c^i should come from whatever our base field is (usually $\mathbb{R}$).

Example Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} -3 \\ 7 \\ 4 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 5 \\ 12 \\ 17 \end{pmatrix}, \qquad v_4 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Are these vectors linearly independent?

No, since $3v_1 + 2v_2 - v_3 + v_4 = 0$, the vectors are linearly *dependent*.

`Worked Example`

In the above example we were given the linear combination $3v_1 + 2v_2 - v_3 + v_4$ seemingly by magic. The next example shows how to find such a linear combination, if it exists.

Example Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Are they linearly independent?

We need to see whether the system

$$c^1 v_1 + c^2 v_2 + c^3 v_3 = 0$$

has any solutions for c^1, c^2, c^3. We can rewrite this as a homogeneous system:

$$\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} c^1 \\ c^2 \\ c^3 \end{pmatrix} = 0.$$

This system has solutions if and only if the matrix $M = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$ is singular, so we should find the determinant of M:

$$\det M = \det \begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 3 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = 0.$$

Therefore nontrivial solutions exist. At this point we know that the vectors are linearly dependent. If we need to, we can find coefficients that demonstrate linear dependence by solving the system of equations:

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Then $c^3 = \mu$, $c^2 = -\mu$, and $c^1 = -2\mu$. Now any choice of μ will produce coefficients c^1, c^2, c^3 that satisfy the linear equation. So we can set $\mu = 1$ and obtain:

$$c^1 v_1 + c^2 v_2 + c^3 v_3 = 0 \Rightarrow -2v_1 - v_2 + v_3 = 0.$$

Reading homework: problem 16.1

Theorem 16.1 (Linear Dependence)**.** *A set of non-zero vectors* $\{v_1, \ldots, v_n\}$ *is linearly dependent if and only if one of the vectors* v_k *is expressible as a linear combination of the preceding vectors.*

Proof. The theorem is an if and only if statement, so there are two things to show.

i. First, we show that if $v_k = c^1 v_1 + \cdots c^{k-1} v_{k-1}$ then the set is linearly dependent.

This is easy. We just rewrite the assumption:

$$c^1 v_1 + \cdots + c^{k-1} v_{k-1} - v_k + 0v_{k+1} + \cdots + 0v_n = 0.$$

This is a vanishing linear combination of the vectors $\{v_1, \ldots, v_n\}$ with not all coefficients equal to zero, so $\{v_1, \ldots, v_n\}$ is a linearly dependent set.

ii. Now, we show that linear dependence implies that there exists k for which v_k is a linear combination of the vectors $\{v_1, \ldots, v_{k-1}\}$.

The assumption says that

$$c^1 v_1 + c^2 v_2 + \cdots + c^n v_n = 0.$$

Take k to be the largest number for which c_k is not equal to zero. So:

$$c^1 v_1 + c^2 v_2 + \cdots + c^{k-1} v_{k-1} + c^k v_k = 0.$$

(Note that $k > 1$, since otherwise we would have $c^1 v_1 = 0 \Rightarrow v_1 = 0$, contradicting the assumption that none of the v_i are the zero vector.)

As such, we can rearrange the equation:

$$\begin{aligned} c^1 v_1 + c^2 v_2 + \cdots + c^{k-1} v_{k-1} &= -c^k v_k \\ \Rightarrow \; -\frac{c^1}{c^k} v_1 - \frac{c^2}{c^k} v_2 - \cdots - \frac{c^{k-1}}{c^k} v_{k-1} &= v_k. \end{aligned}$$

Therefore we have expressed v_k as a linear combination of the previous vectors, and we are done.

$\square$

Worked proof

Example Consider the vector space $P_2(t)$ of polynomials of degree less than or equal to 2. Set:

$$\begin{aligned}
v_1 &= 1+t \\
v_2 &= 1+t^2 \\
v_3 &= t+t^2 \\
v_4 &= 2+t+t^2 \\
v_5 &= 1+t+t^2.
\end{aligned}$$

The set $\{v_1, \ldots, v_5\}$ is linearly dependent, because $v_4 = v_1 + v_2$.

We have seen two different ways to show a set of vectors is linearly dependent: we can either find a linear combination of the vectors which is equal to zero, or we can express one of the vectors as a linear combination of the other vectors. On the other hand, to check that a set of vectors is linearly *independent*, we must check that every linear combination of our vectors with non-vanishing coefficients gives something other than the zero vector. Equivalently, to show that the set $v_1, v_2, \ldots, v_n$ is linearly independent, we must show that the equation $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$ has no solutions other than $c_1 = c_2 = \cdots = c_n = 0$.

Example Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}.$$

Are they linearly independent?

We need to see whether the system

$$c^1v_1 + c^2v_2 + c^3v_3 = 0$$

has any solutions for c^1, c^2, c^3. We can rewrite this as a homogeneous system:

$$\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} c^1 \\ c^2 \\ c^3 \end{pmatrix} = 0.$$

This system has solutions if and only if the matrix $M = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$ is singular, so we should find the determinant of M:

$$\det M = \det \begin{pmatrix} 0 & 2 & 1 \\ 0 & 2 & 4 \\ 2 & 1 & 3 \end{pmatrix} = 2 \det \begin{pmatrix} 2 & 1 \\ 2 & 4 \end{pmatrix} = 12.$$

Since the matrix M has non-zero determinant, the only solution to the system of equations

$$\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} c^1 \\ c^2 \\ c^3 \end{pmatrix} = 0$$

is $c_1 = c_2 = c_3 = 0$. (Why?) So the vectors v_1, v_2, v_3 are linearly independent.

Reading homework: problem 16.2

Now suppose vectors $v_1, \ldots, v_n$ are linearly dependent,

$$c^1 v_1 + c^2 v_2 + \cdots + c^n v_n = 0$$

with $c^1 \neq 0$. Then:

$$\operatorname{span}\{v_1, \ldots, v_n\} = \operatorname{span}\{v_2, \ldots, v_n\}$$

because any $x \in \operatorname{span}\{v_1, \ldots, v_n\}$ is given by

$$\begin{aligned} x &= a^1 v_1 + \cdots a^n v_n \\ &= a^1 \left(-\frac{c^2}{c_1} v_2 - \cdots - \frac{c^n}{c_1} v_n \right) + a^2 v_2 + \cdots + a^n v_n \\ &= \left(a^2 - a^1 \frac{c^2}{c_1} \right) v_2 + \cdots + \left(a^n - a^1 \frac{c^n}{c_1} \right) v_n. \end{aligned}$$

Then x is in $\operatorname{span}\{v_2, \ldots, v_n\}$.

When we write a vector space as the span of a list of vectors, we would like that list to be as short as possible (we will explore this idea further in lecture 17). This can be achieved by iterating the above procedure.

Example In the above example, we found that $v_4 = v_1 + v_2$. In this case, any expression for a vector as a linear combination involving v_4 can be turned into a combination without v_4 by making the substitution $v_4 = v_1 + v_2$.

Then:

$$\begin{aligned} S &= \operatorname{span}\{1+t, 1+t^2, t+t^2, 2+t+t^2, 1+t+t^2\} \\ &= \operatorname{span}\{1+t, 1+t^2, t+t^2, 1+t+t^2\}. \end{aligned}$$

Now we notice that $1+t+t^2 = \frac{1}{2}(1+t) + \frac{1}{2}(1+t^2) + \frac{1}{2}(t+t^2)$. So the vector $1+t+t^2 = v_5$ is also extraneous, since it can be expressed as a linear combination of the remaining three vectors, v_1, v_2, v_3. Therefore

$$S = \operatorname{span}\{1+t, 1+t^2, t+t^2\}.$$

In fact, you can check that there are no (non-zero) solutions to the linear system

$$c^1(1+t) + c^2(1+t^2) + c^3(t+t^2) = 0.$$

Therefore the remaining vectors $\{1+t, 1+t^2, t+t^2\}$ are linearly independent, and span the vector space S. Then these vectors are a minimal spanning set, in the sense that no more vectors can be removed since the vectors are linearly independent. Such a set is called a *basis* for S.

Example Let B^3 be the space of 3×1 bit-valued matrices (i.e., column vectors). Is the following subset linearly independent?

$$\left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\}$$

If the set is linearly dependent, then we can find non-zero solutions to the system:

$$c^1 \begin{pmatrix}1\\1\\0\end{pmatrix} + c^2 \begin{pmatrix}1\\0\\1\end{pmatrix} + c^3 \begin{pmatrix}0\\1\\1\end{pmatrix} = 0,$$

which becomes the linear system

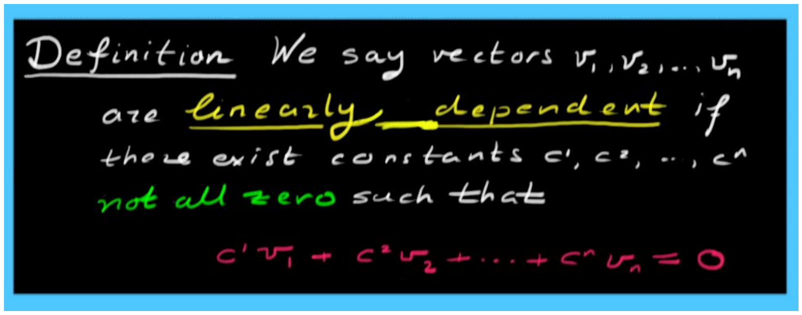
$$\begin{pmatrix}1 & 1 & 0\\1 & 0 & 1\\0 & 1 & 1\end{pmatrix} \begin{pmatrix}c^1\\c^2\\c^3\end{pmatrix} = 0.$$

Solutions exist if and only if the determinant of the matrix is non-zero. But:

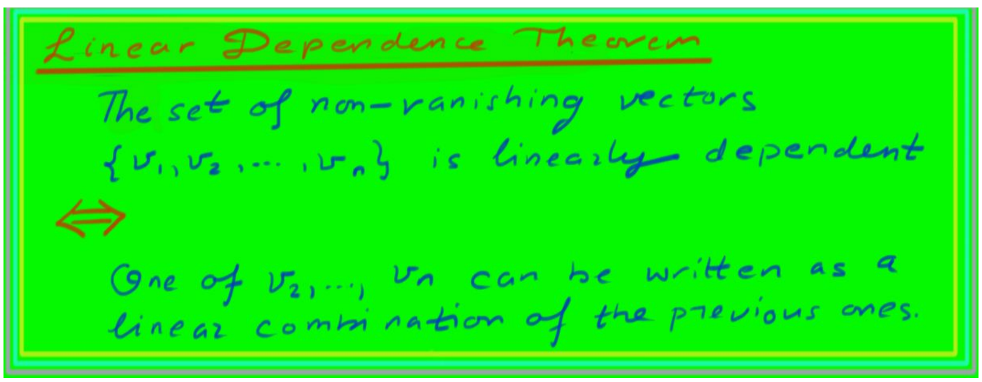
$$\det \begin{pmatrix}1 & 1 & 0\\1 & 0 & 1\\0 & 1 & 1\end{pmatrix} = 1 \det \begin{pmatrix}0 & 1\\1 & 1\end{pmatrix} - 1 \det \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix} = -1 - 1 = 1 + 1 = 0$$

Therefore non-trivial solutions exist, and the set is not linearly independent.

To summarize, the key definition in this lecture was:



Perhaps the most useful Theorem was:



References

Hefferon, Chapter Two, Section II: Linear Independence
Hefferon, Chapter Two, Section III.1: Basis
Beezer, Chapter V, Section LI
Beezer, Chapter V, Section LDS
Beezer, Chapter VS, Section LISS, Subsection LI
Wikipedia:

- Linear Independence
- Basis

Review Problems

1. Let B^n be the space of $n \times 1$ bit-valued matrices (*i.e.*, column vectors) over the field $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$. Remember that this means that the coefficients in any linear combination can be only 0 or 1, with rules for adding and multiplying coefficients given here.

   (a) How many different vectors are there in B^n?

   (b) Find a collection S of vectors that span B^3 and are linearly independent. In other words, find a basis of B^3.

   (c) Write each other vector in B^3 as a linear combination of the vectors in the set S that you chose.

   (d) Would it be possible to span B^3 with only two vectors?

`Hint for Problem 1`

2. Let e_i be the vector in $\mathbb{R}^n$ with a 1 in the ith position and 0's in every other position. Let v be an arbitrary vector in $\mathbb{R}^n$.

   (a) Show that the collection $\{e_1, \dots, e_n\}$ is linearly independent.

   (b) Demonstrate that $v = \sum_{i=1}^{n}(v \cdot e_i)e_i$.

   (c) The span$\{e_1, \dots, e_n\}$ is the same as what vector space?

17 Basis and Dimension

In Lecture 16, we established the notion of a linearly independent set of vectors in a vector space V, and of a set of vectors that span V. We saw that any set of vectors that span V can be reduced to some minimal collection of linearly independent vectors; such a set is called a *basis* of the subspace V.

Definition Let V be a vector space. Then a set S is a *basis* for V if S is linearly independent and $V = \operatorname{span} S$.

If S is a basis of V and S has only finitely many elements, then we say that V is *finite-dimensional*. The number of vectors in S is the *dimension* of V.

Suppose V is a *finite-dimensional* vector space, and S and T are two different bases for V. One might worry that S and T have a different number of vectors; then we would have to talk about the dimension of V in terms of the basis S or in terms of the basis T. Luckily this isn't what happens. Later in this section, we will show that S and T must have the same number of vectors. This means that the dimension of a vector space does not depend on the basis. In fact, dimension is a very important way to characterize of any vector space V.

Example $P_n(t)$ has a basis $\{1, t, \ldots, t^n\}$, since every polynomial of degree less than or equal to n is a sum

$$a^0\, 1 + a^1\, t + \cdots + a^n\, t^n, \qquad a^i \in \mathbb{R}$$

so $P_n(t) = \operatorname{span}\{1, t, \ldots, t^n\}$. This set of vectors is linearly independent: If the polynomial $p(t) = c^0 1 + c^1 t + \cdots + c^n t^n = 0$, then $c^0 = c^1 = \cdots = c^n = 0$, so $p(t)$ is the zero polynomial.

Then $P_n(t)$ is finite dimensional, and $\dim P_n(t) = n + 1$.

Theorem 17.1. *Let $S = \{v_1, \ldots, v_n\}$ be a basis for a vector space V. Then every vector $w \in V$ can be written* uniquely *as a linear combination of vectors in the basis S:*

$$w = c^1 v_1 + \cdots + c^n v_n.$$

Proof. Since S is a basis for V, then $\operatorname{span} S = V$, and so there exist constants c^i such that $w = c^1 v_1 + \cdots + c^n v_n$.

Suppose there exists a second set of constants d^i such that

$$w = d^1 v_1 + \cdots + d^n v_n \,.$$

Then:

$$\begin{aligned}
0_V &= w - w \\
&= c^1 v_1 + \cdots + c^n v_n - d^1 v_1 + \cdots + d^n v_n \\
&= (c^1 - d^1) v_1 + \cdots + (c^n - d^n) v_n.
\end{aligned}$$

If it occurs exactly once that $c^i \neq d^i$, then the equation reduces to $0 = (c^i - d^i) v_i$, which is a contradiction since the vectors v_i are assumed to be non-zero.

If we have more than one i for which $c^i \neq d^i$, we can use this last equation to write one of the vectors in S as a linear combination of other vectors in S, which contradicts the assumption that S is linearly independent. Then for every i, $c^i = d^i$. $\square$

`Proof of Theorem`

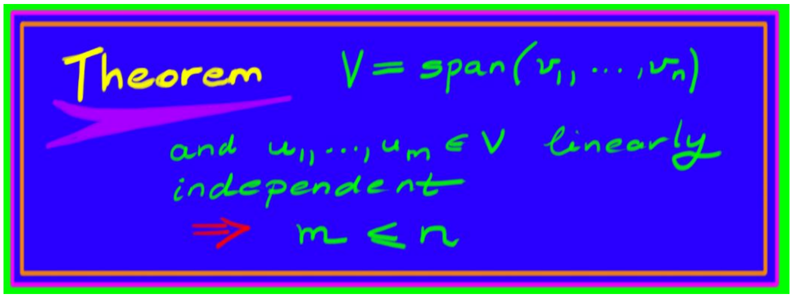
Next, we would like to establish a method for determining whether a collection of vectors forms a basis for $\mathbb{R}^n$. But first, we need to show that any two bases for a finite-dimensional vector space has the same number of vectors.

Lemma 17.2. *If $S = \{v_1, \ldots, v_n\}$ is a basis for a vector space V and $T = \{w_1, \ldots, w_m\}$ is a linearly independent set of vectors in V, then $m \leq n$.*

The idea of the proof is to start with the set S and replace vectors in S one at a time with vectors from T, such that after each replacement we still have a basis for V.

Reading homework: problem 17.1

Proof. Since S spans V, then the set $\{w_1, v_1, \ldots, v_n\}$ is linearly dependent. Then we can write w_1 as a linear combination of the v_i; using that equation, we can express one of the v_i in terms of w_1 and the remaining v_j with $j \neq$



i. Then we can discard one of the v_i from this set to obtain a linearly independent set that still spans V. Now we need to prove that S_1 is a basis; we need to show that S_1 is linearly independent and that S_1 spans V.

The set $S_1 = \{w_1, v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}$ is linearly independent: By the previous theorem, there was a unique way to express w_1 in terms of the set S. Now, to obtain a contradiction, suppose there is some k and constants c^i such that

$$v_k = c^0 w_1 + c^1 v_1 + \cdots + c^{i-1} v_{i-1} + c^{i+1} v_{i+1} + \cdots + c^n v_n.$$

Then replacing w_1 with its expression in terms of the collection S gives a way to express the vector v_k as a linear combination of the vectors in S, which contradicts the linear independence of S. On the other hand, we cannot express w_1 as a linear combination of the vectors in $\{v_j | j \neq i\}$, since the expression of w_1 in terms of S was unique, and had a non-zero coefficient on the vector v_i. Then no vector in S_1 can be expressed as a combination of other vectors in S_1, which demonstrates that S_1 is linearly independent.

The set S_1 spans V: For any $u \in V$, we can express u as a linear combination of vectors in S. But we can express v_i as a linear combination of vectors in the collection S_1; rewriting v_i as such allows us to express u as a linear combination of the vectors in S_1.

Then S_1 is a basis of V with n vectors.

We can now iterate this process, replacing one of the v_i in S_1 with w_2, and so on. If $m \leq n$, this process ends with the set $S_m = \{w_1, \dots, w_m, v_{i_1}, \dots, v_{i_{n-m}}\}$, which is fine.

Otherwise, we have $m > n$, and the set $S_n = \{w_1, \dots, w_n\}$ is a basis for V. But we still have some vector w_{n+1} in T that is not in S_n. Since S_n is a basis, we can write w_{n+1} as a combination of the vectors in S_n, which contradicts the linear independence of the set T. Then it must be the case that $m \leq n$, as desired. $\square$

`Worked Example`

Corollary 17.3. *For a finite-dimensional vector space V, any two bases for V have the same number of vectors.*

Proof. Let S and T be two bases for V. Then both are linearly independent sets that span V. Suppose S has n vectors and T has m vectors. Then by the previous lemma, we have that $m \leq n$. But (exchanging the roles of S and T in application of the lemma) we also see that $n \leq m$. Then $m = n$, as desired. $\square$

Reading homework: problem 17.2

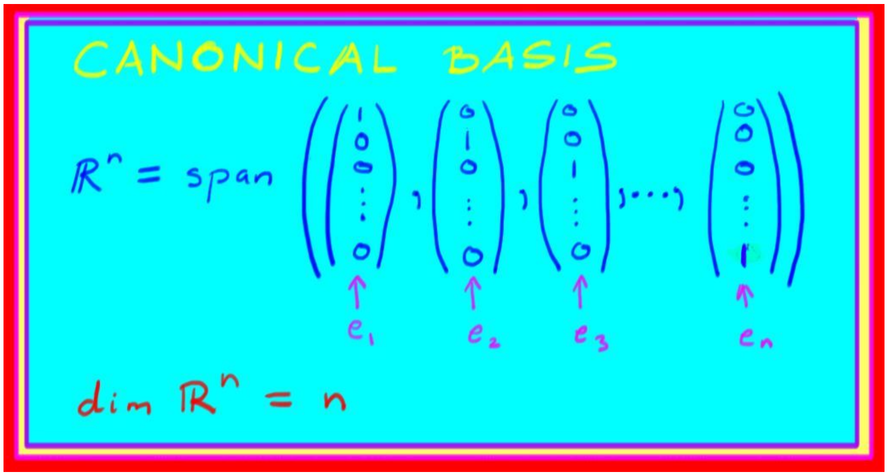
17.1 Bases in $\mathbb{R}^n$.

From one of the review questions, we know that

$$\mathbb{R}^n = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\},$$

and that this set of vectors is linearly independent. So this set of vectors is a basis for $\mathbb{R}^n$, and $\dim \mathbb{R}^n = n$. This basis is often called the *standard* or *canonical basis* for $\mathbb{R}^n$. The vector with a one in the ith position and zeros everywhere else is written e_i. It points in the direction of the ith coordinate axis, and has unit length. In multivariable calculus classes, this basis is often written $\{i, j, k\}$ for $\mathbb{R}^3$.

Note that it is often convenient to order basis elements, so rather than writing a set of vectors, we would write a list. This is called an ordered basis. For example, the canonical ordered basis for $\mathbb{R}^n$ is $(e_1, e_2, \ldots, e_n)$. The possibility to reorder basis vectors is not the only way in which bases are non-unique:



Bases are not unique. While there exists a unique way to express a vector in terms of any particular basis, bases themselves are far from unique. For example, both of the sets:

$$\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\} \text{ and } \left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix} \right\}$$

are bases for $\mathbb{R}^2$. Rescaling any vector in one of these sets is already enough to show that $\mathbb{R}^2$ has infinitely many bases. But even if we require that all of the basis vectors have unit length, it turns out that there are still infinitely many bases for $\mathbb{R}^2$. (See Review Question 3.)

To see whether a collection of vectors $S = \{v_1, \ldots, v_m\}$ is a basis for $\mathbb{R}^n$, we have to check that they are linearly independent and that they span $\mathbb{R}^n$. From the previous discussion, we also know that m must equal n, so assume S has n vectors.

If S is linearly independent, then there is no non-trivial solution of the equation

$$0 = x^1 v_1 + \cdots + x^n v_n.$$

Let M be a matrix whose columns are the vectors v_i. Then the above equation is equivalent to requiring that there is a unique solution to

$$MX = 0\,.$$

To see if S spans $\mathbb{R}^n$, we take an arbitrary vector w and solve the linear system

$$w = x^1 v_1 + \cdots + x^n v_n$$

in the unknowns c^i. For this, we need to find a unique solution for the linear system $MX = w$.

Thus, we need to show that M^{-1} exists, so that

$$X = M^{-1}w$$

is the unique solution we desire. Then we see that S is a basis for V if and only if $\det M \neq 0$.

Theorem 17.4. *Let $S = \{v_1, \ldots, v_m\}$ be a collection of vectors in $\mathbb{R}^n$. Let M be the matrix whose columns are the vectors in S. Then S is a basis for V if and only if m is the dimension of V and*

$$\det M \neq 0.$$

Example Let

$$S = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\} \text{ and } T = \left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix} \right\}.$$

Then set $M_S = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$. Since $\det M_S = 1 \neq 0$, then S is a basis for $\mathbb{R}^2$.

Likewise, set $M_T = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$. Since $\det M_T = -2 \neq 0$, then T is a basis for $\mathbb{R}^2$.

References

Hefferon, Chapter Two, Section II: Linear Independence
Hefferon, Chapter Two, Section III.1: Basis
Beezer, Chapter VS, Section B, Subsections B-BNM
Beezer, Chapter VS, Section D, Subsections D-DVS
Wikipedia:

- Linear Independence
- Basis

Review Problems

1. (a) Draw the collection of all unit vectors in $\mathbb{R}^2$.

(b) Let $S_x = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, x \right\}$, where x is a unit vector in $\mathbb{R}^2$. For which x is S_x a basis of $\mathbb{R}^2$?

2. Let B^n be the vector space of column vectors with bit entries $0, 1$. Write down every basis for B^1 and B^2. How many bases are there for B^3? B^4? Can you make a conjecture for the number of bases for B^n?

   (Hint: You can build up a basis for B^n by choosing one vector at a time, such that the vector you choose is not in the span of the previous vectors you've chosen. How many vectors are in the span of any one vector? Any two vectors? How many vectors are in the span of any k vectors, for $k \leq n$?)

`Hint for Problem 2`

3. Suppose that V is an n-dimensional vector space.

   (a) Show that any n linearly independent vectors in V form a basis.

   (Hint: Let $\{w_1, \ldots, w_m\}$ be a collection of n linearly independent vectors in V, and let $\{v_1, \ldots, v_n\}$ be a basis for V. Apply the method of Lemma 17.2 to these two sets of vectors.)

   (b) Show that any set of n vectors in V which span V forms a basis for V.

   (Hint: Suppose that you have a set of n vectors which span V but do not form a basis. What must be true about them? How could you get a basis from this set? Use Corollary 17.3 to derive a contradiction.)

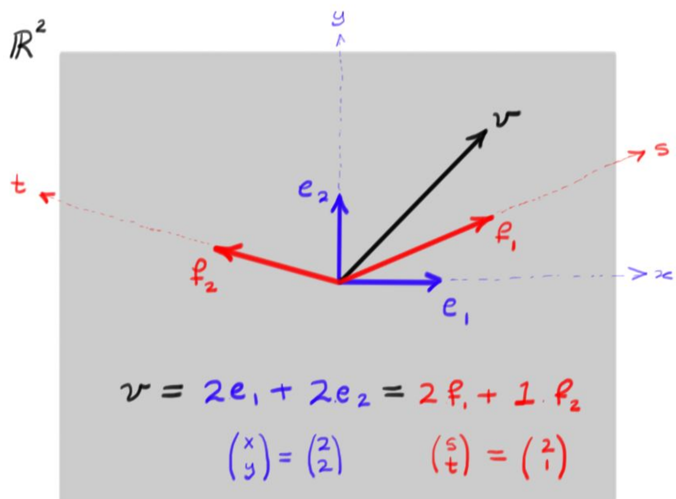
4. Let S be a collection of vectors $\{v_1, \ldots, v_n\}$ in a vector space V. Show that if every vector w in V can be expressed uniquely as a linear combination of vectors in S, then S is a basis of V. In other words: suppose that for every vector w in V, there is exactly one set of constants $c^1, \ldots, c^n$ so that $c^1 v_1 + \cdots + c^n v_n = w$. Show that this means that the set S is linearly independent and spans V. (This is the converse to the theorem in the lecture.)

5. Vectors are objects that you can add together; show that the set of all linear transformations mapping $\mathbb{R}^3 \to \mathbb{R}$ is itself a vector space. Find a basis for this vector space. Do you think your proof could be modified to work for linear transformations $\mathbb{R}^n \to \mathbb{R}$?

   Hint: Represent $\mathbb{R}^3$ *as column vectors, and argue that a linear transformation* $T\colon \mathbb{R}^3 \to \mathbb{R}$ *is just a row vector. If you are stuck or just curious, see dual space.*

18 Eigenvalues and Eigenvectors

Before discussing eigenvalues and eigenvectors, we need to have a better understanding of the relationship between linear transformations and matrices. Consider, as an example the plane $\mathbb{R}^2$



The information of the vector v can be transmitted in many ways. In the basis $\{e_1, e_2\}$ it is the ordered pair $(x, y) = (2, 2)$ while in the basis $\{f_1, f_2\}$ is corresponds to $(s, t) = (2, 1)$. This can be confusing, the idea to keep firm in your mind is that the vector space and its elements—vectors—are what really "exist". Typically they will correspond to configurations of the real world system you are trying to describe. On the other hand, things like coordinate axes and "components of a vector" (x, y) are just mathematical tools used to label vectors.

18.1 Matrix of a Linear Transformation

Let V and W be vector spaces, with bases $S = \{e_1, \ldots, e_n\}$ and $T = \{f_1, \ldots, f_m\}$ respectively. Since these are bases, there exist constants v^i and w^j such that any vectors $v \in V$ and $w \in W$ can be written as:

$$\begin{aligned} v &= v^1 e_1 + v^2 e_2 + \cdots + v^n e_n \\ w &= w^1 f_1 + w^2 f_2 + \cdots + w^m f_m \end{aligned}$$

We call the coefficients $v^1, \dots, v^n$ the *components* of v in the basis[10] $\{e_1, \dots, e_n\}$. It is often convenient to arrange the components v^i in a column vector and the basis vector in a row vector by writing

$$v = \begin{pmatrix} e_1 & e_2 & \cdots & e_n \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{pmatrix}.$$

Worked Example

Example Consider the basis $S = \{1-t, 1+t\}$ for the vector space $P_1(t)$. The vector $v = 2t$ has components $v^1 = -1, v^2 = 1$, because

$$v = -1(1-t) + 1(1+t) = \begin{pmatrix} 1-t & 1+t \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

We may consider these components as vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$:

$$\begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} \in \mathbb{R}^n, \qquad \begin{pmatrix} w^1 \\ \vdots \\ w^m \end{pmatrix} \in \mathbb{R}^m.$$

Now suppose we have a linear transformation $L \colon V \to W$. Then we can expect to write L as an $m \times n$ matrix, turning an n-dimensional vector of coefficients corresponding to v into an m-dimensional vector of coefficients for w.

Using linearity, we write:

$$\begin{aligned} L(v) &= L(v^1 e_1 + v^2 e_2 + \cdots + v^n e_n) \\ &= v^1 L(e_1) + v^2 L(e_2) + \cdots + v^n L(e_n) \\ &= \begin{pmatrix} L(e_1) & L(e_2) & \cdots & L(e_n) \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{pmatrix}. \end{aligned}$$

[10] To avoid confusion, it helps to notice that components of a vector are almost always labeled by a superscript, while basis vectors are labeled by subscripts in the conventions of these lecture notes.

This is a vector in W. Let's compute its components in W.

We know that for each e_j, $L(e_j)$ is a vector in W, and can thus be written uniquely as a linear combination of vectors in the basis T. Then we can find coefficients M^i_j such that:

$$L(e_j) = f_1 M^1_j + \cdots + f_m M^m_j = \sum_{i=1}^{m} f_i M^i_j = \begin{pmatrix} f_1 & f_2 & \cdots & f_m \end{pmatrix} \begin{pmatrix} M^1_j \\ M^2_j \\ \vdots \\ M^m_j \end{pmatrix} .$$

We've written the M^i_j on the right side of the f's to agree with our previous notation for matrix multiplication. We have an "up-hill rule" where the matching indices for the multiplied objects run up and to the right, like so: $f_i M^i_j$.

Now M^i_j is the ith component of $L(e_j)$. Regarding the coefficients M^i_j as a matrix, we can see that the jth column of M is the coefficients of $L(e_j)$ in the basis T.

Then we can write:

$$\begin{aligned} L(v) &= L(v^1 e_1 + v^2 e_2 + \cdots + v^n e_n) \\ &= v^1 L(e_1) + v^2 L(e_2) + \cdots + v^n L(e_n) \\ &= \sum_{i=1}^{m} L(e_j) v^j \\ &= \sum_{i=1}^{m} (M^1_j f_1 + \cdots + M^m_j f_m) v^j \\ &= \sum_{i=1}^{m} f_i \left[\sum_{j=1}^{n} M^i_j v^j \right] \\ &= \begin{pmatrix} f_1 & f_2 & \cdots & f_m \end{pmatrix} \begin{pmatrix} M^1_1 & M^1_2 & \cdots & M^1_n \\ M^2_1 & M^2_2 & & \\ \vdots & & & \vdots \\ M^m_1 & & \cdots & M^m_n \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{pmatrix} \end{aligned}$$

The second last equality is the definition of matrix multiplication which is obvious from the last line. Thus:

$$\begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} \stackrel{L}{\mapsto} \begin{pmatrix} M_1^1 & \dots & M_n^1 \\ \vdots & & \vdots \\ M_1^m & \dots & M_n^m \end{pmatrix} \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix},$$

and $M = (M_j^i)$ is called the matrix of L. Notice that this matrix depends on a *choice* of bases for both V and W. Also observe that the columns of M are computed by examining L acting on each basis vector in V expanded in the basis vectors of W.

Example Let $L\colon P_1(t) \mapsto P_1(t)$, such that $L(a+bt) = (a+b)t$. Since $V = P_1(t) = W$, let's choose the same basis for V and W. We'll choose the basis $\{1-t, 1+t\}$ for this example.

Thus:

$$\begin{aligned} L(1-t) &= (1-1)t = \ 0 \ = (1-t)\cdot 0 + (1+t)\cdot 0 = \big((1-t) \quad (1+t)\big)\begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ L(1+t) &= (1+1)t = 2t \ = (1-t)\cdot -1 + (1+t)\cdot 1 = \big((1-t) \quad (1+t)\big)\begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ \Rightarrow M &= \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

To obtain the last line we used that fact that the columns of M are just the coefficients of L on each of the basis vectors; this always makes it easy to write down M in terms of the basis we have chosen.

Reading homework: problem 20.1

Example Consider a linear transformation

$$L\colon \mathbb{R}^2 \to \mathbb{R}^2 .$$

Suppose we know that $L\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$ and $L\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$. Then, because of linearity, we can determine what L does to any vector $\begin{pmatrix} x \\ y \end{pmatrix}$:

$$L\begin{pmatrix} x \\ y \end{pmatrix} = L(x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = xL\begin{pmatrix} 1 \\ 0 \end{pmatrix} + yL\begin{pmatrix} 0 \\ 1 \end{pmatrix} = x\begin{pmatrix} a \\ c \end{pmatrix} + y\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}.$$

Now notice that for any vector $\begin{pmatrix} x \\ y \end{pmatrix}$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = L \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts by matrix multiplication in the same way that L does. This is the *matrix of* L in the *basis* $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

Example Any vector in $\mathbb{R}^n$ can be written as a linear combination of the *standard basis vectors* $\{e_i | i \in \{1, \ldots, n\}\}$. The vector e_i has a one in the ith position, and zeros everywhere else. *I.e.*

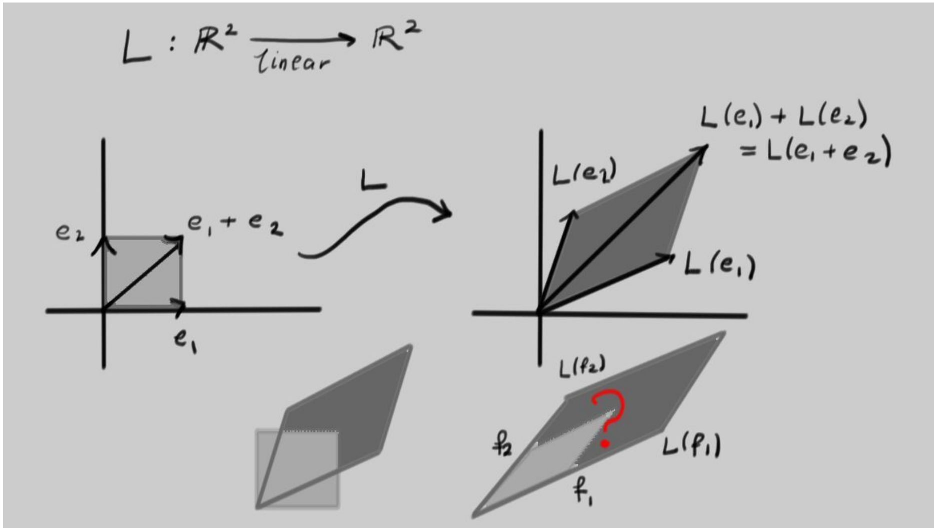
$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Then to find the matrix of any linear transformation $L \colon \mathbb{R}^n \to \mathbb{R}^n$, it suffices to know what $L(e_i)$ is for every i.

For any matrix M, observe that Me_i is equal to the ith column of M. Then if the ith column of M equals $L(e_i)$ for every i, then $Mv = L(v)$ for every $v \in \mathbb{R}^n$. Then the matrix representing L in the standard basis is just the matrix whose ith column is $L(e_i)$.

18.2 Invariant Directions

Have a look at the linear transformation L depicted below:



It was picked at random by choosing a pair of vectors $L(e_1)$ and $L(e_2)$ as the outputs of L acting on the canonical basis vectors. Notice how the unit square with a corner at the origin get mapped to a parallelogram. The second line of the picture shows these superimposed on one another. Now look at the second picture on that line. There, two vectors f_1 and f_2 have been carefully chosen such that if the inputs into L are in the parallelogram spanned by f_1 and f_2, the outputs also form a parallelogram with edges lying along the same two directions. Clearly this is a very special situation that should correspond to a interesting properties of L.

Now lets try an explicit example to see if we can achieve the last picture:

Example Consider the linear transformation L such that

$$L\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-4\\-10\end{pmatrix} \text{ and } L\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}3\\7\end{pmatrix},$$

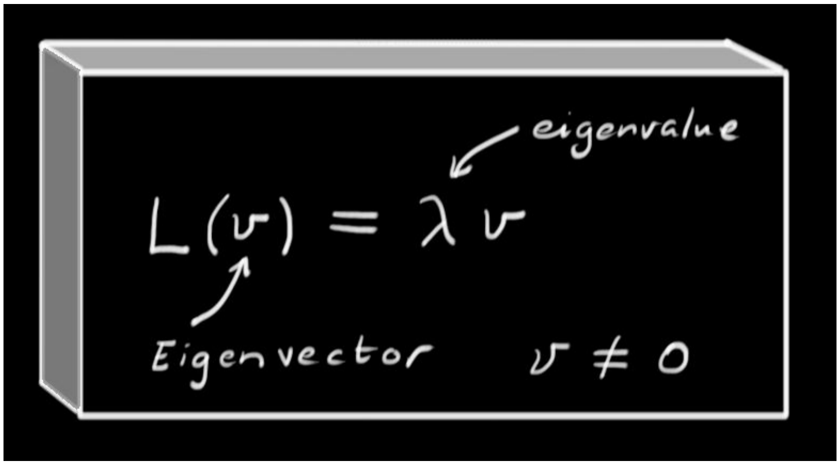
so that the matrix of L is

$$\begin{pmatrix}-4 & 3\\-10 & 7\end{pmatrix}.$$

Recall that a vector is a direction and a magnitude; L applied to $\begin{pmatrix}1\\0\end{pmatrix}$ or $\begin{pmatrix}0\\1\end{pmatrix}$ changes both the direction and the magnitude of the vectors given to it.

Notice that

$$L\begin{pmatrix}3\\5\end{pmatrix} = \begin{pmatrix}-4\cdot 3+3\cdot 5\\-10\cdot 3+7\cdot 5\end{pmatrix} = \begin{pmatrix}3\\5\end{pmatrix}.$$



Then L fixes the direction (and actually also the magnitude) of the vector $v_1 = \begin{pmatrix}3\\5\end{pmatrix}$.

In fact also the vector $v_2 = \begin{pmatrix}1\\2\end{pmatrix}$ has its direction fixed by M.

Reading homework: problem 18.1

Now, notice that any vector with the same direction as v_1 can be written as cv_1 for some constant c. Then $L(cv_1) = cL(v_1) = cv_1$, so L fixes every vector pointing in the same direction as v_1.

Also notice that

$$L\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-4\cdot 1 + 3\cdot 2\\ -10\cdot 1 + 7\cdot 2\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix} = 2\begin{pmatrix}1\\2\end{pmatrix}\,.$$

Then L fixes the direction of the vector $v_2 = \begin{pmatrix}1\\2\end{pmatrix}$ but stretches v_2 by a factor of 2. Now notice that for any constant c, $L(cv_2) = cL(v_2) = 2cv_2$. Then L stretches every vector pointing in the same direction as v_2 by a factor of 2.

In short, given a linear transformation L it is sometimes possible to find a vector $v \neq 0$ and constant $\lambda \neq 0$ such that

$$L(v) = \lambda v.$$

We call the direction of the vector v an *invariant direction*. In fact, any vector pointing in the same direction also satisfies the equation: $L(cv) = cL(v) = \lambda cv$. The vector v is called an *eigenvector* of L, and λ is an *eigenvalue*. Since the direction is all we really care about here, then any other vector cv (so long as $c \neq 0$) is an equally good choice of eigenvector. Notice that the relation "u and v point in the same direction" is an equivalence relation.

In our example of the linear transformation L with matrix

$$\begin{pmatrix} -4 & 3 \\ -10 & 7 \end{pmatrix},$$

we have seen that L enjoys the property of having two invariant directions, represented by eigenvectors v_1 and v_2 with eigenvalues 1 and 2, respectively.

It would be very convenient if we could write any vector w as a linear combination of v_1 and v_2. Suppose $w = rv_1 + sv_2$ for some constants r and s. Then:

$$L(w) = L(rv_1 + sv_2) = rL(v_1) + sL(v_2) = rv_1 + 2sv_2.$$

Now L just multiplies the number r by 1 and the number s by 2. If we could write this as a matrix, it would look like:

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}$$

which is much slicker than the usual scenario

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

Here, s and t give the coordinates of w in terms of the vectors v_1 and v_2. In the previous example, we multiplied the vector by the matrix L and came up with a complicated expression. In these coordinates, we can see that L is a very simple *diagonal matrix*, whose diagonal entries are exactly the *eigenvalues* of L.

This process is called *diagonalization*. It makes complicated linear systems much easier to analyze.

Reading homework: problem 18.2

Now that we've seen what eigenvalues and eigenvectors are, there are a number of questions that need to be answered.

- How do we find eigenvectors and their eigenvalues?
- How many eigenvalues and (independent) eigenvectors does a given linear transformation have?
- When can a linear transformation be diagonalized?

We'll start by trying to find the eigenvectors for a linear transformation.

2×2 Example

Example Let $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $L(x, y) = (2x + 2y, 16x + 6y)$. First, we can find the matrix of L:

$$\begin{pmatrix} x \\ y \end{pmatrix} \stackrel{L}{\longmapsto} \begin{pmatrix} 2 & 2 \\ 16 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

We want to find an invariant direction $v = \begin{pmatrix} x \\ y \end{pmatrix}$ such that

$$L(v) = \lambda v$$

or, in matrix notation,

$$\begin{aligned} \begin{pmatrix} 2 & 2 \\ 16 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \lambda \begin{pmatrix} x \\ y \end{pmatrix} \\ \Leftrightarrow \begin{pmatrix} 2 & 2 \\ 16 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ \Leftrightarrow \begin{pmatrix} 2-\lambda & 2 \\ 16 & 6-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

This is a homogeneous system, so it only has solutions when the matrix $\begin{pmatrix} 2-\lambda & 2 \\ 16 & 6-\lambda \end{pmatrix}$ is singular. In other words,

$$\begin{aligned} \det \begin{pmatrix} 2-\lambda & 2 \\ 16 & 6-\lambda \end{pmatrix} &= 0 \\ \Leftrightarrow (2-\lambda)(6-\lambda) - 32 &= 0 \\ \Leftrightarrow \lambda^2 - 8\lambda - 20 &= 0 \\ \Leftrightarrow (\lambda - 10)(\lambda + 2) &= 0 \end{aligned}$$

For any square $n \times n$ matrix M, the polynomial in λ given by

$$P_M(\lambda) = \det(\lambda I - M) = (-1)^n \det(M - \lambda I)$$

is called the *characteristic polynomial* of M, and its roots are the eigenvalues of M.

In this case, we see that L has two eigenvalues, $\lambda_1 = 10$ and $\lambda_2 = -2$. To find the eigenvectors, we need to deal with these two cases separately. To do so, we solve the linear system $\begin{pmatrix} 2-\lambda & 2 \\ 16 & 6-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ with the particular eigenvalue λ plugged in to the matrix.

$\underline{\lambda = 10}$: We solve the linear system

$$\begin{pmatrix} -8 & 2 \\ 16 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Both equations say that $y = 4x$, so any vector $\begin{pmatrix} x \\ 4x \end{pmatrix}$ will do. Since we only need the direction of the eigenvector, we can pick a value for x. Setting $x = 1$ is convenient, and gives the eigenvector $v_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

$\underline{\lambda = -2}$: We solve the linear system

$$\begin{pmatrix} 4 & 2 \\ 16 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Here again both equations agree, because we chose λ to make the system singular. We see that $y = -2x$ works, so we can choose $v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

In short, our process was the following:

- Find the characteristic polynomial of the matrix M for L, given by[11] $\det(\lambda I - M)$.
- Find the roots of the characteristic polynomial; these are the eigenvalues of L.
- For each eigenvalue λ_i, solve the linear system $(M - \lambda_i I)v = 0$ to obtain an eigenvector v associated to λ_i.

Jordan block example

[11] It is often easier (and equivalent if you only need the roots) to compute $\det(M - \lambda I)$.

References

Hefferon, Chapter Three, Section III.1: Representing Linear Maps with Matrices
Hefferon, Chapter Five, Section II.3: Eigenvalues and Eigenvectors
Beezer, Chapter E, Section EE
Wikipedia:

- Eigen*
- Characteristic Polynomial
- Linear Transformations (and matrices thereof)

Review Problems

1. Let $M = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Find all eigenvalues of M. Does M have two independent[12] eigenvectors? Can M be diagonalized?

2. Consider $L: \mathbb{R}^2 \to \mathbb{R}^2$ with $L(x, y) = (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$.

   (a) Write the matrix of L in the basis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

   (b) When $\theta \neq 0$, explain how L acts on the plane. Draw a picture.

   (c) Do you expect L to have invariant directions?

   (d) Try to find real eigenvalues for L by solving the equation
   $$L(v) = \lambda v.$$

   (e) Are there complex eigenvalues for L, assuming that $i = \sqrt{-1}$ exists?

3. Let L be the linear transformation $L: \mathbb{R}^3 \to \mathbb{R}^3$ given by $L(x, y, z) = (x + y, x + z, y + z)$. Let e_i be the vector with a one in the ith position and zeros in all other positions.

   (a) Find Le_i for each i.

[12] Independence of vectors is explained here.

(b) Given a matrix $M = \begin{pmatrix} m^1_1 & m^1_2 & m^1_3 \\ m^2_1 & m^2_2 & m^2_3 \\ m^3_1 & m^3_2 & m^3_3 \end{pmatrix}$, what can you say about Me_i for each i?

(c) Find a 3×3 matrix M representing L. Choose three nonzero vectors pointing in different directions and show that $Mv = Lv$ for each of your choices.

(d) Find the eigenvectors and eigenvalues of M.

4. Let A be a matrix with eigenvector v with eigenvalue λ. Show that v is also an eigenvector for A^2 and what is its eigenvalue? How about for A^n where $n \in \mathbb{N}$? Suppose that A is invertible, show that v is also an eigenvector for A^{-1}.

5. A *projection* is a linear operator P such that $P^2 = P$. Let v be an eigenvector with eigenvalue λ for a projection P, what are all possible values of λ? Show that every projection P has at least one eigenvector.

   Note that every complex matrix has at least 1 eigenvector, but you need to prove the above for *any* field.

19 Eigenvalues and Eigenvectors II

In Lecture 18, we developed the idea of eigenvalues and eigenvectors in the case of linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$. In this section, we will develop the idea more generally.

Eigenvalues

Definition For a linear transformation $L\colon V \to V$, then λ is an *eigenvalue* of L with *eigenvector* $v \neq 0_V$ if

$$Lv = \lambda v.$$

This equation says that the direction of v is invariant (unchanged) under L.

Let's try to understand this equation better in terms of matrices. Let V be a finite-dimensional vector space and let $L\colon V \to V$. Since we can represent L by a square matrix M, we find eigenvalues λ and associated eigenvectors v by solving the homogeneous system

$$(M - \lambda I)v = 0.$$

This system has non-zero solutions if and only if the matrix

$$M - \lambda I$$

is singular, and so we require that

$$\det(\lambda I - M) = 0.$$

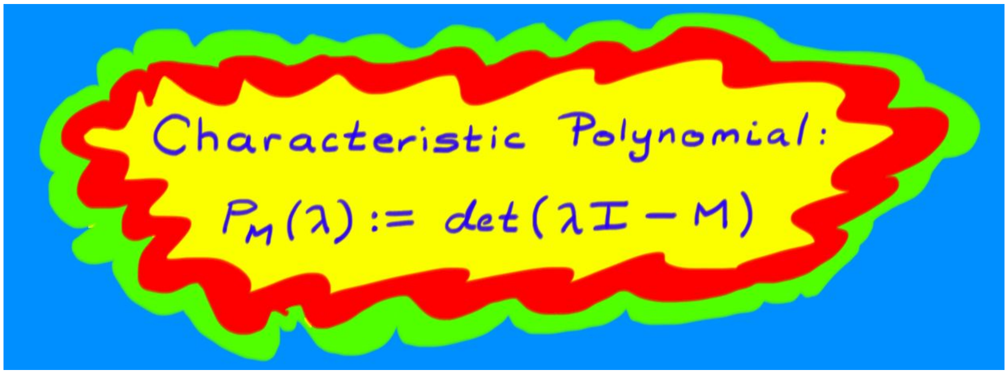
The left hand side of this equation is a polynomial in the variable λ called the *characteristic polynomial* $P_M(\lambda)$ of M. For an $n \times n$ matrix, the characteristic polynomial has degree n. Then

$$P_M(\lambda) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_n.$$

Notice that $P_M(0) = \det(-M) = (-1)^n \det M$.

The *fundamental theorem of algebra* states that any polynomial can be factored into a product of linear terms over $\mathbb{C}$. Then there exists a collection of n complex numbers λ_i (possibly with repetition) such that

$$P_M(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n), \qquad P_M(\lambda_i) = 0$$



The eigenvalues λ_i of M are exactly the roots of $P_M(\lambda)$. These eigenvalues could be real or complex or zero, and they need not all be different. The number of times that any given root λ_i appears in the collection of eigenvalues is called its *multiplicity*.

Example Let L be the linear transformation $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$L(x, y, z) = (2x + y - z, x + 2y - z, -x - y + 2z)\,.$$

The matrix M representing L has columns Le_i for each i, so:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{L}{\mapsto} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Then the characteristic polynomial of L is[13]

$$\begin{aligned} P_M(\lambda) &= \det \begin{pmatrix} \lambda - 2 & -1 & 1 \\ -1 & \lambda - 2 & 1 \\ 1 & 1 & \lambda - 2 \end{pmatrix} \\ &= (\lambda - 2)[(\lambda - 2)^2 - 1] + [-(\lambda - 2) - 1] + [-(\lambda - 2) - 1] \\ &= (\lambda - 1)^2(\lambda - 4) \end{aligned}$$

Then L has eigenvalues $\lambda_1 = 1$ (with multiplicity 2), and $\lambda_2 = 4$ (with multiplicity 1).

To find the eigenvectors associated to each eigenvalue, we solve the homogeneous system $(M - \lambda_i I)X = 0$ for each i.

[13] It is often easier (and equivalent) to solve $\det(M - \lambda I) = 0$.

$\underline{\lambda = 4:}$ We set up the augmented matrix for the linear system:

$$\left(\begin{array}{ccc|c} -2 & 1 & -1 & 0 \\ 1 & -2 & -1 & 0 \\ -1 & -1 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

So we see that $z = t$, $y = -t$, and $x = -t$ gives a formula for eigenvectors in terms of the free parameter t. Any such eigenvector is of the form $t\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$; thus L leaves a line through the origin invariant.

$\underline{\lambda = 1:}$ Again we set up an augmented matrix and find the solution set:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Then the solution set has two free parameters, s and t, such that $z = t$, $y = s$, and $x = -s + t$. Then L leaves invariant the set:

$$\left\{ s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \middle| s, t \in \mathbb{R} \right\}.$$

This set is a plane through the origin. So the multiplicity two eigenvalue has two independent eigenvectors, $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ that determine an invariant plane.

Example Let V be the vector space of smooth (*i.e.* infinitely differentiable) functions $f\colon \mathbb{R} \to \mathbb{R}$. Then the derivative is a linear operator $\frac{d}{dx}\colon V \to V$. What are the eigenvectors of the derivative? In this case, we don't have a matrix to work with, so we have to make do.

A function f is an eigenvector of $\frac{d}{dx}$ if there exists some number λ such that $\frac{d}{dx}f = \lambda f$. An obvious candidate is the exponential function, $e^{\lambda x}$; indeed, $\frac{d}{dx}e^{\lambda x} = \lambda e^{\lambda x}$. As such, the operator $\frac{d}{dx}$ has an eigenvector $e^{\lambda x}$ for every $\lambda \in \mathbb{R}$.

This is actually the whole collection of eigenvectors for $\frac{d}{dx}$; this can be proved using the fact that every infinitely differentiable function has a Taylor series with infinite radius of convergence, and then using the Taylor series to show that if two functions are eigenvectors of $\frac{d}{dx}$ with eigenvalues λ, then they are scalar multiples of each other.

19.1 Eigenspaces

In the previous example, we found two eigenvectors $\begin{pmatrix}-1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\0\\1\end{pmatrix}$ for L with eigenvalue 1. Notice that $\begin{pmatrix}-1\\1\\0\end{pmatrix}+\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}0\\1\\1\end{pmatrix}$ is also an eigenvector of L with eigenvalue 1. In fact, any linear combination $r\begin{pmatrix}-1\\1\\0\end{pmatrix}+s\begin{pmatrix}1\\0\\1\end{pmatrix}$ of these two eigenvectors will be another eigenvector with the same eigenvalue.

More generally, let $\{v_1, v_2, \ldots\}$ be eigenvectors of some linear transformation L with the same eigenvalue λ. A *linear combination* of the v_i can be written $c_1v_1 + c_2v_2 + \cdots$ for some constants $\{c_1, c_2, \ldots\}$. Then:

$$\begin{aligned} L(c_1v_1 + c_2v_2 + \cdots) &= c_1Lv_1 + c_2Lv_2 + \cdots \text{ by linearity of } L \\ &= c_1\lambda v_1 + c_2\lambda v_2 + \cdots \text{ since } Lv_i = \lambda v_i \\ &= \lambda(c_1v_1 + c_2v_2 + \cdots). \end{aligned}$$

So every linear combination of the v_i is an eigenvector of L with the same eigenvalue λ. In simple terms, any sum of eigenvectors is again an eigenvector *if they share the same eigenvalue.*

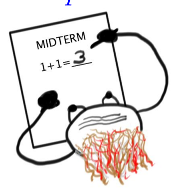
The space of all vectors with eigenvalue λ is called an *eigenspace*. It is, in fact, a vector space contained within the larger vector space V: It contains 0_V, since $L0_V = 0_V = \lambda 0_V$, and is closed under addition and scalar multiplication by the above calculation. All other vector space properties are inherited from the fact that V itself is a vector space.

An eigenspace is an example of a *subspace* of V, a notion explored in Lecture 15.

More on eigenspaces

 Reading homework: problem 19.1

You are now ready to attempt the second sample midterm.



References

Hefferon, Chapter Three, Section III.1: Representing Linear Maps with Matrices
Hefferon, Chapter Five, Section II.3: Eigenvalues and Eigenvectors
Beezer, Chapter E, Section EE
Wikipedia:

- Eigen*
- Characteristic Polynomial
- Linear Transformations (and matrices thereof)

Review Problems

1. Explain why the characteristic polynomial of an $n \times n$ matrix has degree n. Make your explanation easy to read by starting with some simple examples, and then use properties of the determinant to give a *general* explanation.

2. Compute the characteristic polynomial $P_M(\lambda)$ of the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Now, since we can evaluate polynomials on square matrices, we can plug M into its characteristic polynomial and find the *matrix* $P_M(M)$. What do you find from this computation? Does something similar hold for 3×3 matrices? What about $n \times n$ matrices?

3. *Discrete dynamical system.* Let M be the matrix given by

$$M = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}.$$

Given any vector $v(0) = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$, we can create an infinite sequence of vectors $v(1), v(2), v(3)$, and so on using the rule

$$v(t+1) = Mv(t) \text{ for all natural numbers } t.$$

(This is known as a *discrete dynamical system* whose *initial condition* is $v(0)$.)

(a) Find all eigenvectors and eigenvalues of M.

(b) Find all vectors $v(0)$ such that

$$v(0) = v(1) = v(2) = v(3) = \cdots$$

(Such a vector is known as a *fixed point* of the dynamical system.)

(c) Find all vectors $v(0)$ such that $v(0), v(1), v(2), v(3), \ldots$ all point in the same direction. (Any such vector describes an *invariant curve* of the dynamical system.)

Hint

20 Diagonalization

Given a linear transformation, we are interested in how to write it as a matrix. We are especially interested in the case that the matrix is written with respect to a basis of eigenvectors, in which case it is a particularly nice matrix.

20.1 Diagonalization

Now suppose we are lucky, and we have $L\colon V \mapsto V$, and the basis $\{v_1, \ldots, v_n\}$ is a set of linearly independent eigenvectors for L, with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then:

$$
\begin{array}{rcl}
L(v_1) & = & \lambda_1 v_1 \\
L(v_2) & = & \lambda_2 v_2 \\
& \vdots & \\
L(v_n) & = & \lambda_n v_n
\end{array}
$$

As a result, the matrix of L in the basis of eigenvectors is diagonal:

$$
\begin{pmatrix}
\lambda_1 & & & \\
& \lambda_2 & & \\
& & \ddots & \\
& & & \lambda_n
\end{pmatrix},
$$

where all entries off of the diagonal are zero.

Suppose that V is any n-dimensional vector space. We call a linear transformation $L\colon V \mapsto V$ *diagonalizable* if there exists a collection of n linearly independent eigenvectors for L. In other words, L is diagonalizable if there exists a basis for V of eigenvectors for L.

In a basis of eigenvectors, the matrix of a linear transformation is diagonal. On the other hand, if an $n \times n$ matrix is diagonal, then the standard basis vectors e_i must already be a set of n linearly independent eigenvectors. We have shown:

Theorem 20.1. *Given a basis S for a vector space V and a linear transformation $L\colon V \to V$, then the matrix for L in the basis S is diagonal if and only if S is a basis of eigenvectors for L.*

`Non-diagonalizable example`

Reading homework: problem 20.2

20.2 Change of Basis

Suppose we have two bases $S = \{v_1, \ldots, v_n\}$ and $T = \{u_1, \ldots, u_n\}$ for a vector space V. (Here v_i and u_i are *vectors*, not components of vectors in a basis!) Then we may write each v_i uniquely as a linear combination of the u_j:

$$v_j = \sum_i u_i P^i_j \,,$$

or in a matrix notation

$$\begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \end{pmatrix} \begin{pmatrix} P^1_1 & P^1_2 & \cdots & P^1_n \\ P^2_1 & P^2_2 & & \\ \vdots & & & \vdots \\ P^n_1 & & \cdots & P^n_n \end{pmatrix} .$$

Here, the P^i_j are constants, which we can regard as entries of a square matrix $P = (P^i_j)$. The matrix P must have an inverse, since we can also write each u_i uniquely as a linear combination of the v_j:

$$u_j = \sum_k v_k Q^k_j .$$

Then we can write:

$$v_j = \sum_k \sum_i v_k Q^k_j P^i_j .$$

But $\sum_i Q^k_j P^i_j$ is the k, j entry of the product of the matrices QP. Since the only expression for v_j in the basis S is v_j itself, then QP fixes each v_j. As a result, each v_j is an eigenvector for QP with eigenvalue 1, so QP is the identity.

The matrix P is called a *change of basis* matrix. There is a quick and dirty trick to obtain it: Look at the formula above relating the new basis vectors $v_1, v_2, \ldots v_n$ to the old ones $u_1, u_2, \ldots, u_n$. In particular focus on v_1 for which

$$v_1 = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \end{pmatrix} \begin{pmatrix} P_1^1 \\ P_1^2 \\ \vdots \\ P_1^n \end{pmatrix} .$$

This says that the first column of the change of basis matrix P is really just the components of the vector v_1 in the basis $u_1, u_2, \ldots, u_n$.

Example Suppose the vectors v_1 and v_2 form a basis for a vector space V and with respect to some other basis u_1, u_2 have, respectively, components

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix} .$$

What is the change of basis matrix P from the old basis u_1, u_2 to the new basis v_1, v_2?

Before answering note that the above statements mean

$$v_1 = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{u_1 + u_2}{\sqrt{2}} \quad \text{and} \quad v_2 = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix} = \frac{u_1 - u_2}{\sqrt{3}} .$$

The change of basis matrix has as its columns just the components of v_1 and v_2, so is just

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{pmatrix} .$$

Changing basis changes the matrix of a linear transformation. However, as a map between vector spaces, the linear transformation is the same no matter which basis we use. Linear transformations are the actual objects of study of this course, not matrices; matrices are merely a convenient way of doing computations.

`Worked Change of Basis Example`

Lets now apply this to our eigenvector problem. To wit, suppose $L\colon V \mapsto V$ has matrix $M = (M^i_j)$ in the basis $T = \{u_1, \ldots, u_n\}$, so

$$L(u_i) = \sum_k u_k M^k_i.$$

Now, let $S = \{v_1, \ldots, v_n\}$ be a basis of eigenvectors for L, with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then

$$L(v_i) = \lambda_i v_i = \sum_k v_k D^k_i$$

where D is the diagonal matrix whose diagonal entries D^k_k are the eigenvalues λ_k; ie, $D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$. Let P be the change of basis matrix from the basis T to the basis S. Then:

$$L(v_j) = L\left(\sum_i u_i P^i_j\right) = \sum_i L(u_i) P^i_j = \sum_i \sum_k u_k M^k_i P^i_j.$$

Meanwhile, we have:

$$L(v_i) = \sum_k v_k D^k_i = \sum_k \sum_j u_j P^j_k D^k_i.$$

Since the expression for a vector in a basis is unique, then we see that the entries of MP are the same as the entries of PD. In other words, we see that

$$MP = PD \qquad \text{or} \qquad D = P^{-1}MP.$$

This motivates the following definition:

Definition A matrix M is *diagonalizable* if there exists an invertible matrix P and a diagonal matrix D such that

$$D = P^{-1}MP.$$

We can summarize as follows:

- Change of basis multiplies vectors by the change of basis matrix P, to give vectors in the new basis.

- To get the matrix of a linear transformation in the new basis, we *conjugate* the matrix of L by the change of basis matrix: $M \to P^{-1}MP$.

If for two matrices N and M there exists an invertible matrix P such that $M = P^{-1}NP$, then we say that M and N are *similar*. Then the above discussion shows that diagonalizable matrices are similar to diagonal matrices.

Corollary 20.2. *A square matrix M is diagonalizable if and only if there exists a basis of eigenvectors for M. Moreover, these eigenvectors are the columns of the change of basis matrix P which diagonalizes M.*

Reading homework: problem 20.3

Example Let's try to diagonalize the matrix

$$M = \begin{pmatrix} -14 & -28 & -44 \\ -7 & -14 & -23 \\ 9 & 18 & 29 \end{pmatrix}.$$

The eigenvalues of M are determined by

$$\det(M - \lambda) = -\lambda^3 + \lambda^2 + 2\lambda = 0.$$

So the eigenvalues of M are $-1, 0,$ and 2, and associated eigenvectors turn out to be $v_1 = \begin{pmatrix} -8 \\ -1 \\ 3 \end{pmatrix}, v_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, and $v_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$. In order for M to be diagonalizable, we need the vectors v_1, v_2, v_3 to be linearly independent. Notice that the matrix

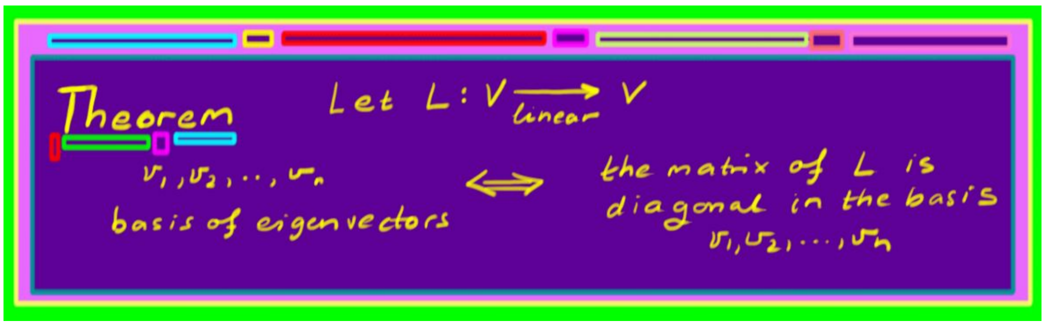
$$P = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} -8 & -2 & -1 \\ -1 & 1 & -1 \\ 3 & 0 & 1 \end{pmatrix}$$

is invertible because its determinant is -1. Therefore, the eigenvectors of M form a basis of $\mathbb{R}$, and so M is diagonalizable. Moreover, the matrix P of eigenvectors is a change of basis matrix which diagonalizes M:

$$P^{-1}MP = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

2×2 Example

As a reminder, here is the key result of this Lecture



References

Hefferon, Chapter Three, Section V: Change of Basis
Beezer, Chapter E, Section SD
Beezer, Chapter R, Sections MR-CB
Wikipedia:

- Change of Basis
- Diagonalizable Matrix
- Similar Matrix

Review Problems

1. Let $P_n(t)$ be the vector space of polynomials of degree n or less, and $\frac{d}{dt} : P_n(t) \mapsto P_{n-1}(t)$ be the derivative operator. Find the matrix of $\frac{d}{dt}$ in the bases $\{1, t, \dots, t^n\}$ for $P_n(t)$ and $\{1, t, \dots, t^{n-1}\}$ for $P_{n-1}(t)$.

   Recall that the derivative operator is linear from Chapter 7.

2. When writing a matrix for a linear transformation, we have seen that the choice of basis matters. In fact, even the order of the basis matters!

- Write all possible reorderings of the standard basis $\{e_1, e_2, e_3\}$ for $\mathbb{R}^3$.
- Write each change of basis matrix between the standard basis $\{e_1, e_2, e_3\}$ and each of its reorderings. Make as many observations as you can about these matrices: what are their entries? Do you notice anything about how many of each type of entry appears in each row and column? What are their determinants? (Note: These matrices are known as *permutation matrices*.)
- Given the linear transformation $L(x, y, z) = (2y - z, 3x, 2z + x + y)$, write the matrix M for L in the standard basis, and two other reorderings of the standard basis. How are these matrices related?

3. When is the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ diagonalizable? Include examples in your answer.

4. Show that similarity of matrices is an *equivalence relation*. (The definition of an equivalence relation is given in Homework 0.)

5. *Jordan form*

   - Can the matrix $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ be diagonalized? Either diagonalize it or explain why this is impossible.
   - Can the matrix $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$ be diagonalized? Either diagonalize it or explain why this is impossible.
   - Can the $n \times n$ matrix $\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$ be diagonalized? Either diagonalize it or explain why this is impossible.

     Note: It turns out that every matrix is similar to a block matrix whose diagonal blocks look like diagonal matrices or the ones above and whose off-diagonal blocks are all zero. This is called the *Jordan form* of the matrix and a (maximal) block that look like

$$\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

is called a *Jordan n-cell* or a *Jordan block* where n is the size of the block.

6. Let A and B be commuting matrices (i.e. $AB = BA$) and suppose that A has an eigenvector v with eigenvalue λ. Show that Bv also has an eigenvalue of λ. Additionally suppose that A is diagonalizable with distinct eigenvalues. Show that v is also an eigenvector of B, and thus showing A and B can be *simultaneously diagonalized* (*i.e.* they have the same eigenvalues and eigenvectors).

21 Orthonormal Bases

You may have noticed that we have only rarely used the dot product. That is because many of the results we have obtained do not require a preferred notion of lengths of vectors. Now let us consider the case of $\mathbb{R}^n$ where the length of a vector $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is $\sqrt{(x_1)^2 + (x_2)^2 + \cdots (x_n)^2}$.

The canonical/standard basis in $\mathbb{R}^n$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

has many useful properties.

- Each of the standard basis vectors has unit length:

$$\|e_i\| = \sqrt{e_i \cdot e_i} = \sqrt{e_i^T e_i} = 1.$$

- The standard basis vectors are *orthogonal* (in other words, at right angles or perpendicular).

$$e_i \cdot e_j = e_i^T e_j = 0 \text{ when } i \neq j$$

This is summarized by

$$e_i^T e_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases},$$

where δ_{ij} is the *Kronecker delta*. Notice that the Kronecker delta gives the entries of the identity matrix.

Given column vectors v and w, we have seen that the dot product $v \cdot w$ is the same as the matrix multiplication $v^T w$. This is the *inner product* on $\mathbb{R}^n$. We can also form the *outer product* vw^T, which gives a square matrix.

The outer product on the standard basis vectors is interesting. Set

$$\begin{aligned}
\Pi_1 &= e_1 e_1^T \\
&= \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \\
&\vdots \\
\Pi_n &= e_n e_n^T \\
&= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}
\end{aligned}$$

In short, Π_i is the diagonal square matrix with a 1 in the ith diagonal position and zeros everywhere else. [14]

Notice that $\Pi_i \Pi_j = e_i e_i^T e_j e_j^T = e_i \delta_{ij} e_j^T$. Then:

$$\Pi_i \Pi_j = \begin{cases} \Pi_i & i = j \\ 0 & i \neq j \end{cases} .$$

Moreover, for a diagonal matrix D with diagonal entries $\lambda_1, \ldots, \lambda_n$, we can write

$$D = \lambda_1 \Pi_1 + \cdots + \lambda_n \Pi_n .$$

[14] This is reminiscent of an older notation, where vectors are written in juxtaposition. This is called a "dyadic tensor", and is still used in some applications.

Other bases that share these properties should behave in many of the same ways as the standard basis. As such, we will study:

- *Orthogonal bases* $\{v_1, \ldots, v_n\}$:

$$v_i \cdot v_j = 0 \text{ if } i \neq j$$

  In other words, all vectors in the basis are perpendicular.

- *Orthonormal bases* $\{u_1, \ldots, u_n\}$:

$$u_i \cdot u_j = \delta_{ij}.$$

  In addition to being orthogonal, each vector has unit length.

Suppose $T = \{u_1, \ldots, u_n\}$ is an orthonormal basis for $\mathbb{R}^n$. Since T is a basis, we can write any vector v uniquely as a linear combination of the vectors in T:

$$v = c^1 u_1 + \cdots c^n u_n.$$

Since T is orthonormal, there is a very easy way to find the coefficients of this linear combination. By taking the dot product of v with any of the vectors in T, we get:

$$\begin{array}{rcl}
v \cdot u_i &=& c^1 u_1 \cdot u_i + \cdots + c^i u_i \cdot u_i + \cdots + c^n u_n \cdot u_i \\
&=& c^1 \cdot 0 + \cdots + c^i \cdot 1 + \cdots + c^n \cdot 0 \\
&=& c^i, \\
\Rightarrow c^i &=& v \cdot u_i \\
\Rightarrow v &=& (v \cdot u_1) u_1 + \cdots + (v \cdot u_n) u_n \\
&=& \displaystyle\sum_i (v \cdot u_i) u_i.
\end{array}$$

This proves the theorem:

Theorem 21.1. *For an orthonormal basis $\{u_1, \ldots, u_n\}$, any vector v can be expressed as*

$$v = \sum_i (v \cdot u_i) u_i.$$

Reading homework: problem 21.1

`All orthonormal bases for` $\mathbb{R}^2$

21.1 Relating Orthonormal Bases

Suppose $T = \{u_1, \dots, u_n\}$ and $R = \{w_1, \dots, w_n\}$ are two orthonormal bases for $\mathbb{R}^n$. Then:

$$\begin{aligned}
w_1 &= (w_1 \cdot u_1)u_1 + \cdots + (w_1 \cdot u_n)u_n \\
&\vdots \\
w_n &= (w_n \cdot u_1)u_1 + \cdots + (w_n \cdot u_n)u_n \\
\Rightarrow w_i &= \sum_j u_j (u_j \cdot w_i)
\end{aligned}$$

As such, the matrix for the change of basis from T to R is given by

$$P = (P_i^j) = (u_j \cdot w_i).$$

Consider the product PP^T in this case.

$$\begin{aligned}
(PP^T)_k^j &= \sum_i (u_j \cdot w_i)(w_i \cdot u_k) \\
&= \sum_i (u_j^T w_i)(w_i^T u_k) \\
&= u_j^T \left[\sum_i (w_i w_i^T) \right] u_k \\
&= u_j^T I_n u_k \qquad (*) \\
&= u_j^T u_k = \delta_{jk}.
\end{aligned}$$

The equality $(*)$ is explained below. So assuming $(*)$ holds, we have shown that $PP^T = I_n$, which implies that

$$P^T = P^{-1}.$$

The equality in the line $(*)$ says that $\sum_i w_i w_i^T = I_n$. To see this, we examine $\left(\sum_i w_i w_i^T \right) v$ for an arbitrary vector v. We can find constants c^j

such that $v = \sum_j c^j w_j$, so that:

$$\begin{aligned}
\left(\sum_i w_i w_i^T\right) v &= \left(\sum_i w_i w_i^T\right)\left(\sum_j c^j w_j\right) \\
&= \sum_j c^j \sum_i w_i w_i^T w_j \\
&= \sum_j c^j \sum_i w_i \delta_{ij} \\
&= \sum_j c^j w_j \text{ since all terms with } i \neq j \text{ vanish} \\
&= v.
\end{aligned}$$

Then as a linear transformation, $\sum_i w_i w_i^T = I_n$ fixes every vector, and thus must be the identity I_n.

Definition A matrix P is *orthogonal* if $P^{-1} = P^T$.

Then to summarize,

Theorem 21.2. *A change of basis matrix P relating two orthonormal bases is an orthogonal matrix. I.e.,*

$$P^{-1} = P^T.$$

Reading homework: problem 21.2

Example Consider $\mathbb{R}^3$ with the orthonormal basis

$$S = \left\{ u_1 = \begin{pmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, u_3 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \right\}.$$

Let R be the standard basis $\{e_1, e_2, e_3\}$. Since we are changing from the standard basis to a new basis, then the columns of the change of basis matrix are exactly the images of the standard basis vectors. Then the change of basis matrix from R to S is given by:

$$
\begin{aligned}
P = (P_i^j) = (e_j u_i) &= \begin{pmatrix} e_1 \cdot u_1 & e_1 \cdot u_2 & e_1 \cdot u_3 \\ e_2 \cdot u_1 & e_2 \cdot u_2 & e_2 \cdot u_3 \\ e_3 \cdot u_1 & e_3 \cdot u_2 & e_3 \cdot u_3 \end{pmatrix} \\
= \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} &= \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}.
\end{aligned}
$$

From our theorem, we observe that:

$$
\begin{aligned}
P^{-1} = P^T &= \begin{pmatrix} u_1^T \\ u_2^T \\ u_3^T \end{pmatrix} \\
&= \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}.
\end{aligned}
$$

We can check that $P^TP = I$ by a lengthy computation, or more simply, notice that

$$
\begin{aligned}
(P^TP)_{ij} &= \begin{pmatrix} u_1^T \\ u_2^T \\ u_3^T \end{pmatrix} \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

We are using orthonormality of the u_i for the matrix multiplication above. It is very important to realize that the columns of an $orthogonal$ matrix are made from an $orthonormal$ set of vectors.

Orthonormal Change of Basis and Diagonal Matrices. Suppose D is a diagonal matrix, and we use an orthogonal matrix P to change to a new basis. Then the matrix M of D in the new basis is:

$$M = PDP^{-1} = PDP^T.$$

Now we calculate the transpose of M.

$$\begin{aligned} M^T &= (PDP^T)^T \\ &= (P^T)^T D^T P^T \\ &= PDP^T \\ &= M \end{aligned}$$

So we see the matrix PDP^T is symmetric!

References

Hefferon, Chapter Three, Section V: Change of Basis
Beezer, Chapter V, Section O, Subsection N
Beezer, Chapter VS, Section B, Subsection OBC
Wikipedia:

- Orthogonal Matrix
- Diagonalizable Matrix
- Similar Matrix

Review Problems

1. Let $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

   (a) Write D in terms of the vectors e_1 and e_2, and their transposes.

   (b) Suppose $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible. Show that D is similar to

   $$M = \frac{1}{ad-bc} \begin{pmatrix} \lambda_1 ad - \lambda_2 bc & -(\lambda_1 - \lambda_2)ab \\ (\lambda_1 - \lambda_2)cd & -\lambda_1 bc + \lambda_2 ad \end{pmatrix}.$$

   (c) Suppose the vectors $\begin{pmatrix} a & b \end{pmatrix}$ and $\begin{pmatrix} c & d \end{pmatrix}$ are orthogonal. What can you say about M in this case? (Hint: think about what M^T is equal to.)

2. Suppose $S = \{v_1, \ldots, v_n\}$ is an *orthogonal* (not orthonormal) basis for $\mathbb{R}^n$. Then we can write any vector v as $v = \sum_i c^i v_i$ for some constants c^i. Find a formula for the constants c^i in terms of v and the vectors in S.

Hint for 2

3. Let u, v be independent vectors in $\mathbb{R}^3$, and $P = \operatorname{span}\{u, v\}$ be the plane spanned by u and v.

   (a) Is the vector $v^\perp = v - \frac{u \cdot v}{u \cdot u} u$ in the plane P?

   (b) What is the angle between $v^\perp$ and u?

   (c) Given your solution to the above, how can you find a third vector perpendicular to both u and $v^\perp$?

   (d) Construct an orthonormal basis for $\mathbb{R}^3$ from u and v.

   (e) Test your abstract formulae starting with

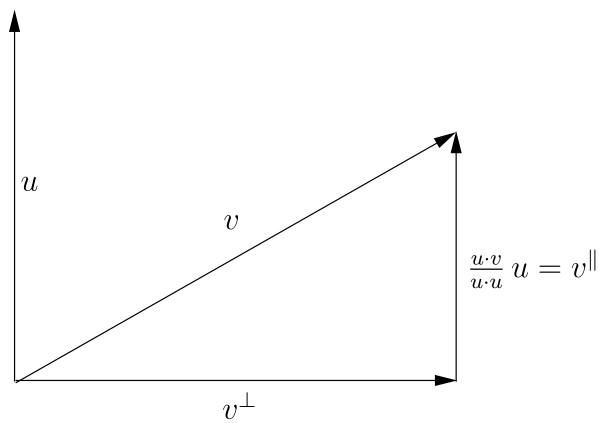
$$u = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix} \text{ and } v = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}.$$

Hint for 3

22 Gram-Schmidt and Orthogonal Complements

Given a vector u and some other vector v not in the span of u, we can construct a new vector:

$$v^{\perp} = v - \frac{u \cdot v}{u \cdot u} u.$$



This new vector $v^{\perp}$ is orthogonal to u because

$$u \cdot v^{\perp} = u \cdot v - \frac{u \cdot v}{u \cdot u} u \cdot u = 0.$$

Hence, $\{u, v^{\perp}\}$ is an orthogonal basis for $\operatorname{span}\{u, v\}$. When v is not parallel to u, $v^{\perp} \neq 0$, and normalizing these vectors we obtain $\left\{ \frac{u}{|u|}, \frac{v^{\perp}}{|v^{\perp}|} \right\}$, an orthonormal basis.

Sometimes we write $v = v^{\perp} + v^{\parallel}$ where:

$$\begin{aligned} v^{\perp} &= v - \frac{u \cdot v}{u \cdot u} u \\ v^{\parallel} &= \frac{u \cdot v}{u \cdot u} u. \end{aligned}$$

This is called an *orthogonal decomposition* because we have decomposed v into a sum of orthogonal vectors. It is significant that we wrote this decomposition with u in mind; $v^{\parallel}$ is parallel to u.

If u, v are linearly independent vectors in $\mathbb{R}^3$, then the set $\{u, v^{\perp}, u \times v^{\perp}\}$ would be an orthogonal basis for $\mathbb{R}^3$. This set could then be normalized by dividing each vector by its length to obtain an orthonormal basis.

However, it often occurs that we are interested in vector spaces with dimension greater than 3, and must resort to craftier means than cross products to obtain an orthogonal basis. [15]

Given a third vector w, we should first check that w does not lie in the span of u and v, *i.e.* check that u, v and w are linearly independent. We then can define:

$$w^{\perp} = w - \frac{u \cdot w}{u \cdot u}\, u - \frac{v^{\perp} \cdot w}{v^{\perp} \cdot v^{\perp}}\, v^{\perp}.$$

We can check that $u \cdot w^{\perp}$ and $v^{\perp} \cdot w^{\perp}$ are both zero:

$$\begin{aligned} u \cdot w^{\perp} &= u \cdot \left(w - \frac{u \cdot w}{u \cdot u}\, u - \frac{v^{\perp} \cdot w}{v^{\perp} \cdot v^{\perp}}\, v^{\perp} \right) \\ &= u \cdot w - \frac{u \cdot w}{u \cdot u} u \cdot u - \frac{v^{\perp} \cdot w}{v^{\perp} \cdot v^{\perp}} u \cdot v^{\perp} \\ &= u \cdot w - u \cdot w - \frac{v^{\perp} \cdot w}{v^{\perp} \cdot v^{\perp}} u \cdot v^{\perp} \ = \ 0 \end{aligned}$$

since u is orthogonal to $v^{\perp}$, and

$$\begin{aligned} v^{\perp} \cdot w^{\perp} &= v^{\perp} \cdot \left(w - \frac{u \cdot w}{u \cdot u}\, u - \frac{v^{\perp} \cdot w}{v^{\perp} \cdot v^{\perp}}\, v^{\perp} \right) \\ &= v^{\perp} \cdot w - \frac{u \cdot w}{u \cdot u} v^{\perp} \cdot u - \frac{v^{\perp} \cdot w}{v^{\perp} \cdot v^{\perp}} v^{\perp} \cdot v^{\perp} \\ &= v^{\perp} \cdot w - \frac{u \cdot w}{u \cdot u} v^{\perp} \cdot u - v^{\perp} \cdot w \ = \ 0 \end{aligned}$$

because u is orthogonal to $v^{\perp}$. Since $w^{\perp}$ is orthogonal to both u and $v^{\perp}$, we have that $\{u, v^{\perp}, w^{\perp}\}$ is an orthogonal basis for $\operatorname{span}\{u, v, w\}$.

In fact, given a collection $\{x, v_2, \ldots\}$ of linearly independent vectors, we can produce an orthogonal basis for $\operatorname{span}\{v_1, v_2, \ldots\}$ consisting of the follow-

[15] Actually, given a set T of $(n-1)$ independent vectors in n-space, one can define an analogue of the cross product that will produce a vector orthogonal to the span of T, using a method exactly analogous to the usual computation for calculating the cross product of two vectors in $\mathbb{R}^3$. This only gets us the *last* orthogonal vector, though; the process in this Section gives a way to get a full orthogonal basis.

ing vectors:

$$
\begin{aligned}
v_1^{\perp} &= v_1 \\
v_2^{\perp} &= v_2 - \frac{v_1^{\perp} \cdot v_2}{v_1^{\perp} \cdot v_1^{\perp}}\, v_1^{\perp} \\
v_3^{\perp} &= v_3 - \frac{v_1^{\perp} \cdot v_3}{v_1^{\perp} \cdot v_1^{\perp}}\, v_1^{\perp} - \frac{v_2^{\perp} \cdot v_3}{v_2^{\perp} \cdot v_2^{\perp}}\, v_2^{\perp} \\
&\vdots \\
v_i^{\perp} &= v_i - \sum_{j<i} \frac{v_j^{\perp} \cdot v_i}{v_j^{\perp} \cdot v_j^{\perp}}\, v_j^{\perp} \\
&= v_i - \frac{v_1^{\perp} \cdot v_i}{v_1^{\perp} \cdot v_1^{\perp}}\, v_1^{\perp} - \cdots - \frac{v_{n-1}^{\perp} \cdot v_i}{v_{n-1}^{\perp} \cdot v_{n-1}^{\perp}}\, v_{n-1}^{\perp} \\
&\vdots
\end{aligned}
$$

Notice that each $v_i^{\perp}$ here depends on the existence of $v_j^{\perp}$ for every $j < i$. This allows us to inductively/algorithmically build up a linearly independent, orthogonal set of vectors whose span is $\operatorname{span}\{v_1, v_2, \ldots\}$. This algorithm bears the name *Gram–Schmidt orthogonalization procedure*.

Example Let $u = \begin{pmatrix}1 & 1 & 0\end{pmatrix}, v = \begin{pmatrix}1 & 1 & 1\end{pmatrix}$, and $w = \begin{pmatrix}3 & 1 & 1\end{pmatrix}$. We'll apply Gram-Schmidt to obtain an orthogonal basis for $\mathbb{R}^3$.

First, we set $u^{\perp} = u$. Then:

$$
\begin{aligned}
v^{\perp} &= \begin{pmatrix}1 & 1 & 1\end{pmatrix} - \frac{2}{2}\begin{pmatrix}1 & 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & 1\end{pmatrix} \\
w^{\perp} &= \begin{pmatrix}3 & 1 & 1\end{pmatrix} - \frac{4}{2}\begin{pmatrix}1 & 1 & 0\end{pmatrix} - \frac{1}{1}\begin{pmatrix}0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & -1 & 0\end{pmatrix}.
\end{aligned}
$$

Then the set

$$\left\{\begin{pmatrix}1 & 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & -1 & 0\end{pmatrix}\right\}$$

is an orthogonal basis for $\mathbb{R}^3$. To obtain an orthonormal basis, as always we simply divide each of these vectors by its length, yielding:

$$\left\{\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 1\end{pmatrix}, \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0\end{pmatrix}\right\}.$$

A 4×4 Gram Schmidt Example

In Lecture 11 we learned how to solve linear systems by decomposing a matrix M into a product of lower and upper triangular matrices

$$M = LU\,.$$

The Gram–Schmidt procedure suggests another matrix decomposition,

$$M = QR$$

where Q is an orthogonal matrix and R is an upper triangular matrix. So-called QR-decompositions are useful for solving linear systems, eigenvalue problems and least squares approximations. You can easily get the idea behind QR decomposition by working through a simple example.

Example Find the QR decomposition of

$$M = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 3 & -2 \\ 0 & 1 & -2 \end{pmatrix}\,.$$

What we will do is to think of the columns of M as three vectors and use Gram–Schmidt to build an orthonormal basis from these that will become the columns of the orthogonal matrix Q. We will use the matrix R to record the steps of the Gram–Schmidt procedure in such a way that the product QR equals M.

To begin with we write

$$M = \begin{pmatrix} 2 & -\frac{7}{5} & 1 \\ 1 & \frac{14}{5} & -2 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{5} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\,.$$

In the first matrix the first two columns are mutually orthogonal because we simpy replaced the second column of M by the vector that the Gram–Schmidt procedure produces from the first two columns of M, namely

$$\begin{pmatrix} -\frac{7}{5} \\ \frac{14}{5} \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}\,.$$

The matrix on the right is almost the identity matrix, save the $+\frac{1}{5}$ in the second entry of the first row, whose effect upon multiplying the two matrices precisely undoes what we we did to the second column of the first matrix.

For the third column of M we use Gram–Schmidt to deduce the third orthogonal vector

$$\begin{pmatrix}-\frac{1}{6}\\ \frac{1}{3}\\ -\frac{7}{6}\end{pmatrix} = \begin{pmatrix}1\\-2\\-2\end{pmatrix} - 0.\begin{pmatrix}2\\1\\0\end{pmatrix} - \frac{-9}{\frac{54}{5}}\begin{pmatrix}-\frac{7}{5}\\ \frac{14}{5}\\ 1\end{pmatrix},$$

and therefore, using exactly the same procedure write

$$M = \begin{pmatrix}2 & -\frac{7}{5} & -\frac{1}{6}\\ 1 & \frac{14}{5} & \frac{1}{3}\\ 0 & 1 & -\frac{7}{6}\end{pmatrix}\begin{pmatrix}1 & \frac{1}{5} & 0\\ 0 & 1 & -\frac{5}{6}\\ 0 & 0 & 1\end{pmatrix}.$$

This is not quite the answer because the first matrix is now made of mutually orthogonal column vectors, but a *bona fide* orthogonal matrix is comprised of *orthonormal* vectors. To achieve that we divide each column of the first matrix by its length and multiply the corresponding row of the second matrix by the same amount:

$$M = \begin{pmatrix}\frac{2\sqrt{5}}{5} & -\frac{7\sqrt{30}}{90} & -\frac{\sqrt{6}}{18}\\ \frac{\sqrt{5}}{5} & \frac{7\sqrt{30}}{45} & \frac{\sqrt{6}}{9}\\ 0 & \frac{\sqrt{30}}{18} & -\frac{7\sqrt{6}}{18}\end{pmatrix}\begin{pmatrix}\sqrt{5} & \frac{\sqrt{5}}{5} & 0\\ 0 & \frac{3\sqrt{30}}{5} & -\frac{\sqrt{30}}{2}\\ 0 & 0 & \frac{\sqrt{6}}{2}\end{pmatrix} = QR\,.$$

A nice check of this result is to verify that entry (i,j) of the matrix R equals the dot product of the i-th column of Q with the j-th column of M. (Some people memorize this fact and use it as a recipe for computing QR deompositions.) *A good test of your own understanding is to work out why this is true!*

Another QR decomposition example

22.1 Orthogonal Complements

Let U and V be subspaces of a vector space W. We saw as a review exercise that $U \cap V$ is a subspace of W, and that $U \cup V$ was not a subspace. However, $\text{span}(U \cup V)$ is certainly a subspace, since the span of *any* subset is a subspace.

Notice that all elements of $\text{span}(U \cup V)$ take the form $u + v$ with $u \in U$ and $v \in V$. We call the subspace

$$U + V = \text{span}(U \cup V) = \{u + v | u \in U, v \in V\}$$

the *sum* of U and V. Here, we are not adding vectors, but vector spaces to produce a new vector space!

Definition Given two subspaces U and V of a space W such that $U \cap V = \{0_W\}$, the *direct sum* of U and V is defined as:

$$U \oplus V = \operatorname{span}(U \cup V) = \{u + v | u \in U, v \in V\}.$$

Notice that when $U \cap V = \{0_W\}$, $U + V = U \oplus V$.

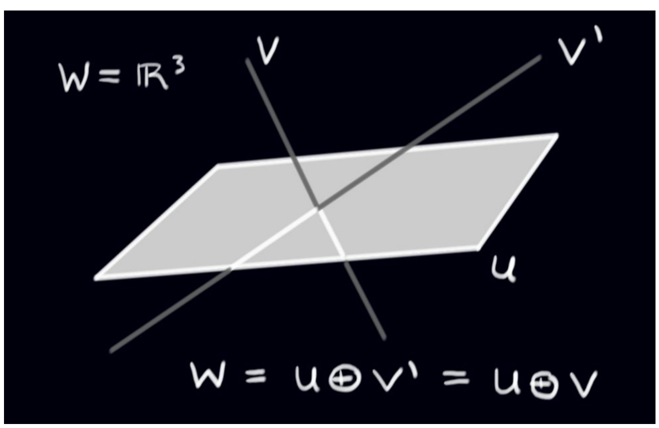
The direct sum has a very nice property.

Theorem 22.1. *Let $w = u + v \in U \oplus V$. Then the expression $w = u + v$ is unique. That is, there is only one way to write w as the sum of a vector in U and a vector in V.*

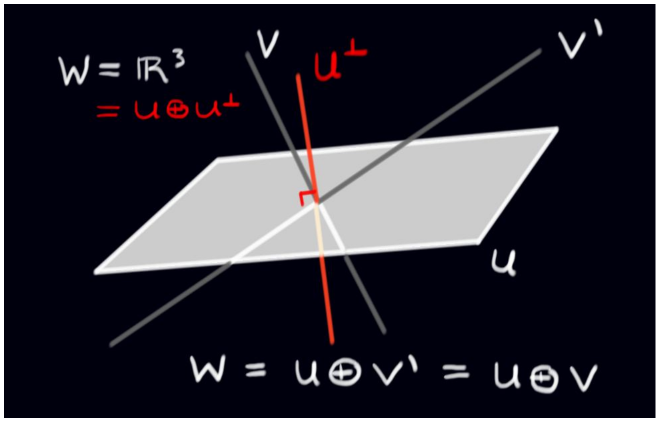
Proof. Suppose that $u + v = u' + v'$, with $u, u' \in U$, and $v, v' \in V$. Then we could express $0 = (u - u') + (v - v')$. Then $(u - u') = -(v - v')$. Since U and V are subspaces, we have $(u - u') \in U$ and $-(v - v') \in V$. But since these elements are equal, we also have $(u - u') \in V$. Since $U \cap V = \{0\}$, then $(u - u') = 0$. Similarly, $(v - v') = 0$, proving the theorem. $\square$

Reading homework: problem 22.1

Given a subspace U in W, we would like to write W as the direct sum of U and *something*. There is not a unique answer to this question as can be seen from this picture of subspaces in $W = \mathbb{R}^3$:



However, using the inner product, there is a natural candidate $U^\perp$ for this second subspace as shown here:



The general definition is as follows:

Definition Given a subspace U of a vector space W, define:

$$U^{\perp} = \{w \in W | w \cdot u = 0 \text{ for all } u \in U\}.$$

The set $U^{\perp}$ (pronounced "U-perp") is the set of all vectors in W orthogonal to *every* vector in U. This is also often called the *orthogonal complement* of U. Probably by now you may be feeling overwhelmed, it may help to watch this quick overview video:

Overview


Example Consider any plane P through the origin in $\mathbb{R}^3$. Then P is a subspace, and $P^{\perp}$ is the line through the origin orthogonal to P. For example, if P is the xy-plane, then

$$\mathbb{R}^3 = P \oplus P^{\perp} = \{(x, y, 0) | x, y \in \mathbb{R}\} \oplus \{(0, 0, z) | z \in \mathbb{R}\}.$$

Theorem 22.2. *Let U be a subspace of a finite-dimensional vector space W. Then the set $U^{\perp}$ is a subspace of W, and $W = U \oplus U^{\perp}$.*

Proof. To see that $U^{\perp}$ is a subspace, we only need to check closure, which requires a simple check.

We have $U \cap U^{\perp} = \{0\}$, since if $u \in U$ and $u \in U^{\perp}$, we have:

$$u \cdot u = 0 \Leftrightarrow u = 0.$$

Finally, we show that any vector $w \in W$ is in $U \oplus U^{\perp}$. (This is where we use the assumption that W is finite-dimensional.) Let $e_1, \ldots, e_n$ be an orthonormal basis for W. Set:

$$\begin{aligned} u &= (w \cdot e_1)e_1 + \cdots + (w \cdot e_n)e_n \in U \\ u^{\perp} &= w - u \end{aligned}$$

It is easy to check that $u^{\perp} \in U^{\perp}$ (see the Gram-Schmidt procedure). Then $w = u + u^{\perp}$, so $w \in U \oplus U^{\perp}$, and we are done. $\square$

Reading homework: problem 22.2

Example Consider any line L through the origin in $\mathbb{R}^4$. Then L is a subspace, and $L^{\perp}$ is a 3-dimensional subspace orthogonal to L. For example, let L be the line spanned by the vector $(1,1,1,1) \in \mathbb{R}^4$. Then $L^{\perp}$ is given by

$$\begin{aligned} L^{\perp} &= \{(x,y,z,w) \mid x,y,z,w \in \mathbb{R} \text{ and } (x,y,z,w) \cdot (1,1,1,1) = 0\} \\ &= \{(x,y,z,w) \mid x,y,z,w \in \mathbb{R} \text{ and } x,y,z,w = 0\}. \end{aligned}$$

It is easy to check that $\{v_1 = (1,-1,0,0), v_2 = (1,0,-1,0), v_3 = (1,0,0,-1)\}$ forms a basis for $L^{\perp}$. We use Gram-Schmidt to find an orthogonal basis for $L^{\perp}$:

First, we set $v_1^{\perp} = v_1$. Then:

$$\begin{aligned} v_2^{\perp} &= (1,0,-1,0) - \frac{1}{2}(1,-1,0,0) = \left(\frac{1}{2}, \frac{1}{2}, -1, 0\right), \\ v_3^{\perp} &= (1,0,0,-1) - \frac{1}{2}(1,-1,0,0) - \frac{1/2}{3/2}\left(\frac{1}{2}, \frac{1}{2}, -1, 0\right) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1\right). \end{aligned}$$

So the set

$$\left\{(1,-1,0,0), \left(\frac{1}{2}, \frac{1}{2}, -1, 0\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1\right)\right\}$$

is an orthogonal basis for $L^{\perp}$. We find an orthonormal basis for $L^{\perp}$ by dividing each basis vector by its length:

$$\left\{\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0\right), \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, 0\right), \left(\frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{6}, -\frac{\sqrt{3}}{2}\right)\right\}.$$

Moreover, we have

$$\mathbb{R}^4 = L \oplus L^{\perp} = \{(c,c,c,c) \mid c \in \mathbb{R}\} \oplus \{(x,y,z,w) \mid x,y,z,w \in \mathbb{R} \text{ and } x+y+z+w = 0\}.$$

Notice that for any subspace U, the subspace $(U^{\perp})^{\perp}$ is just U again. As such, $\perp$ is an involution on the set of subspaces of a vector space.

References

Hefferon, Chapter Three, Section VI.2: Gram-Schmidt Orthogonalization
Beezer, Chapter V, Section O, Subsection GSP
Wikipedia:

- Gram-Schmidt Process
- QR Decomposition
- Orthonormal Basis
- Direct Sum

Review Problems

1. Find the QR factorization of

$$M = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ -1 & -2 & 2 \end{pmatrix}.$$

Hint

2. Suppose u and v are linearly independent. Show that u and $v^{\perp}$ are also linearly independent. Explain why $\{u, v^{\perp}\}$ are a basis for $\operatorname{span}\{u, v\}$.

3. Repeat the previous problem, but with three independent vectors u, v, w, and $v^{\perp}$ and $w^{\perp}$ as defined in the lecture.

4. Given any three vectors u, v, w, when do $v^{\perp}$ or $w^{\perp}$ vanish?

5. For U a subspace of W, use the subspace theorem to check that $U^{\perp}$ is a subspace of W.

6. This question will answer the question, "If I choose a bit vector *at random*, what is the probability that it lies in the span of some other vectors?"

i. Given a collection S of k bit vectors in B^3, consider the bit matrix M whose columns are the vectors in S. Show that S is linearly independent if and only if the kernel of M is trivial.

ii. Give some method for choosing a random bit vector v in B^3. Suppose S is a collection of 2 linearly independent bit vectors in B^3. How can we tell whether $S \cup \{v\}$ is linearly independent? Do you think it is likely or unlikely that $S \cup \{v\}$ is linearly independent? Explain your reasoning.

iii. If P is the characteristic polynomial of a 3×3 bit matrix, what must the degree of P be? Given that each coefficient must be either 0 or 1, how many possibilities are there for P? How many of these possible characteristic polynomials have 0 as a root? If M is a 3×3 bit matrix chosen at random, what is the probability that it has 0 as an eigenvalue? (Assume that you are choosing a random matrix M in such a way as to make each characteristic polynomial equally likely.) What is the probability that the columns of M form a basis for B^3? (Hint: what is the relationship between the kernel of M and its eigenvalues?)

Note: We could ask the same question for real vectors: If I choose a real vector at random, what is the probability that it lies in the span of some other vectors? In fact, once we write down a reasonable way of choosing a random real vector, if I choose a real vector in $\mathbb{R}^n$ at random, the probability that it lies in the span of $n-1$ other real vectors is 0!

23 Diagonalizing Symmetric Matrices

Symmetric matrices have many applications. For example, if we consider the shortest distance between pairs of important cities, we might get a table like this:

	Davis	Seattle	San Francisco
Davis	0	2000	80
Seattle	2000	0	2010
San Francisco	80	2010	0

Encoded as a matrix, we obtain:

$$M = \begin{pmatrix} 0 & 2000 & 80 \\ 2000 & 0 & 2010 \\ 80 & 2010 & 0 \end{pmatrix} = M^T.$$

Definition A matrix is *symmetric* if it obeys

$$M = M^T.$$

One very nice property of symmetric matrices is that they always have real eigenvalues. The general proof is an exercise, but here's an example for 2×2 matrices.

Example For a general symmetric 2×2 matrix, we have:

$$\begin{aligned} P_\lambda \begin{pmatrix} a & b \\ b & d \end{pmatrix} &= \det \begin{pmatrix} \lambda - a & -b \\ -b & \lambda - d \end{pmatrix} \\ &= (\lambda - a)(\lambda - d) - b^2 \\ &= \lambda^2 - (a+d)\lambda - b^2 + ad \\ \Rightarrow \lambda &= \frac{a+d}{2} \pm \sqrt{b^2 + \left(\frac{a-d}{2}\right)^2}. \end{aligned}$$

Notice that the discriminant $4b^2 + (a-d)^2$ is always positive, so that the eigenvalues must be real.

Now, suppose a symmetric matrix M has two distinct eigenvalues $\lambda \neq \mu$ and eigenvectors x and y:

$$Mx = \lambda x, \qquad My = \mu y.$$

Consider the dot product $x \cdot y = x^T y = y^T x$. And now calculate:

$$\begin{aligned} x^T M y &= x^T \mu y = \mu x \cdot y, \text{ and} \\ x^T M y &= (y^T M x)^T \text{ (by transposing a } 1 \times 1 \text{ matrix)} \\ &= x^T M^T y \\ &= x^T M y \\ &= x^T \lambda y \\ &= \lambda x \cdot y. \end{aligned}$$

Subtracting these two results tells us that:

$$0 = x^T M y - x^T M y = (\mu - \lambda)\, x \cdot y.$$

Since μ and λ were assumed to be distinct eigenvalues, $\lambda - \mu$ is non-zero, and so $x \cdot y = 0$. Then we have proved the following theorem.

Theorem 23.1. *Eigenvectors of a symmetric matrix with distinct eigenvalues are orthogonal.*

Reading homework: problem 23.1

Example The matrix $M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has eigenvalues determined by

$$\det(M - \lambda) = (2 - \lambda)^2 - 1 = 0.$$

Then the eigenvalues of M are 3 and 1, and the associated eigenvectors turn out to be $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. It is easily seen that these eigenvectors are orthogonal:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$$

In Lecture 21 we saw that the matrix P built from orthonormal basis vectors $\{v_1, \ldots, v_n\}$

$$P = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$$

was an orthogonal matrix:

$$P^{-1} = P^T, \text{ or } PP^T = I = P^T P.$$

Moreover, given any (unit) vector x_1, one can always find vectors $x_2, \dots, x_n$ such that $\{x_1, \dots, x_n\}$ is an orthonormal basis. (Such a basis can be obtained using the Gram-Schmidt procedure.)

Now suppose M is a symmetric $n \times n$ matrix and λ_1 is an eigenvalue with eigenvector x_1. Let the square matrix of column vectors P be the following:

$$P = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix},$$

where x_1 through x_n are orthonormal, and x_1 is an eigenvector for M, but the others are not necessarily eigenvectors for M. Then

$$MP = \begin{pmatrix} \lambda_1 x_1 & M x_2 & \cdots & M x_n \end{pmatrix}.$$

But P is an orthogonal matrix, so $P^{-1} = P^T$. Then:

$$\begin{aligned}
P^{-1} = P^T &= \begin{pmatrix} x_1^T \\ \vdots \\ x_n^T \end{pmatrix} \\
\Rightarrow P^T M P &= \begin{pmatrix} x_1^T \lambda_1 x_1 & * & \cdots & * \\ x_2^T \lambda_1 x_1 & * & \cdots & * \\ \vdots & & & \vdots \\ x_n^T \lambda_1 x_1 & * & \cdots & * \end{pmatrix} \\
&= \begin{pmatrix} \lambda_1 & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & * & & \vdots \\ 0 & * & \cdots & * \end{pmatrix} \\
&= \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \hat{M} & \\ 0 & & & \end{pmatrix}
\end{aligned}$$

The last equality follows since $P^T M P$ is symmetric. The asterisks in the matrix are where "stuff" happens; this extra information is denoted by $\hat{M}$ in the final equation. We know nothing about $\hat{M}$ except that it is an $(n-1) \times (n-1)$ matrix and that it is symmetric. But then, by finding an (unit) eigenvector for $\hat{M}$, we could repeat this procedure successively. The end result would be a diagonal matrix with eigenvalues of M on the diagonal. Then we have proved a theorem.

Theorem 23.2. *Every symmetric matrix is similar to a diagonal matrix of its eigenvalues. In other words,*

$$M = M^T \Rightarrow M = PDP^T$$

where P is an orthogonal matrix and D is a diagonal matrix whose entries are the eigenvalues of M.

Reading homework: problem 23.2

To diagonalize a real symmetric matrix, begin by building an orthogonal matrix from an orthonormal basis of eigenvectors.

Example The symmetric matrix $M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has eigenvalues 3 and 1 with eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ respectively. From these eigenvectors, we normalize and build the orthogonal matrix:

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

Notice that $P^T P = I_2$. Then:

$$MP = \begin{pmatrix} \frac{3}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}.$$

In short, $MP = DP$, so $D = P^T MP$. Then D is the diagonalized form of M and P the associated change-of-basis matrix from the standard basis to the basis of eigenvectors.

3×3 `Example`

References

Hefferon, Chapter Three, Section V: Change of Basis
Beezer, Chapter E, Section PEE, Subsection EHM
Beezer, Chapter E, Section SD, Subsection D
Wikipedia:

- Symmetric Matrix
- Diagonalizable Matrix
- Similar Matrix

Review Problems

1. (On Reality of Eigenvalues)

   (a) Suppose $z = x + iy$ where $x, y \in \mathbb{R}, i = \sqrt{-1}$, and $\overline{z} = x - iy$. Compute $z\overline{z}$ and $\overline{z}z$ in terms of x and y. What kind of numbers are $z\overline{z}$ and $\overline{z}z$? (The complex number $\overline{z}$ is called the *complex conjugate* of z).

   (b) Suppose that $\lambda = x + iy$ is a complex number with $x, y \in \mathbb{R}$, and that $\lambda = \overline{\lambda}$. Does this determine the value of x or y? What kind of number must λ be?

   (c) Let $x = \begin{pmatrix} z^1 \\ \vdots \\ z^n \end{pmatrix} \in \mathbb{C}^n$. Let $x^\dagger = \begin{pmatrix} \overline{z^1} & \cdots & \overline{z^n} \end{pmatrix} \in \mathbb{C}^n$ (a $1 \times n$ complex matrix or a row vector). Compute $x^\dagger x$. Using the result of part 1a, what can you say about the number $x^\dagger x$? (E.g., is it real, imaginary, positive, negative, etc.)

   (d) Suppose $M = M^T$ is an $n \times n$ symmetric matrix with real entries. Let λ be an eigenvalue of M with eigenvector x, so $Mx = \lambda x$. Compute:
   $$\frac{x^\dagger M x}{x^\dagger x}$$

   (e) Suppose Λ is a 1×1 matrix. What is Λ^T?

   (f) What is the size of the matrix $x^\dagger M x$?

   (g) For any matrix (or vector) N, we can compute $\overline{N}$ by applying complex conjugation to each entry of N. Compute $\overline{(x^\dagger)^T}$. Then compute $\overline{(x^\dagger M x)^T}$. Note that for matrices $\overline{AB + C} = \overline{A}\,\overline{B} + \overline{C}$.

   (h) Show that $\lambda = \overline{\lambda}$. Using the result of a previous part of this problem, what does this say about λ?

Problem 1 hint

2. Let $x_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, where $a^2 + b^2 + c^2 = 1$. Find vectors x_2 and x_3 such that $\{x_1, x_2, x_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

3. (Dimensions of Eigenspaces)

(a) Let $A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$. Find all eigenvalues of A.

(b) Find a basis for each eigenspace of A. What is the sum of the dimensions of the eigenspaces of A?

(c) Based on your answer to the previous part, guess a formula for the sum of the dimensions of the eigenspaces of a real $n \times n$ symmetric matrix. Explain why your formula must work for any real $n \times n$ symmetric matrix.

24 Kernel, Range, Nullity, Rank

Given a linear transformation $L: V \to W$, we would like to know whether it has an inverse. That is, we would like to know whether there exists a linear transformation $M: W \to V$ such that for any vector $v \in V$, we have $M(L(v)) = v$, and for any vector $w \in W$, we have $L(M(w)) = w$. A linear transformation is just a special kind of function from one vector space to another. So before we discuss which linear transformations have inverses, let us first discuss inverses of arbitrary functions. When we later specialize to linear transformations, we'll also find some nice ways of creating subspaces.

Let $f: S \to T$ be a function from a set S to a set T. Recall that S is called the *domain* of f, T is called the *codomain* of f, and the set

$$\operatorname{ran}(f) = \operatorname{im}(f) = f(S) = \{f(s) | s \in S\} \subset T$$

is called the *range* or *image* of f. The image of f is the set of elements of T to which the function f maps, i.e., the things in T which you can get to by starting in S and applying f. We can also talk about the pre-image of any subset $U \subset T$:

$$f^{-1}(U) = \{s \in S | f(s) \in U\} \subset S.$$

The pre-image of a set U is the set of all elements of S which map to U.

The function f is *one-to-one* if different elements in S always map to different elements in T. That is, f is one-to-one if for any elements $x \neq y \in S$, we have that $f(x) \neq f(y)$. One-to-one functions are also called *injective* functions. Notice that injectivity is a condition on the pre-image of f.

The function f is *onto* if every element of T is mapped to by some element of S. That is, f is onto if for any $t \in T$, there exists some $s \in S$ such that $f(s) = t$. Onto functions are also called *surjective* functions. Notice that surjectivity is a condition on the image of f.

If f is both injective and surjective, it is *bijective*.

Theorem 24.1. *A function $f: S \to T$ has an inverse function $g: T \to S$ if and only if it is bijective.*

Proof. Suppose that f is bijective. Since f is surjective, every element $t \in T$ has at least one pre-image, and since f is injective, every t has no more than one pre-image. Therefore, to construct an inverse function g, we simply define $g(t)$ to be the unique pre-image $f^{-1}(t)$ of t.

Conversely, suppose that f has an inverse function g.

- The function f is injective:

  Suppose that we have $x, y \in S$ such that $f(x) = f(y)$. We must have that $g(f(s)) = s$ for any $s \in S$, so in particular $g(f(x)) = x$ and $g(f(y)) = y$. But since $f(x) = f(y)$, we have $g(f(x)) = g(f(y))$ so $x = y$. Therefore, f is injective.

- The function f is surjective:

  Let t be any element of T. We must have that $f(g(t)) = t$. Thus, $g(t)$ is an element of S which maps to t. So f is surjective.

□

Now let us restrict to the case that our function f is not just an arbitrary function, but a linear transformation between two vector spaces. Everything we said above for arbitrary functions is exactly the same for linear transformations. However, the linear structure of vector spaces lets us say much more about one-to-one and onto functions than we can say about functions on general sets. For example, we always know that a linear function sends 0_V to 0_W. You will show that a linear transformation is one-to-one if and only if 0_V is the only vector that is sent to 0_W: by looking at just one (very special) vector, we can figure out whether f is one-to-one. For arbitrary functions between arbitrary sets, things aren't nearly so convenient!

Let $L\colon V \to W$ be a linear transformation. Suppose L is *not* injective. Then we can find $v_1 \neq v_2$ such that $Lv_1 = Lv_2$. Then $v_1 - v_2 \neq 0$, but

$$L(v_1 - v_2) = 0.$$

Definition Let $L\colon V \to W$ be a linear transformation. The set of all vectors v such that $Lv = 0_W$ is called the *kernel of* L:

$$\ker L = \{v \in V | Lv = 0_W\}.$$

Theorem 24.2. *A linear transformation L is injective if and only if*

$$\ker L = \{0_V\}\,.$$

Proof. The proof of this theorem is an exercise. □

Notice that if L has matrix M in some basis, then finding the kernel of L is equivalent to solving the homogeneous system

$$MX = 0.$$

Example Let $L(x, y) = (x + y, x + 2y, y)$. Is L one-to-one?

To find out, we can solve the linear system:

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Then all solutions of $MX = 0$ are of the form $x = y = 0$. In other words, $\ker L = 0$, and so L is injective.

Reading homework: problem 24.1

Theorem 24.3. *Let $L \colon V \to W$. Then $\ker L$ is a subspace of V.*

Proof. Notice that if $L(v) = 0$ and $L(u) = 0$, then for any constants c, d, $L(cu + dv) = 0$. Then by the subspace theorem, the kernel of L is a subspace of V. □

This theorem has an interpretation in terms of the eigenspaces of $L \colon V \to V$. Suppose L has a zero eigenvalue. Then the associated eigenspace consists of all vectors v such that $Lv = 0v = 0$; in other words, the 0-eigenspace of L is exactly the kernel of L.

Returning to the previous example, let $L(x, y) = (x + y, x + 2y, y)$. L is clearly not surjective, since L sends $\mathbb{R}^2$ to a plane in $\mathbb{R}^3$.

Example Let $L \colon \mathbb{R}^3 \to \mathbb{R}$ be the linear transformation defined by $L(x, y, z) = (x + y + z)$. Then $\ker L$ consists of all vectors $(x, y, z) \in \mathbb{R}^3$ such that $x + y + z = 0$. Therefore, the set

$$V = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$$

is a subspace of $\mathbb{R}^3$.

Notice that if $x = L(v)$ and $y = L(u)$, then for any constants c, d, $cx + dy = L(cv + du)$. Now the subspace theorem strikes again, and we have the following theorem.

Theorem 24.4. *Let $L \colon V \to W$. Then the image $L(V)$ is a subspace of W.*

To find a basis of the image of L, we can start with a basis $S = \{v_1, \ldots, v_n\}$ for V, and conclude (see the Review Exercises) that

$$L(V) = \operatorname{span} L(S) = \operatorname{span}\{L(v_1), \ldots, L(v_n)\}.$$

However, the set $\{L(v_1), \ldots, L(v_n)\}$ may not be linearly independent, so we solve

$$c^1 L(v_1) + \cdots + c^n L(v_n) = 0.$$

By finding relations amongst $L(S)$, we can discard vectors until a basis is arrived at. The size of this basis is the dimension of the image of L, which is known as the *rank* of L.

Definition The *rank* of a linear transformation L is the dimension of its image, written $\operatorname{rank} L = \dim L(V) = \dim \ \operatorname{ran}\ L$.

The *nullity* of a linear transformation is the dimension of the kernel, written $\operatorname{null} L = \dim \ker L$.

Theorem 24.5 (Dimension Formula)**.** *Let $L \colon V \to W$ be a linear transformation, with V a finite-dimensional vector space*[16]*. Then:*

$$\begin{aligned} \dim V &= \dim \ker V + \dim L(V) \\ &= \operatorname{null} L + \operatorname{rank} L. \end{aligned}$$

Proof. Pick a basis for V:

$$\{v_1, \ldots, v_p, u_1, \ldots, u_q\},$$

where $v_1, \ldots, v_p$ is also a basis for $\ker L$. This can always be done, for example, by finding a basis for the kernel of L and then extending to a basis for V. Then $p = \operatorname{null} L$ and $p + q = \dim V$. Then we need to show that $q = \operatorname{rank} L$. To accomplish this, we show that $\{L(u_1), \ldots, L(u_q)\}$ is a basis for $L(V)$.

To see that $\{L(u_1), \ldots, L(u_q)\}$ spans $L(V)$, consider any vector w in $L(V)$. Then we can find constants c^i, d^j such that:

$$\begin{aligned} w &= L(c^1 v_1 + \cdots + c^p v_p + d^1 u_1 + \cdots + d^q u_q) \\ &= c^1 L(v_1) + \cdots + c^p L(v_p) + d^1 L(u_1) + \cdots + d^q L(u_q) \\ &= d^1 L(u_1) + \cdots + d^q L(u_q) \text{ since } L(v_i) = 0, \\ \Rightarrow L(V) &= \operatorname{span}\{L(u_1), \ldots, L(u_q)\}. \end{aligned}$$

[16] The formula still makes sense for infinite dimensional vector spaces, such as the space of all polynomials, but the notion of a basis for an infinite dimensional space is more sticky than in the finite-dimensional case. Furthermore, the dimension formula for infinite dimensional vector spaces isn't useful for computing the rank of a linear transformation, since an equation like $\infty = \infty + x$ cannot be solved for x. As such, the proof presented assumes a finite basis for V.

Now we show that $\{L(u_1), \ldots, L(u_q)\}$ is linearly independent. We argue by contradiction: Suppose there exist constants d^j (not all zero) such that

$$\begin{aligned} 0 &= d^1 L(u_1) + \cdots + d^q L(u_q) \\ &= L(d^1 u_1 + \cdots + d^q u_q). \end{aligned}$$

But since the u^j are linearly independent, then $d^1 u_1 + \cdots + d^q u_q \neq 0$, and so $d^1 u_1 + \cdots + d^q u_q$ is in the kernel of L. But then $d^1 u_1 + \cdots + d^q u_q$ must be in the span of $\{v_1, \ldots, v_p\}$, since this was a basis for the kernel. This contradicts the assumption that $\{v_1, \ldots, v_p, u_1, \ldots, u_q\}$ was a basis for V, so we are done. □

Reading homework: problem 24.2

24.1 Summary

We have seen that a linear transformation has an inverse if and only if it is bijective (i.e., one-to-one and onto). We also know that linear transformations can be represented by matrices, and we have seen many ways to tell whether a matrix is invertible. Here is a list of them.

Theorem 24.6 (Invertibility)**.** *Let M be an $n \times n$ matrix, and let $L\colon \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation defined by $L(v) = Mv$. Then the following statements are equivalent:*

1. *If V is any vector in $\mathbb{R}^n$, then the system $MX = V$ has exactly one solution.*
2. *The matrix M is row-equivalent to the identity matrix.*
3. *If v is any vector in $\mathbb{R}^n$, then $L(x) = v$ has exactly one solution.*
4. *The matrix M is invertible.*
5. *The homogeneous system $MX = 0$ has no non-zero solutions.*
6. *The determinant of M is not equal to 0.*
7. *The transpose matrix M^T is invertible.*
8. *The matrix M does not have 0 as an eigenvalue.*
9. *The linear transformation L does not have 0 as an eigenvalue.*
10. *The characteristic polynomial $\det(\lambda I - M)$ does not have 0 as a root.*
11. *The columns (or rows) of M span $\mathbb{R}^n$.*
12. *The columns (or rows) of M are linearly independent.*
13. *The columns (or rows) of M are a basis for $\mathbb{R}^n$.*
14. *The linear transformation L is injective.*
15. *The linear transformation L is surjective.*
16. *The linear transformation L is bijective.*

Note: it is important that M be an $n \times n$ matrix! If M is not square, then it can't be invertible, and many of the statements above are no longer equivalent to each other.

Proof. Many of these equivalences were proved earlier in these notes. Some were left as review questions or sample final questions. The rest are left as exercises for the reader. $\square$

Discussion on Theorem 24.6.

References

Hefferon, Chapter Three, Section II.2: Rangespace and Nullspace (Recall that "homomorphism" is is used instead of "linear transformation" in Hefferon.)
Beezer, Chapter LT, Sections ILT-IVLT
Wikipedia:

- Rank
- Dimension Theorem
- Kernel of a Linear Operator

Review Problems

1. Let $L\colon V \to W$ be a linear transformation. Show that $\ker L = \{0_V\}$ if and only if L is one-to-one:

   (a) First, suppose that $\ker L = \{0_V\}$. Show that L is one-to-one. Think about methods of proof–does a proof by contradiction, a proof by induction, or a direct proof seem most appropriate?

   (b) Now, suppose that L is one-to-one. Show that $\ker L = \{0_V\}$. That is, show that 0_V is in $\ker L$, and then show that there are no other vectors in $\ker L$.

   Hint for 1

2. Let $\{v_1, \ldots, v_n\}$ be a basis for V. Explain why

$$L(V) = \operatorname{span}\{L(v_1), \ldots, L(v_n)\}.$$

3. Suppose $L\colon \mathbb{R}^4 \to \mathbb{R}^3$ whose matrix M in the standard basis is row equivalent to the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

   Explain why the first three columns of the original matrix M form a basis for $L(\mathbb{R}^4)$.
   Find and describe and algorithm (*i.e.* a general procedure) for finding a basis for $L(\mathbb{R}^n)$ when $L\colon \mathbb{R}^n \to \mathbb{R}^m$.
   Finally, use your algorithm to find a basis for $L(\mathbb{R}^4)$ when $L\colon \mathbb{R}^4 \to \mathbb{R}^3$ is the linear transformation whose matrix M in the standard basis is

$$\begin{pmatrix} 2 & 1 & 1 & 4 \\ 0 & 1 & 0 & 5 \\ 4 & 1 & 1 & 6 \end{pmatrix}.$$

4. Claim: If $\{v_1, \ldots, v_n\}$ is a basis for $\ker L$, where $L\colon V \to W$, then it is always possible to extend this set to a basis for V.

   Choose a simple yet non-trivial linear transformation with a non-trivial kernel and verify the above claim for the transformation you choose.

5. Let $P_n(x)$ be the space of polynomials in x of degree less than or equal to n, and consider the derivative operator $\frac{\partial}{\partial x}$. Find the dimension of the kernel and image of $\frac{\partial}{\partial x}$.

   Now, consider $P_2(x, y)$, the space of polynomials of degree two or less in x and y. (Recall that xy is degree two, y is degree one and x^2y is degree three, for example.) Let $L = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. (For example, $L(xy) = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(xy) = y + x$.) Find a basis for the kernel of L. Verify the dimension formula in this case.

6. (Extra credit) We will show some ways the dimension formula can break down with the vector space is infinite dimensional.

   (a) Let $\mathbb{R}[x]$ be the vector space of all polynomials with coefficients in $\mathbb{R}$ in the variable x. Let $D = \frac{d}{dx}$ be the usual derivative operator. Show that the range of D is $\mathbb{R}[x]$. What is $\ker D$?

   Hint: Use the basis $\{x^n \mid n \in \mathbb{N}\}$.

(b) Consider $\mathbb{R}[x]$ and let $M\colon \mathbb{R}[x] \to \mathbb{R}[x]$ be the linear map $M(p(x)) = xp(x)$ which multiplies by x. What is the kernel and range of M?

(c) Let ℓ^1 be the vector space of all absolutely convergent sequences $s = \{s_i\}$. Define the map $P\colon \ell^1 \to \ell^1$ by

$$s_i \mapsto \begin{cases} s_i & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd,} \end{cases}$$

and for example

$$P((1,0,0,\ldots)) = (0,0,0,\ldots), \qquad P((0,1,0,\ldots)) = (0,1,0,\ldots).$$

Show that P is linear, is a projection (i.e. $P^2 = P$), has infinite kernel, and has infinite range.

Hint: Define e_i as the sequence with a 1 in the i-th position and 0 everywhere else which you can think of as a standard basis vector. What is $P(e_i)$ when i is even? When i is odd?

(d) Let V be an infinite dimensional vector space and $L\colon V \to V$ be a linear operator. Suppose that $\dim \ker L < \infty$, show that $\dim L(V)$ is infinite. Also show when $\dim L(V) < \infty$ that $\dim \ker L$ is infinite.

25 Least Squares

Consider the linear system $L(x) = v$, where $L\colon U \xrightarrow{\text{linear}} W$, and $v \in W$ is given. As we have seen, this system may have no solutions, a unique solution, or a space of solutions. But if v is not in the range of L then there will *never* be any solutions for $L(x) = v$.

However, for many applications we do not need a exact solution of the system; instead, we try to find the best approximation possible. To do this, we try to find x that minimizes $||L(x) - v||$.

> "My work always tried to unite the Truth with the Beautiful, but when I had to choose one or the other, I usually chose the Beautiful."
>
> – Hermann Weyl.

This method has many applications, such as when trying to fit a (perhaps linear) function to a "noisy" set of observations. For example, suppose we measured the position of a bicycle on a racetrack once every five seconds. Our observations won't be exact, but so long as the observations are right on average, we can figure out a best-possible linear function of position of the bicycle in terms of time.

Suppose M is the matrix for L in some bases for U and W, and v and x are given by column vectors V and X in these bases. Then we need to approximate

$$MX - V \approx 0\,.$$

Note that if $\dim U = n$ and $\dim W = m$ then M can be represented by an $m \times n$ matrix and x and v as vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Thus, we can write $W = L(U) \oplus L(U)^\perp$. Then we can uniquely write $v = v^\parallel + v^\perp$, with $v^\parallel \in L(U)$ and $v^\perp \in L(U)^\perp$.

Then we should solve $L(u) = v^\parallel$. In components, $v^\perp$ is just $V - MX$, and is the part we will eventually wish to minimize.

In terms of M, recall that $L(V)$ is spanned by the columns of M. (In the natural basis, the columns of M are Me_1, …, Me_n.) Then $v^\perp$ must be perpendicular to the columns of M. *i.e.*, $M^T(V - MX) = 0$, or

$$M^T MX = M^T V.$$

Solutions X to $M^T MX = M^T V$ are called *least squares* solutions to $MX = V$.

Notice that any solution X to $MX = V$ is a least squares solution. However, the converse is often false. In fact, the equation $MX = V$ may have no solutions at all, but still have least squares solutions to $M^T MX = M^T V$.

Observe that since M is an $m \times n$ matrix, then M^T is an $n \times m$ matrix. Then $M^T M$ is an $n \times n$ matrix, and is symmetric, since $(M^T M)^T = M^T M$. Then, for any vector X, we can evaluate $X^T M^T MX$ to obtain a number. This is a very nice number, though! It is just the length $|MX|^2 = (MX)^T(MX) = X^T M^T MX$.

Reading homework: problem 25.1

Now suppose that $\ker L = \{0\}$, so that the only solution to $MX = 0$ is $X = 0$. (This need not mean that M is invertible because M is an $n \times m$ matrix, so not necessarily square.) However the square matrix $M^T M$ *is* invertible. To see this, suppose there was a vector X such that $M^T MX = 0$. Then it would follow that $X^T M^T MX = |MX|^2 = 0$. In other words the vector MX would have zero length, so could only be the zero vector. But we are assuming that $\ker L = \{0\}$ so $MX = 0$ implies $X = 0$. Thus the kernel of $M^T M$ is $\{0\}$ so this matrix is invertible. So, in this case, the least squares solution (the X that solves $M^T MX = MV$) is unique, and is equal to

$$X = (M^T M)^{-1} M^T V.$$

In a nutshell, this is the least squares method.

- Compute $M^T M$ and $M^T V$.
- Solve $(M^T M)X = M^T V$ by Gaussian elimination.

Example Captain Conundrum falls off of the leaning tower of Pisa and makes three (rather shaky) measurements of his velocity at three different times.

t s	v m/s
1	11
2	19
3	31

Having taken some calculus[17], he believes that his data are best approximated by a straight line

$$v = at + b.$$

Then he should find a and b to best fit the data.

$$\begin{aligned} 11 &= a \cdot 1 + b \\ 19 &= a \cdot 2 + b \\ 31 &= a \cdot 3 + b. \end{aligned}$$

As a system of linear equations, this becomes:

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 11 \\ 19 \\ 31 \end{pmatrix}.$$

There is likely no actual straight line solution, so instead solve $M^T M X = M^T V$.

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 11 \\ 19 \\ 31 \end{pmatrix}.$$

This simplifies to the system:

$$\left(\begin{array}{cc|c} 14 & 6 & 142 \\ 6 & 3 & 61 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 1 & \frac{1}{3} \end{array}\right).$$

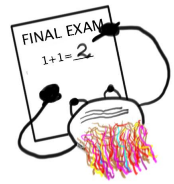
Then the least-squares fit is the line

$$v = 10\ t + \frac{1}{3}.$$

Notice that this equation implies that Captain Conundrum accelerates towards Italian soil at 10 m/s^2 (which is an excellent approximation to reality) and that he started at a downward velocity of $\frac{1}{3}$ m/s (perhaps somebody gave him a shove...)!

[17] In fact, he is a *Calculus Superhero*.

Congratulations, you have reached the end of the notes!



Now test your skills on the sample final exam.

References

Hefferon, Chapter Three, Section VI.2: Gram-Schmidt Orthogonalization
Beezer, Part A, Section CF, Subsection DF
Wikipedia:

- Linear Least Squares
- Least Squares

Review Problems

1. Let $L : U \to V$ be a linear transformation. Suppose $v \in L(U)$ and you have found a vector u_{ps} that obeys $L(u_{\mathrm{ps}}) = v$.

   Explain why you need to compute $\ker L$ to describe the solution space of the linear system $L(u) = v$.

`Hint for Problem 1`

2. Suppose that M is an $m \times n$ matrix with trivial kernel. Show that for any vectors u and v in $\mathbb{R}^m$:

   - $u^T M^T M v = v^T M^T M u$.
   - $v^T M^T M v \geq 0$. In case you are concerned (you don't need to be) and for future reference, the notation $v \geq 0$ means each entry $v^i \geq 0$.

- If $v^T M^T M v = 0$, then $v = 0$.

(Hint: Think about the dot product in $\mathbb{R}^n$.)

`Hint for Problem 2`

A Sample Midterm I Problems and Solutions

1. Solve the following linear system. Write the solution set in vector form. Check your solution. Write one particular solution and one homogeneous solution, if they exist. What does the solution set look like geometrically?

$$\begin{array}{rcrcrl} x & + & 3y & & & = 4 \\ x & - & 2y & + & z & = 1 \\ 2x & + & y & + & z & = 5 \end{array}$$

2. Consider the system

$$\left\{ \begin{array}{rcrcrcrcr} x & & & - & z & + & 2w & = & -1 \\ x & + & y & + & z & - & w & = & 2 \\ & - & y & - & 2z & + & 3w & = & -3 \\ 5x & + & 2y & - & z & + & 4w & = & 1 \end{array} \right.$$

   (a) Write an augmented matrix for this system.

   (b) Use elementary row operations to find its reduced row echelon form.

   (c) Write the solution set for the system in the form

$$S = \{X_0 + \sum_i \mu_i Y_i : \mu_i \in \mathbb{R}\}.$$

   (d) What are the vectors X_0 and Y_i called *and* which matrix equations do they solve?

   (e) Check separately that X_0 and each Y_i solve the matrix systems you claimed they solved in part (d).

3. Use row operations to invert the matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 11 \\ 3 & 7 & 14 & 25 \\ 4 & 11 & 25 & 50 \end{pmatrix}$$

4. Let $M = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$. Calculate $M^T M^{-1}$. Is M symmetric? What is the trace of the transpose of $f(M)$, where $f(x) = x^2 - 1$?

5. In this problem M is the matrix

$$M = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and X is the vector

$$X = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Calculate all possible dot products between the vectors X and MX. Compute the lengths of X and MX. What is the angle between the vectors MX and X. Draw a picture of these vectors in the plane. For what values of θ do you expect equality in the triangle and Cauchy–Schwartz inequalities?

6. Let M be the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Find a formula for M^k for any positive integer power k. Try some simple examples like $k = 2, 3$ if confused.

7. *Determinants:* The determinant $\det M$ of a 2×2 matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined by

$$\det M = ad - bc\,.$$

   (a) For which values of $\det M$ does M have an inverse?

   (b) Write down all 2×2 bit matrices with determinant 1. (Remember bits are either 0 or 1 and $1 + 1 = 0$.)

   (c) Write down all 2×2 bit matrices with determinant 0.

(d) Use one of the above examples to show why the following statement is FALSE.

Square matrices with the same determinant are always row equivalent.

8. What does it mean for a function to be linear? Check that integration is a linear function from V to V, where $V = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is integrable}\}$ is a vector space over $\mathbb{R}$ with usual addition and scalar multiplication.

9. What are the four main things we need to define for a vector space? Which of the following is a vector space over $\mathbb{R}$? For those that are not vector spaces, modify one part of the definition to make it into a vector space.

   (a) $V = \{\ 2 \times 2$ matrices with entries in $\mathbb{R}\}$, usual matrix addition, and $k \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & b \\ kc & d \end{pmatrix}$ for $k \in \mathbb{R}$.

   (b) $V = \{$polynomials with complex coefficients of degree $\leq 3\}$, with usual addition and scalar multiplication of polynomials.

   (c) $V = \{$vectors in $\mathbb{R}^3$ with at least one entry containing a $1\}$, with usual addition and scalar multiplication.

10. *Subspaces:* If V is a vector space, we say that U is a *subspace* of V when the set U is also a vector space, using the vector addition and scalar multiplication rules of the vector space V. (Remember that $U \subset V$ says that "U is a subset of V", *i.e.*, all elements of U are also elements of V. The symbol $\forall$ means "for all" and $\in$ means "is an element of".)

    Explain why additive closure ($u + w \in U\ \forall\ u, v \in U$) and multiplicative closure ($r.u \in U\ \forall\ r \in \mathbb{R},\ u \in V$) ensure that (i) the zero vector $0 \in U$ and (ii) every $u \in U$ has an additive inverse.

    In fact it suffices to check closure under addition and scalar multiplication to verify that U is a vector space. Check whether the following choices of U are vector spaces:

    (a) $U = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} : x, y \in \mathbb{R} \right\}$

(b) $U = \left\{ \begin{pmatrix} 1 \\ 0 \\ z \end{pmatrix} : z \in \mathbb{R} \right\}$

Solutions

1. *As an additional exercise, write out the row operations above the* $\sim$ *signs below:*

$$\left(\begin{array}{rrr|r} 1 & 3 & 0 & 4 \\ 1 & -2 & 1 & 1 \\ 2 & 1 & 1 & 5 \end{array}\right) \sim \left(\begin{array}{rrr|r} 1 & 3 & 0 & 4 \\ 0 & -5 & 1 & -3 \\ 0 & -5 & 1 & -3 \end{array}\right) \sim \left(\begin{array}{rrr|r} 1 & 0 & \frac{3}{5} & \frac{11}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{3}{5} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Solution set

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{11}{5} \\ \frac{3}{5} \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -\frac{3}{5} \\ \frac{1}{5} \\ 1 \end{pmatrix} : \mu \in \mathbb{R} \right\}$$

Geometrically this represents a line in $\mathbb{R}^3$ through the point $\begin{pmatrix} \frac{11}{5} \\ \frac{3}{5} \\ 0 \end{pmatrix}$ and running parallel to the vector $\begin{pmatrix} -\frac{3}{5} \\ \frac{1}{5} \\ 1 \end{pmatrix}$.

A particular solution is $\begin{pmatrix} \frac{11}{5} \\ \frac{3}{5} \\ 0 \end{pmatrix}$ and a homogeneous solution is $\begin{pmatrix} -\frac{3}{5} \\ \frac{1}{5} \\ 1 \end{pmatrix}$.

As a double check note that

$$\begin{pmatrix} 1 & 3 & 0 \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{11}{5} \\ \frac{3}{5} \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 3 & 0 \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{3}{5} \\ \frac{1}{5} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

2. (a) *Again, write out the row operations as an additional exercise.*

$$\left(\begin{array}{rrrr|r} 1 & 0 & -1 & 2 & -1 \\ 1 & 1 & 1 & -1 & 2 \\ 0 & -1 & -2 & 3 & -3 \\ 5 & 2 & -1 & 4 & 1 \end{array}\right)$$

(b)

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & -1 & 2 & -1 \\ 0 & 1 & 2 & -3 & 3 \\ 0 & -1 & -2 & 3 & -3 \\ 0 & 2 & 4 & -6 & 6 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & -1 & 2 & -1 \\ 0 & 1 & 2 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

(c) Solution set

$$\left\{ X = \begin{pmatrix} -1 \\ 3 \\ 0 \\ 0 \end{pmatrix} + \mu_1 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix} : \mu_1, \mu_2 \in \mathbb{R} \right\} .$$

(d) The vector $X_0 = \begin{pmatrix} -1 \\ 3 \\ 0 \\ 0 \end{pmatrix}$ is *a* particular solution and the vectors $Y_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}$ and $Y_2 = \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix}$ are homogeneous solutions. Calling $M = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 1 & 1 & 1 & -1 \\ 0 & -1 & -2 & 3 \\ 5 & 2 & -1 & 4 \end{pmatrix}$ and $V = \begin{pmatrix} -1 \\ 2 \\ -3 \\ 1 \end{pmatrix}$, they obey

$$MX = V \,, \qquad MY_1 = 0 = MY_2 \,.$$

(e) This amounts to performing explicitly the matrix manipulations $MX - V$, MY_1, MY_2 and checking they all return the zero vector.

3. *As usual, be sure to write out the row operations above the* $\sim$*'s so your work can be easily checked.*

$$\left(\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 2 & 4 & 7 & 11 & 0 & 1 & 0 & 0 \\ 3 & 7 & 14 & 25 & 0 & 0 & 1 & 0 \\ 4 & 11 & 25 & 50 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 1 & 5 & 13 & -3 & 0 & 1 & 0 \\ 0 & 3 & 13 & 34 & -4 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 0 & -7 & -22 & 7 & 0 & -2 & 0 \\ 0 & 1 & 5 & 13 & -3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 0 & -2 & -5 & 5 & 0 & -3 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & -7 & 7 & -2 & 0 \\ 0 & 1 & 0 & -2 & 7 & -5 & 1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & -3 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -6 & 9 & -5 & 1 \\ 0 & 1 & 0 & 0 & 9 & -1 & -5 & 2 \\ 0 & 0 & 1 & 0 & -5 & -5 & 9 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 & -3 & 1 \end{array}\right).$$

Check

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 11 \\ 3 & 7 & 14 & 25 \\ 4 & 11 & 25 & 50 \end{pmatrix} \begin{pmatrix} -6 & 9 & -5 & 1 \\ 9 & -1 & -5 & 2 \\ -5 & -5 & 9 & -3 \\ 1 & 2 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

4.

$$M^T M^{-1} = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix} = \begin{pmatrix} \frac{11}{5} & -\frac{4}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{pmatrix}.$$

Since $M^T M^{-1} \neq I$, it follows $M^T \neq M$ so M is *not* symmetric. Finally

$$\operatorname{tr} f(M)^T = \operatorname{tr} f(M) = \operatorname{tr}(M^2 - I) = \operatorname{tr} \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} - \operatorname{tr} I$$

$$= (2 \cdot 2 + 1 \cdot 3) + (3 \cdot 1 + (-1) \cdot (-1)) - 2 = 9\,.$$

5. First

$$X \cdot (MX) = X^T M X = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{pmatrix} = (x^2+y^2)\cos\theta\,.$$

Now $||X|| = \sqrt{X \cdot X} = \sqrt{x^2+y^2}$ and $(MX)\cdot(MX) = XM^TMX$. But

$$M^TM = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{pmatrix} = I\,.$$

Hence $||MX|| = ||X|| = \sqrt{x^2+y^2}$. Thus the cosine of the angle between X and MX is given by

$$\frac{X \cdot (MX)}{||X||\,||MX||} = \frac{(x^2+y^2)\cos\theta}{\sqrt{x^2+y^2}\,\sqrt{x^2+y^2}} = \cos\theta\,.$$

In other words, the angle is θ OR $-\theta$. You should draw two pictures, one where the angle between X and MX is θ, the other where it is $-\theta$.

For Cauchy–Schwartz, $\frac{|X\cdot(MX)|}{||X||\,||MX||} = |\cos\theta| = 1$ when $\theta = 0, \pi$. For the triangle equality $MX = X$ achieves $||X + MX|| = ||X|| + ||MX||$, which requires $\theta = 0$.

6. This is a block matrix problem. Notice the that matrix M is really just $M = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$, where I and 0 are the 3×3 identity zero matrices, respectively. But

$$M^2 = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 2I \\ 0 & I \end{pmatrix}$$

and

$$M^3 = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 2I \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 3I \\ 0 & I \end{pmatrix}$$

so, $M^k = \begin{pmatrix} I & kI \\ 0 & I \end{pmatrix}$, or explicitly

$$M^k = \begin{pmatrix} 1 & 0 & 0 & k & 0 & 0 \\ 0 & 1 & 0 & 0 & k & 0 \\ 0 & 0 & 1 & 0 & 0 & k \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\,.$$

7. (a) Whenever $\det M = ad - bc \neq 0$.

   (b) Unit determinant bit matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

   (c) Bit matrices with vanishing determinant:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

   As a check, count that the total number of 2×2 *bit matrices is* $2^{(\text{number of entries})} = 2^4 = 16$.

   (d) To disprove this statement, we just need to find a single counterexample. All the unit determinant examples above are actually row equivalent to the identity matrix, so focus on the bit matrices with vanishing determinant. Then notice (for example), that

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \not\sim \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

   So we have found a pair of matrices that are not row equivalent but do have the same determinant. It follows that the statement is false.

8. We can call a function $f\colon V \longrightarrow W$ *linear* if the sets V and W are vector spaces and f obeys

$$f(\alpha u + \beta v) = \alpha f(u) + \beta f(v),$$

   for all $u, v \in V$ and $\alpha, \beta \in \mathbb{R}$.

   Now, integration is a linear transformation from the space V of all integrable functions (don't be confused between the definition of a linear function above, and integrable functions $f(x)$ which here are the vectors in V) to the real numbers $\mathbb{R}$, because $\int_{-\infty}^{\infty}(\alpha f(x) + \beta g(x))dx = \alpha \int_{-\infty}^{\infty} f(x)dx + \beta \int_{-\infty}^{\infty} g(x)dx$.

9. The four main ingredients are (i) a set V of vectors, (ii) a number field K (usually $K = \mathbb{R}$), (iii) a rule for adding vectors (vector addition) and (iv) a way to multiply vectors by a number to produce a new vector (scalar multiplication). There are, of course, ten rules that these four ingredients must obey.

   (a) This is not a vector space. Notice that distributivity of scalar multiplication requires $2u = (1+1)u = u+u$ for any vector u but

   $$2 \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2a & b \\ 2c & d \end{pmatrix}$$

   which does *not* equal

   $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix} .$$

   This could be repaired by taking

   $$k \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix} .$$

   (b) This is a vector space. *Although, the question does not ask you to, it is a useful exercise to verify that all ten vector space rules are satisfied.*

   (c) This is not a vector space for many reasons. An easy one is that $(1, -1, 0)$ and $(-1, 1, 0)$ are both in the space, but their sum $(0, 0, 0)$ is not (*i.e.*, additive closure fails). The easiest way to repair this would be to drop the requirement that there be at least one entry equaling 1.

10. (i) Thanks to multiplicative closure, if $u \in U$, so is $(-1) \cdot u$. But $(-1) \cdot u + u = (-1) \cdot u + 1 \cdot u = (-1+1) \cdot u = 0.u = 0$ (at each step in this chain of equalities we have used the fact that V is a vector space and therefore can use its vector space rules). In particular, this means that the zero vector of V is in U and is its zero vector also. (ii) Also, in V, for each u there is an element $-u$ such that $u + (-u) = 0$. But by additive close, $(-u)$ must also be in U, thus every $u \in U$ has an additive inverse.

(a) This is a vector space. First we check additive closure: let $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ and $\begin{pmatrix} z \\ w \\ 0 \end{pmatrix}$ be arbitrary vectors in U. But since $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} z \\ w \\ 0 \end{pmatrix} = \begin{pmatrix} x+z \\ y+w \\ 0 \end{pmatrix}$, so is their sum (because vectors in U are those whose third component vanishes). Multiplicative closure is similar: for any $\alpha \in \mathbb{R}$, $\alpha \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \\ 0 \end{pmatrix}$, which also has no third component, so is in U.

(b) This is not a vector space for various reasons. A simple one is that $u = \begin{pmatrix} 1 \\ 0 \\ z \end{pmatrix}$ is in U but the vector $u + u = \begin{pmatrix} 2 \\ 0 \\ 2z \end{pmatrix}$ is not in U (it has a 2 in the first component, but vectors in U always have a 1 there).

B Sample Midterm II Problems and Solutions

1. Find an LU decomposition for the matrix
$$\begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 3 & 2 & 2 \\ -1 & -3 & -4 & 6 \\ 0 & 4 & 7 & -2 \end{pmatrix}$$
Use your result to solve the system
$$\left\{ \begin{array}{rcrcrcrl} x & + & y & - & z & + & 2w & = 7 \\ x & + & 3y & + & 2z & + & 2w & = 6 \\ -x & - & 3y & - & 4z & + & 6w & = 12 \\ & & 4y & + & 7z & - & 2w & = -7 \end{array} \right.$$

2. Let
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$
Compute $\det A$. Find all solutions to (i) $AX = 0$ and (ii) $AX = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ for the vector $X \in \mathbb{R}^3$. Find, but do not solve, the characteristic polynomial of A.

3. Let M be any 2×2 matrix. Show
$$\det M = -\frac{1}{2}\mathrm{tr}M^2 + \frac{1}{2}(\mathrm{tr}M)^2 \,.$$

4. *The permanent:* Let $M = (M^i_j)$ be an $n \times n$ matrix. An operation producing a single number from M similar to the determinant is the "permanent"
$$\mathrm{perm}\, M = \sum_{\sigma} M^1_{\sigma(1)} M^2_{\sigma(2)} \cdots M^n_{\sigma(n)} \,.$$

For example

$$\operatorname{perm}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad + bc\,.$$

Calculate

$$\operatorname{perm}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\,.$$

What do you think would happen to the permanent of an $n \times n$ matrix M if (include a *brief* explanation with each answer):

(a) You multiplied M by a number λ.

(b) You multiplied a row of M by a number λ.

(c) You took the transpose of M.

(d) You swapped two rows of M.

5. Let X be an $n \times 1$ matrix subject to

$$X^T X = (1)\,,$$

and define

$$H = I - 2XX^T\,,$$

(where I is the $n \times n$ identity matrix). Show

$$H = H^T = H^{-1}.$$

6. Suppose λ is an eigenvalue of the matrix M with associated eigenvector v. Is v an eigenvector of M^k (where k is any positive integer)? If so, what would the associated eigenvalue be?

Now suppose that the matrix N is *nilpotent*, *i.e.*

$$N^k = 0$$

for some integer $k \geq 2$. Show that 0 is the only eigenvalue of N.

7. Let $M = \begin{pmatrix} 3 & -5 \\ 1 & -3 \end{pmatrix}$. Compute M^{12}. (Hint: $2^{12} = 4096$.)

8. *The Cayley Hamilton Theorem*: Calculate the characteristic polynomial $P_M(\lambda)$ of the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Now compute the matrix polynomial $P_M(M)$. What do you observe? Now suppose the $n \times n$ matrix A is "similar" to a diagonal matrix D, in other words

$$A = P^{-1} D P$$

for some invertible matrix P and D is a matrix with values $\lambda_1, \lambda_2, \ldots \lambda_n$ along its diagonal. Show that the two matrix polynomials $P_A(A)$ and $P_A(D)$ are similar (*i.e.* $P_A(A) = P^{-1} P_A(D) P$). Finally, compute $P_A(D)$, what can you say about $P_A(A)$?

9. *Define* what it means for a set U to be a subspace of a vector space V. Now let U and W be subspaces of V. Are the following also subspaces? (Remember that $\cup$ means "union" and $\cap$ means "intersection".)

   (a) $U \cup W$

   (b) $U \cap W$

   In each case *draw* examples in $\mathbb{R}^3$ that justify your answers. If you answered "yes" to either part also give a general explanation why this is the case.

10. *Define* what it means for a set of vectors $\{v_1, v_2, \ldots, v_n\}$ to (i) be linearly independent, (ii) span a vector space V and (iii) be a basis for a vector space V.

    Consider the following vectors in $\mathbb{R}^3$

$$u = \begin{pmatrix} -1 \\ -4 \\ 3 \end{pmatrix}, \qquad v = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}, \qquad w = \begin{pmatrix} 10 \\ 7 \\ h+3 \end{pmatrix}.$$

    For which values of h is $\{u, v, w\}$ a basis for $\mathbb{R}^3$?

Solutions

1.

$$\begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 3 & 2 & 2 \\ -1 & -3 & -4 & 6 \\ 0 & 4 & 7 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 2 & 3 & 0 \\ 0 & -2 & -5 & 8 \\ 0 & 4 & 7 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & -2 & 8 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 2 & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & -2 & 8 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

To solve $MX = V$ using $M = LU$ we first solve $LW = V$ whose augmented matrix reads

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 7 \\ 1 & 1 & 0 & 0 & 6 \\ -1 & -1 & 1 & 0 & 12 \\ 0 & 2 & -\frac{1}{2} & 1 & -7 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 18 \\ 0 & 2 & -\frac{1}{2} & 1 & -7 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 18 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right),$$

from which we can read off W. Now we compute X by solving $UX = W$ with the augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 7 \\ 0 & 2 & 3 & 0 & -1 \\ 0 & 0 & -2 & 8 & 18 \\ 0 & 0 & 0 & 2 & 4 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 7 \\ 0 & 2 & 3 & 0 & -1 \\ 0 & 0 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 7 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

So $x = 1$, $y = 1$, $z = -1$ and $w = 2$.

2.
$$\det A = 1.(2.6 - 3.5) - 1.(2.6 - 3.4) + 1.(2.5 - 2.4) = -1\,.$$

(i) Since det$A \neq 0$, the homogeneous system $AX = 0$ only has the solution $X = 0$. (ii) It is efficient to compute the adjoint

$$\text{adj } A = \begin{pmatrix} -3 & 0 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{pmatrix}^T = \begin{pmatrix} -3 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

Hence

$$A^{-1} = \begin{pmatrix} 3 & 1 & -1 \\ 0 & -2 & 1 \\ -2 & 1 & 0 \end{pmatrix}.$$

Thus

$$X = \begin{pmatrix} 3 & 1 & -1 \\ 0 & -2 & 1 \\ -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}.$$

Finally,

$$P_A(\lambda) = -\det \begin{pmatrix} 1-\lambda & 1 & 1 \\ 2 & 2-\lambda & 3 \\ 4 & 5 & 6-\lambda \end{pmatrix}$$

$$= -\Big[(1-\lambda)[(2-\lambda)(6-\lambda) - 15] - [2.(6-\lambda) - 12] + [10 - 4.(2-\lambda)]\Big]$$

$$= \lambda^3 - 9\lambda^2 - \lambda + 1\,.$$

3. Call $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then det$M = ad - bc$, yet

$$-\frac{1}{2}\operatorname{tr} M^2 + \frac{1}{2}(\operatorname{tr} M)^2 = -\frac{1}{2}\operatorname{tr}\begin{pmatrix} a^2+bc & * \\ * & bc+d^2 \end{pmatrix} - \frac{1}{2}(a+d)^2$$

$$= -\frac{1}{2}(a^2 + 2bc + d^2) + \frac{1}{2}(a^2 + 2ad + d^2) = ad - bc\,,$$

which is what we were asked to show.

4.

$$\operatorname{perm}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 1.(5.9 + 6.8) + 2.(4.9 + 6.7) + 3.(4.8 + 5.7) = 450\,.$$

(a) Multiplying M by λ replaces every matrix element $M^i_{\sigma(j)}$ in the formula for the permanent by $\lambda M^i_{\sigma(j)}$, and therefore produces an overall factor λ^n.

(b) Multiplying the i^{th} row by λ replaces $M^i_{\sigma(j)}$ in the formula for the permanent by $\lambda M^i_{\sigma(j)}$. Therefore the permanent is multiplied by an overall factor λ.

(c) The permanent of a matrix transposed equals the permanent of the original matrix, because in the formula for the permanent this amounts to summing over permutations of rows rather than columns. But we could then sort the product $M_1^{\sigma(1)} M_2^{\sigma(2)} \ldots M_n^{\sigma(n)}$ back into its original order using the inverse permutation σ^{-1}. But summing over permutations is equivalent to summing over inverse permutations, and therefore the permanent is unchanged.

(d) Swapping two rows also leaves the permanent unchanged. The argument is almost the same as in the previous part, except that we need only reshuffle two matrix elements $M^j_{\sigma(i)}$ and $M^i_{\sigma(j)}$ (in the case where rows i and j were swapped). Then we use the fact that summing over all permutations σ or over all permutations $\widetilde{\sigma}$ obtained by swapping a pair in σ are equivalent operations.

5. Firstly, lets call $(1) = 1$ (the 1×1 identity matrix). Then we calculate

$$H^T = (I-2XX^T)^T = I^T-2(XX^T)^T = I-2(X^T)^T X^T = I-2XX^T = H\,,$$

which demonstrates the first equality. Now we compute

$$H^2 = (I-2XX^T)(I-2XX^T) = I - 4XX^T + 4XX^TXX^T$$

$$= I - 4XX^T + 4X(X^TX)X^T = I - 4XX^T + 4X.1.X^T = I\,.$$

So, since $HH = I$, we have $H^{-1} = H$.

6. We know $Mv = \lambda v$. Hence

$$M^2 v = MMv = M\lambda v = \lambda M v = \lambda^2 v\,,$$

and similarly

$$M^k v = \lambda M^{k-1} v = \ldots = \lambda^k v\,.$$

So v is an eigenvector of M^k with eigenvalue λ^k.

Now let us assume v is an eigenvector of the nilpotent matrix N with eigenvalue λ. Then from above

$$N^k v = \lambda^k v$$

but by nilpotence, we also have

$$N^k v = 0$$

Hence $\lambda^k v = 0$ and v (being an eigenvector) cannot vanish. Thus $\lambda^k = 0$ and in turn $\lambda = 0$.

7. Let us think about the eigenvalue problem $Mv = \lambda v$. This has solutions when

$$0 = \det \begin{pmatrix} 3-\lambda & -5 \\ 1 & -3-\lambda \end{pmatrix} = \lambda^2 - 4 \Rightarrow \lambda = \pm 2\,.$$

The associated eigenvalues solve the homogeneous systems (in augmented matrix form)

$$\left(\begin{array}{cc|c} 1 & -5 & 0 \\ 1 & -5 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right) \text{ and } \left(\begin{array}{cc|c} 5 & -5 & 0 \\ 1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right),$$

respectively, so are $v_2 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ and $v_{-2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Hence $M^{12} v_2 = 2^{12} v_2$ and $M^{12} v_{-2} = (-2)^{12} v_{-2}$. Now, $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{x-y}{4} \begin{pmatrix} 5 \\ 1 \end{pmatrix} - \frac{x-5y}{4} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (this was obtained by solving the linear system $av_2 + bv_{-2} =$ for a and b). Thus

$$M \begin{pmatrix} x \\ y \end{pmatrix} = \frac{x-y}{4} M v_2 - \frac{x-5y}{4} M v_{-2}$$

$$= 2^{12} \Big(\frac{x-y}{4} v_2 - \frac{x-5y}{4} v_{-2} \Big) = 2^{12} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Thus

$$M^{12} = \begin{pmatrix} 4096 & 0 \\ 0 & 4096 \end{pmatrix}.$$

If you understand the above explanation, then you have a good understanding of diagonalization. A quicker route is simply to observe that $M^2 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$.

8.
$$P_M(\lambda) = (-1)^2 \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (\lambda - a)(\lambda - d) - bc\,.$$

Thus
$$P_M(M) = (M - aI)(M - dI) - bcI$$
$$= \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \right) - \begin{pmatrix} bc & 0 \\ 0 & bc \end{pmatrix}$$
$$= \begin{pmatrix} 0 & b \\ c & d-a \end{pmatrix} \begin{pmatrix} a-d & b \\ c & 0 \end{pmatrix} - \begin{pmatrix} bc & 0 \\ 0 & bc \end{pmatrix} = 0\,.$$

Observe that any 2×2 matrix is a zero of its own characteristic polynomial (*in fact this holds for square matrices of any size*).

Now if $A = P^{-1}DP$ then $A^2 = P^{-1}DPP^{-1}DP = P^{-1}D^2P$. Similarly $A^k = P^{-1}D^kP$. So for *any* matrix polynomial we have

$$\begin{aligned} & A^n + c_1 A^{n-1} + \cdots c_{n-1}A + c_n I \\ = \;& P^{-1}D^nP + c_1 P^{-1}D^{n-1}P + \cdots c_{n-1}P^{-1}DP + c_n P^{-1}P \\ = \;& P^{-1}(D^n + c_1 D^{n-1} + \cdots c_{n-1}D + c_n I)P\,. \end{aligned}$$

Thus we may conclude $P_A(A) = P^{-1}P_A(D)P$.

Now suppose $D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \lambda_n \end{pmatrix}$. Then

$$P_A(\lambda) = \det(\lambda I - A) = \det(\lambda P^{-1}IP - P^{-1}DP) = \det P . \det(\lambda I - D) . \det P$$

$$= \det(\lambda I - D) = \det \begin{pmatrix} \lambda - \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda - \lambda_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \lambda - \lambda_n \end{pmatrix}$$
$$= (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n)\,.$$

Thus we see that λ_1, $\lambda_2, \ldots, \lambda_n$ are the eigenvalues of M. Finally we compute
$$P_A(D) = (D - \lambda_1)(D - \lambda_2)\ldots(D - \lambda_n)$$

$$= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \lambda_n \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \lambda_n \end{pmatrix} \cdots \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 0 \end{pmatrix} = 0\,.$$

We conclude the $P_M(M) = 0$.

9. A subset of a vector space is called a subspace if it itself is a vector space, using the rules for vector addition and scalar multiplication inherited from the original vector space.

   (a) So long as $U \neq U \cup W \neq W$ the answer is *no*. Take, for example, U to be the x-axis in $\mathbb{R}^2$ and W to be the y-axis. Then $\begin{pmatrix} 1 & 0 \end{pmatrix} \in U$ and $\begin{pmatrix} 0 & 1 \end{pmatrix} \in W$, but $\begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix} \notin U \cup W$. So $U \cup W$ is not additively closed and is not a vector space (and thus not a subspace). It is easy to draw the example described.

   (b) Here the answer is always *yes*. The proof is not difficult. Take a vector u and w such that $u \in U \cap W \ni w$. This means that *both* u and w are in *both* U and W. But, since U is a vector space, $\alpha u + \beta w$ is also in U. Similarly, $\alpha u + \beta w \in W$. Hence $\alpha u + \beta w \in U \cap W$. So closure holds in $U \cap W$ and this set is a subspace by the subspace theorem. Here, a good picture to draw is two planes through the origin in $\mathbb{R}^3$ intersecting at a line (also through the origin).

10. (i) We say that the vectors $\{v_1, v_2, \dots v_n\}$ are linearly independent if there exist *no* constants c^1, $c^2, \dots c^n$ (all non-vanishing) such that $c^1 v_1 + c^2 v_2 + \cdots + c^n v_n = 0$. Alternatively, we can require that there is no non-trivial solution for scalars c^1, $c^2, \dots, c^n$ to the linear system $c^1 v_1 + c^2 v_2 + \cdots + c^n v_n = 0$. (ii) We say that these vectors span a vector space V if the set $\operatorname{span}\{v_1, v_2, \dots v_n\} = \{c^1 v_1 + c^2 v_2 + \cdots + c^n v_n : c^1, c^2, \dots c^n \in \mathbb{R}\} = V$. (iii) We call $\{v_1, v_2, \dots v_n\}$ a basis for V if $\{v_1, v_2, \dots v_n\}$ are linearly independent *and* $\operatorname{span}\{v_1, v_2, \dots v_n\} = V$.

    For u, v, w to be a basis for $\mathbb{R}^3$, we firstly need (the spanning requirement) that any vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ can be written as a linear combination of

u, v and w

$$c^1 \begin{pmatrix} -1 \\ -4 \\ 3 \end{pmatrix} + c^2 \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} + c^3 \begin{pmatrix} 10 \\ 7 \\ h+3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} .$$

The linear independence requirement implies that when $x = y = z = 0$, the only solution to the above system is $c^1 = c^2 = c^3 = 0$. But the above system in matrix language reads

$$\begin{pmatrix} -1 & 4 & 10 \\ -4 & 5 & 7 \\ 3 & 0 & h+3 \end{pmatrix} \begin{pmatrix} c^1 \\ c^2 \\ c^3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} .$$

Both requirements mean that the matrix on the left hand side must be invertible, so we examine its determinant

$$\det \begin{pmatrix} -1 & 4 & 10 \\ -4 & 5 & 7 \\ 3 & 0 & h+3 \end{pmatrix} = -4.(-4.(h+3) - 7.3) + 5.(-1.(h+3) - 10.3)$$

$$= 11(h-3) .$$

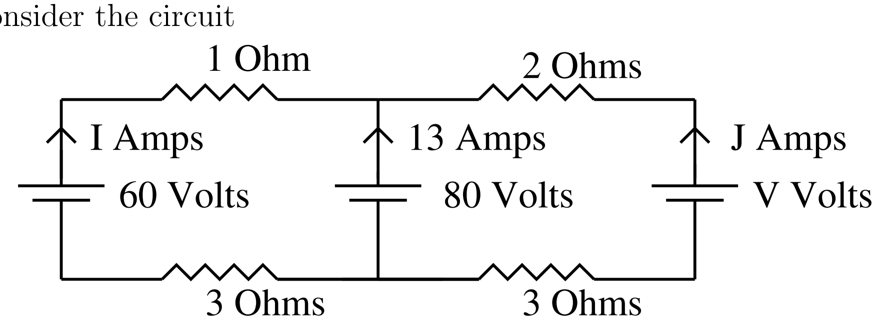
Hence we obtain a basis whenever $h \neq 3$.

C Sample Final Problems and Solutions

1. Define the following terms:

   (a) An *orthogonal matrix*.
   (b) A *basis* for a vector space.
   (c) The *span* of a set of vectors.
   (d) The *dimension* of a vector space.
   (e) An *eigenvector*.
   (f) A *subspace* of a vector space.
   (g) The *kernel* of a linear transformation.
   (h) The *nullity* of a linear transformation.
   (i) The *image* of a linear transformation.
   (j) The *rank* of a linear transformation.
   (k) The *characteristic polynomial* of a square matrix.
   (l) An *equivalence relation*.
   (m) A *homogeneous solution* to a linear system of equations.
   (n) A *particular solution* to a linear system of equations.
   (o) The *general solution* to a linear system of equations.
   (p) The *direct sum* of a pair of subspaces of a vector space.
   (q) The *orthogonal complement* to a subspace of a vector space.

2. *Kirchoff's laws*: Electrical circuits are easy to analyze using systems of equations. The change in voltage (measured in Volts) around any loop due to batteries || and resistors /\/\/\/\ (given by the product of the current measured in Amps and resistance measured in Ohms) equals zero. Also, the sum of currents entering any junction vanishes. Consider the circuit



Find all possible equations for the unknowns I, J and V and then solve for I, J and V. Give your answers with correct units.

3. Suppose M is the matrix of a linear transformation

$$L : U \to V$$

and the vector spaces U and V have dimensions

$$\dim U = n \,, \qquad \dim V = m \,,$$

and

$$m \neq n \,.$$

Also assume

$$\ker L = \{0_U\} \,.$$

(a) How many rows does M have?

(b) How many columns does M have?

(c) Are the columns of M linearly independent?

(d) What size matrix is $M^T M$?

(e) What size matrix is $M M^T$?

(f) Is $M^T M$ invertible?

(g) is $M^T M$ symmetric?

(h) Is $M^T M$ diagonalizable?

(i) Does $M^T M$ have a zero eigenvalue?

(j) Suppose $U = V$ and $\ker L \neq \{0_U\}$. Find an eigenvalue of M.

(k) Suppose $U = V$ and $\ker L \neq \{0_U\}$. Find $\det M$.

4. Consider the system of equations

$$\begin{array}{ccccccccc} x & + & y & + & z & + & w & = & 1 \\ x & + & 2y & + & 2z & + & 2w & = & 1 \\ x & + & 2y & + & 3z & + & 3w & = & 1 \end{array}$$

Express this system as a matrix equation $MX = V$ and then find the solution set by computing an LU decomposition for the matrix M (be sure to use back and forward substitution).

5. Compute the following determinants

$$\det\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix},\ \det\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix},\ \det\begin{pmatrix}1 & 2 & 3 & 4\\ 5 & 6 & 7 & 8\\ 9 & 10 & 11 & 12\\ 13 & 14 & 15 & 16\end{pmatrix},$$

$$\det\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 6 & 7 & 8 & 9 & 10\\ 11 & 12 & 13 & 14 & 15\\ 16 & 17 & 18 & 19 & 20\\ 21 & 22 & 23 & 24 & 25\end{pmatrix}.$$

Now test your skills on

$$\det\begin{pmatrix}1 & 2 & 3 & \cdots & n\\ n+1 & n+2 & n+3 & \cdots & 2n\\ 2n+1 & 2n+2 & 2n+3 & & 3n\\ \vdots & & & \ddots & \vdots\\ n^2-n+1 & n^2-1+2 & n^2-n+3 & \cdots & n^2\end{pmatrix}.$$

Make sure to jot down a few brief notes explaining any clever tricks you use.

6. For which values of a does

$$U = \operatorname{span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\2\\-3\end{pmatrix}, \begin{pmatrix}a\\1\\0\end{pmatrix}\right\} = \mathbb{R}^3\ ?$$

For any special values of a at which $U \neq \mathbb{R}^3$, express the subspace U as the span of the least number of vectors possible. Give the dimension of U for these cases and draw a picture showing U inside $\mathbb{R}^3$.

7. *Vandermonde determinant:* Calculate the following determinants

$$\det\begin{pmatrix}1 & x\\ 1 & y\end{pmatrix},\quad \det\begin{pmatrix}1 & x & x^2\\ 1 & y & y^2\\ 1 & z & z^2\end{pmatrix},\quad \det\begin{pmatrix}1 & x & x^2 & x^3\\ 1 & y & y^2 & y^3\\ 1 & z & z^2 & z^3\\ 1 & w & w^2 & w^3\end{pmatrix}.$$

Be sure to factorize you answers, if possible.

Challenging: Compute the determinant

$$\det\begin{pmatrix}1 & x_1 & (x_1)^2 & \cdots & (x_1)^{n-1}\\ 1 & x_2 & (x_2)^2 & \cdots & (x_2)^{n-1}\\ 1 & x_3 & (x_3)^2 & \cdots & (x_3)^{n-1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & x_n & (x_n)^2 & \cdots & (x_n)^{n-1}\end{pmatrix}.$$

8. (a) Do the vectors $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}3\\2\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ form a basis for $\mathbb{R}^3$? *Be sure to justify your answer.*

   (b) Find a basis for $\mathbb{R}^4$ that includes the vectors $\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$ and $\begin{pmatrix}4\\3\\2\\1\end{pmatrix}$.

   (c) Explain in words how to generalize your computation in part (b) to obtain a basis for $\mathbb{R}^n$ that includes a given pair of (linearly independent) vectors u and v.

9. Elite NASA engineers determine that if a satellite is placed in orbit starting at a point $\mathcal{O}$, it will return exactly to that same point after one orbit of the earth. Unfortunately, if there is a small mistake in the original location of the satellite, which the engineers label by a vector X in $\mathbb{R}^3$ with origin[18] at $\mathcal{O}$, after one orbit the satellite will instead return to some other point $Y \in \mathbb{R}^3$. The engineer's computations show that Y is related to X by a matrix

$$Y = \begin{pmatrix}0 & \frac{1}{2} & 1\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\ 1 & \frac{1}{2} & 0\end{pmatrix} X\,.$$

   (a) Find all eigenvalues of the above matrix.

[18]This is a spy satellite. The exact location of $\mathcal{O}$, the orientation of the coordinate axes in $\mathbb{R}^3$ and the unit system employed by the engineers are CIA secrets.

(b) Determine *all* possible eigenvectors associated with each eigenvalue.

Let us assume that the rule found by the engineers applies to all subsequent orbits. Discuss case by case, what will happen to the satellite if the initial mistake in its location is in a direction given by an eigenvector.

10. In this problem the scalars in the vector spaces are bits ($0, 1$ with $1+1 = 0$). The space B^k is the vector space of bit-valued, k-component column vectors.

(a) Find a basis for B^3.

(b) Your answer to part (a) should be a list of vectors v_1, $v_2, \ldots v_n$. What number did you find for n?

(c) How many elements are there in the *set* B^3.

(d) What is the dimension of the vector space B^3.

(e) Suppose $L : B^3 \to B = \{0, 1\}$ is a linear transformation. Explain why specifying $L(v_1)$, $L(v_2), \ldots, L(v_n)$ completely determines L.

(f) Use the notation of part (e) to list all linear transformations

$$L : B^3 \to B\,.$$

How many different linear transformations did you find? Compare your answer to part (c).

(g) Suppose $L_1 : B^3 \to B$ and $L_2 : B^3 \to B$ are linear transformations, and α and β are bits. Define a new map $(\alpha L_1 + \beta L_2) : B^3 \to B$ by

$$(\alpha L_1 + \beta L_2)(v) = \alpha L_1(v) + \beta L_2(v).$$

Is this map a linear transformation? Explain.

(h) Do you think the set of all linear transformations from B^3 to B is a vector space using the addition rule above? If you answer yes, give a basis for this vector space and state its dimension.

11. A team of distinguished, post-doctoral engineers analyzes the design for a bridge across the English channel. They notice that the force on the center of the bridge when it is displaced by an amount $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is given by

$$F = \begin{pmatrix} -x - y \\ -x - 2y - z \\ -y - z \end{pmatrix} .$$

Moreover, having read Newton's Principiæ, they know that force is proportional to acceleration so that[19]

$$F = \frac{d^2X}{dt^2} .$$

Since the engineers are worried the bridge might start swaying in the heavy channel winds, they search for an oscillatory solution to this equation of the form[20]

$$X = \cos(\omega t) \begin{pmatrix} a \\ b \\ c \end{pmatrix} .$$

(a) By plugging their proposed solution in the above equations the engineers find an eigenvalue problem

$$M \begin{pmatrix} a \\ b \\ c \end{pmatrix} = -\omega^2 \begin{pmatrix} a \\ b \\ c \end{pmatrix} .$$

Here M is a 3×3 matrix. Which 3×3 matrix M did the engineers find? *Justify your answer.*

(b) Find the eigenvalues and eigenvectors of the matrix M.

(c) The number $|\omega|$ is often called a *characteristic frequency*. What characteristic frequencies do you find for the proposed bridge?

[19]The bridge is intended for French and English military vehicles, so the exact units, coordinate system and constant of proportionality are state secrets.

[20]Here, a, b, c and ω are constants which we aim to calculate.

(d) Find an orthogonal matrix P such that $MP = PD$ where D is a diagonal matrix. *Be sure to also state your result for D.*

(e) Is there a direction in which displacing the bridge yields no force? If so give a vector in that direction. *Briefly* evaluate the quality of this bridge design.

12. *Conic Sections*: The equation for the most general conic section is given by

$$ax^2 + 2bxy + dy^2 + 2cx + 2ey + f = 0\,.$$

Our aim is to analyze the solutions to this equation using matrices.

(a) Rewrite the above quadratic equation as one of the form

$$X^T MX + X^T C + C^T X + f = 0$$

relating an unknown column vector $X = \begin{pmatrix} x \\ y \end{pmatrix}$, its transpose X^T, a 2×2 matrix M, a constant column vector C and the constant f.

(b) Does your matrix M obey any special properties? Find its eigenvalues. You may call your answers λ and μ for the rest of the problem to save writing.

For the rest of this problem we will focus on central conics for which the matrix M is invertible.

(c) Your equation in part (a) above should be be quadratic in X. Recall that if $m \neq 0$, the quadratic equation $mx^2 + 2cx + f = 0$ can be rewritten by *completing the square*

$$m\Big(x + \frac{c}{m}\Big)^2 = \frac{c^2}{m} - f\,.$$

Being very careful that you are now dealing with matrices, use the same trick to rewrite your answer to part (a) in the form

$$Y^T MY = g.$$

Make sure you give formulas for the new unknown column vector Y and constant g in terms of X, M, C and f. You need not multiply out any of the matrix expressions you find.

If all has gone well, you have found a way to shift coordinates for the original conic equation to a new coordinate system with its origin at the center of symmetry. Our next aim is to rotate the coordinate axes to produce a readily recognizable equation.

(d) Why is the angle between vectors V and W is not changed when you replace them by PV and PW for P any orthogonal matrix?

(e) Explain how to choose an orthogonal matrix P such that $MP = PD$ where D is a diagonal matrix.

(f) For the choice of P above, define our final unknown vector Z by $Y = PZ$. Find an expression for $Y^T MY$ in terms of Z and the eigenvalues of M.

(g) Call $Z = \begin{pmatrix} z \\ w \end{pmatrix}$. What equation do z and w obey? (Hint, write your answer using λ, μ and g.)

(h) Central conics are circles, ellipses, hyperbolae or a pair of straight lines. Give examples of values of (λ, μ, g) which produce each of these cases.

13. Let $L\colon V \to W$ be a linear transformation between finite-dimensional vector spaces V and W, and let M be a matrix for L (with respect to some basis for V and some basis for W). We know that L has an inverse if and only if it is bijective, and we know a lot of ways to tell whether M has an inverse. In fact, L has an inverse if and only if M has an inverse:

    (a) Suppose that L is bijective (i.e., one-to-one and onto).

        i. Show that $\dim V = \operatorname{rank} L = \dim W$.
        ii. Show that 0 is not an eigenvalue of M.
        iii. Show that M is an invertible matrix.

    (b) Now, suppose that M is an invertible matrix.

        i. Show that 0 is not an eigenvalue of M.
        ii. Show that L is injective.
        iii. Show that L is surjective.

14. Captain Conundrum gives Queen Quandary a pair of newborn doves, male and female for her birthday. After one year, this pair of doves breed and produce a pair of dove eggs. One year later these eggs hatch yielding a new pair of doves while the original pair of doves breed again and an additional pair of eggs are laid. Captain Conundrum is very happy because now he will never need to buy the Queen a present ever again!

    Let us say that in year zero, the Queen has no doves. In year one she has one pair of doves, in year two she has two pairs of doves *etc...* Call F_n the number of pairs of doves in years n. For example, $F_0 = 0$, $F_1 = 1$ and $F_2 = 1$. Assume no doves die and that the same breeding pattern continues well into the future. Then $F_3 = 2$ because the eggs laid by the first pair of doves in year two hatch. Notice also that in year three, two pairs of eggs are laid (by the first and second pair of doves). Thus $F_4 = 3$.

    (a) Compute F_5 and F_6.

    (b) Explain why (for any $n \geq 2$) the following *recursion relation* holds
    $$F_n = F_{n-1} + F_{n-2}\,.$$

    (c) Let us introduce a column vector $X_n = \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}$. Compute X_1 and X_2. Verify that these vectors obey the relationship
    $$X_2 = MX_1 \text{ where } M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\,.$$

    (d) Show that $X_{n+1} = MX_n$.

    (e) Diagonalize M. (*I.e.*, write M as a product $M = PDP^{-1}$ where D is diagonal.)

    (f) Find a simple expression for M^n in terms of P, D and P^{-1}.

    (g) Show that $X_{n+1} = M^n X_1$.

    (h) The number
    $$\varphi = \frac{1+\sqrt{5}}{2}$$
    is called the *golden ratio*. Write the eigenvalues of M in terms of φ.

(i) Put your results from parts (c), (f) and (g) together (along with a short matrix computation) to find the formula for the number of doves F_n in year n expressed in terms of φ, $1-\varphi$ and n.

15. Use Gram–Schmidt to find an orthonormal basis for

$$\operatorname{span}\left\{\begin{pmatrix}1\\1\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\\2\end{pmatrix}\right\}.$$

16. Let M be the matrix of a linear transformation $L : V \to W$ in given bases for V and W. Fill in the blanks below with one of the following six vector spaces: V, W, $\ker L$, $\big(\ker L\big)^{\perp}$, $\operatorname{im} L$, $\big(\operatorname{im} L\big)^{\perp}$.

(a) The columns of M span ______ in the basis given for ______.

(b) The rows of M span ______ in the basis given for ______.

Suppose

$$M = \begin{pmatrix}1&2&1&3\\2&1&-1&2\\1&0&0&-1\\4&1&-1&0\end{pmatrix}$$

is the matrix of L in the bases $\{v_1, v_2, v_3, v_4\}$ for V and $\{w_1, w_2, w_3, w_4\}$ for W. Find bases for $\ker L$ and $\operatorname{im} L$. Use the dimension formula to check your result.

17. Captain Conundrum collects the following data set

y	x
5	-2
2	-1
0	1
3	2

which he believes to be well-approximated by a parabola

$$y = ax^2 + bx + c\,.$$

(a) Write down a system of four linear equations for the unknown coefficients a, b and c.

(b) Write the augmented matrix for this system of equations.

(c) Find the reduced row echelon form for this augmented matrix.

(d) Are there any solutions to this system?

(e) Find the least squares solution to the system.

(f) What value does Captain Conundrum predict for y when $x = 2$?

18. Suppose you have collected the following data for an experiment

x	y
x_1	y_1
x_2	y_2
x_3	y_3

and believe that the result is well modeled by a straight line

$$y = mx + b\,.$$

(a) Write down a linear system of equations you could use to find the slope m and constant term b.

(b) Arrange the unknowns (m, b) in a column vector X and write your answer to (a) as a matrix equation

$$MX = V\,.$$

Be sure to give explicit expressions for the matrix M and column vector V.

(c) For a generic data set, would you expect your system of equations to have a solution? *Briefly* explain your answer.

(d) Calculate $M^T M$ and $(M^T M)^{-1}$ (for the latter computation, state the condition required for the inverse to exist).

(e) Compute the least squares solution for m and b.

(f) The least squares method determines a vector X that minimizes the length of the vector $V - MX$. Draw a rough sketch of the three data points in the (x, y)-plane as well as their least squares fit. Indicate how the components of $V - MX$ could be obtained from your picture.

Solutions

1. You can find the definitions for all these terms by consulting the index of these notes.

2. Both junctions give the same equation for the currents

$$I + J + 13 = 0\,.$$

   There are three voltage loops (one on the left, one on the right and one going around the outside of the circuit). Respectively, they give the equations

$$\begin{aligned}60 - I - 80 - 3I &= 0\\ 80 + 2J - V + 3J &= 0\\ 60 - I + 2J - V + 3J - 3I &= 0 \quad . \end{aligned} \tag{2}$$

   The above equations are easily solved (either using an augmented matrix and row reducing, or by substitution). The result is $I = -5$ Amps, $J = -8$ Amps, $V = 40$ Volts.

3. (a) m.
   (b) n.
   (c) Yes.
   (d) $n \times n$.
   (e) $m \times m$.
   (f) Yes. This relies on $\ker M = 0$ because if $M^T M$ had a non-trivial kernel, then there would be a non-zero solution X to $M^T M X = 0$. But then by multiplying on the left by X^T we see that $||MX|| = 0$. This in turn implies $MX = 0$ which contradicts the triviality of the kernel of M.
   (g) Yes because $\big(M^T M\big)^T = M^T (M^T)^T = M^T M$.
   (h) Yes, all symmetric matrices have a basis of eigenvectors.
   (i) No, because otherwise it would not be invertible.
   (j) Since the kernel of L is non-trivial, M must have 0 as an eigenvalue.

(k) Since M has a zero eigenvalue in this case, its determinant must vanish. I.e., $\det M = 0$.

4. To begin with the system becomes

$$\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 2 & 2 & 2\\ 1 & 2 & 3 & 3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\\ w\end{pmatrix} = \begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$$

Then

$$M = \begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 2 & 2 & 2\\ 1 & 2 & 3 & 3\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 1 & 1\\ 0 & 1 & 2 & 2\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 1 & 1\\ 0 & 0 & 1 & 1\end{pmatrix} = LU$$

So now $MX = V$ becomes $LW = V$ where $W = UX = \begin{pmatrix}a\\ b\\ c\end{pmatrix}$ (say).

Thus we solve $LW = V$ by forward substitution

$$a = 1,\ a + b = 1,\ a + b + c = 1 \Rightarrow a = 1, b = 0, c = 0\,.$$

Now solve $UX = W$ by back substitution

$$x + y + z + w = 1,\ y + z + w = 0,\ z + w = 0$$

$$\Rightarrow w = \mu \text{ (arbitrary)}, z = -\mu, y = 0, x = 1\,.$$

The solution set is $\left\{\begin{pmatrix}x\\ y\\ z\\ y\end{pmatrix} = \begin{pmatrix}1\\ 0\\ -\mu\\ \mu\end{pmatrix} : \mu \in \mathbb{R}\right\}$

5. ...

6. If U spans $\mathbb{R}^3$, then we must be able to express any vector $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\in \mathbb{R}^3$ as

$$X = c^1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c^2 \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + c^3 \begin{pmatrix} a \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & a \\ 0 & 2 & 1 \\ 1 & -3 & 0 \end{pmatrix} \begin{pmatrix} c^1 \\ c^2 \\ c^3 \end{pmatrix},$$

for some coefficients c^1, c^2 and c^3. This is a linear system. We could solve for c^1, c^2 and c^3 using an augmented matrix and row operations. However, since we know that $\dim \mathbb{R}^3 = 3$, if U spans $\mathbb{R}^3$, it will also be a basis. Then the solution for c^1, c^2 and c^3 would be unique. Hence, the 3×3 matrix above must be invertible, so we examine its determinant

$$\det \begin{pmatrix} 1 & 1 & a \\ 0 & 2 & 1 \\ 1 & -3 & 0 \end{pmatrix} = 1.(2.0 - 1.(-3)) + 1.(1.1 - a.2) = 4 - 2a\,.$$

Thus U spans $\mathbb{R}^3$ whenever $a \neq 2$. When $a = 2$ we can write the third vector in U in terms of the preceding ones as

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \frac{3}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}.$$

(*You can obtain this result, or an equivalent one by studying the above linear system with $X = 0$, i.e., the associated homogeneous system.*)

The two vectors $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ are clearly linearly independent, so this is the least number of vectors spanning U for this value of a. Also we see that $\dim U = 2$ in this case. Your picture should be a plane in $\mathbb{R}^3$ though the origin containing the vectors $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$.

7.

$$\det \begin{pmatrix} 1 & x \\ 1 & y \end{pmatrix} = y - x\,,$$

$$\det\begin{pmatrix}1 & x & x^2\\ 1 & y & y^2\\ 1 & z & z^2\end{pmatrix} = \det\begin{pmatrix}1 & x & x^2\\ 0 & y-x & y^2-x^2\\ 0 & z-x & z^2-x^2\end{pmatrix}$$
$$= (y-x)(z^2-x^2)-(y^2-x^2)(z-x) = (y-x)(z-x)(z-y)\,.$$

$$\det\begin{pmatrix}1 & x & x^2 & x^3\\ 1 & y & y^2 & y^3\\ 1 & z & z^2 & z^3\\ 1 & w & w^2 & w^3\end{pmatrix} = \det\begin{pmatrix}1 & x & x^2 & x^3\\ 0 & y-x & y^2-x^2 & y^3-x^3\\ 0 & z-x & z^2-x^2 & z^3-x^3\\ 0 & w-x & w^2-x^2 & w^3-x^3\end{pmatrix}$$
$$= \det\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & y-x & y(y-x) & y^2(y-x)\\ 0 & z-x & z(z-x) & z^2(z-x)\\ 0 & w-x & w(w-x) & w^2(w-x)\end{pmatrix}$$
$$= (y-x)(z-x)(w-x)\det\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & y & y^2\\ 0 & 1 & z & z^2\\ 0 & 1 & w & w^2\end{pmatrix}$$
$$= (y-x)(z-x)(w-x)\det\begin{pmatrix}1 & x & x^2\\ 1 & y & y^2\\ 1 & z & z^2\end{pmatrix}$$
$$= (y-x)(z-x)(w-x)(y-x)(z-x)(z-y)\,.$$

From the 4×4 case above, you can see all the tricks required for a general Vandermonde matrix. First zero out the first column by subtracting the first row from all other rows (which leaves the determinant unchanged). Now zero out the top row by subtracting x_1 times the first column from the second column, x_1 times the second column from the third column *etc*. Again these column operations do not change the determinant. Now factor out x_2-x_1 from the second row, x_3-x_1 from the third row, *etc*. This does change the determinant so we write these factors outside the remaining determinant, which is just the same problem but for the $(n-1)\times(n-1)$ case. Iterating the same procedure gives the result

$$\det \begin{pmatrix} 1 & x_1 & (x_1)^2 & \cdots & (x_1)^{n-1} \\ 1 & x_2 & (x_2)^2 & \cdots & (x_2)^{n-1} \\ 1 & x_3 & (x_3)^2 & \cdots & (x_3)^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & (x_n)^2 & \cdots & (x_n)^{n-1} \end{pmatrix} = \prod_{i>j}(x_i - x_j)\,.$$

(Here $\prod$ stands for a multiple product, just like Σ stands for a multiple sum.)

8. ...

9. (a)

$$\det \begin{pmatrix} \lambda & -\frac{1}{2} & -1 \\ -\frac{1}{2} & \lambda-\frac{1}{2} & -\frac{1}{2} \\ -1 & -\frac{1}{2} & \lambda \end{pmatrix} = \lambda\Big((\lambda-\frac{1}{2}\Big)\lambda-\frac{1}{4})+\frac{1}{2}\Big(-\frac{\lambda}{2}-\frac{1}{2}\Big)-\Big(-\frac{1}{4}+\lambda\Big)$$

$$= \lambda^3 - \frac{1}{2}\lambda^2 - \frac{3}{2}\lambda = \lambda(\lambda+1)(\lambda-\frac{3}{2})\,.$$

Hence the eigenvalues are $0, -1, \frac{3}{2}$.

(b) When $\lambda = 0$ we must solve the homogenous system

$$\left(\begin{array}{ccc|c} 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & \frac{1}{2} & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

So we find the eigenvector $\begin{pmatrix} s \\ -2s \\ s \end{pmatrix}$ where $s \neq 0$ is arbitrary.

For $\lambda = -1$

$$\left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 0 \\ 1 & \frac{1}{2} & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

So we find the eigenvector $\begin{pmatrix} -s \\ 0 \\ s \end{pmatrix}$ where $s \neq 0$ is arbitrary.

Finally, for $\lambda = \frac{3}{2}$

$$\left(\begin{array}{ccc|c} -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ 1 & \frac{1}{2} & -\frac{3}{2} & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & -\frac{5}{4} & \frac{5}{4} & 0 \\ 0 & \frac{5}{4} & -\frac{5}{4} & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

So we find the eigenvector $\begin{pmatrix} s \\ s \\ s \end{pmatrix}$ where $s \neq 0$ is arbitrary.

If the mistake X is in the direction of the eigenvector $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, then $Y = 0$. *I.e.*, the satellite returns to the origin $\mathcal{O}$. For all subsequent orbits it will again return to the origin. NASA would be very pleased in this case.

If the mistake X is in the direction $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, then $Y = -X$. Hence the satellite will move to the point opposite to X. After next orbit will move back to X. It will continue this wobbling motion indefinitely. Since this is a stable situation, again, the elite engineers will pat themselves on the back.

Finally, if the mistake X is in the direction $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, the satellite will move to a point $Y = \frac{3}{2}X$ which is further away from the origin. The same will happen for all subsequent orbits, with the satellite moving a factor 3/2 further away from $\mathcal{O}$ each orbit (in reality, after several orbits, the approximations used by the engineers in their calculations probably fail and a new computation will be needed). In this case, the satellite will be lost in outer space and the engineers will likely lose their jobs!

10. (a) A basis for B^3 is $\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$

(b) 3.

(c) $2^3 = 8$.

(d) $\dim B^3 = 3$.

(e) Because the vectors $\{v_1, v_2, v_3\}$ are a basis any element $v \in B^3$ can be written uniquely as $v = b^1 v_1 + b^2 v_2 + b^3 v_3$ for some triplet of bits $\begin{pmatrix}b^1\\b^2\\b^3\end{pmatrix}$. Hence, to compute $L(v)$ we use linearity of L

$$\begin{aligned} L(v) = L(b^1 v_1 + b^2 v_2 + b^3 v_3) &= b^1 L(v_1) + b^2 L(v_2) + b^3 L(v_3) \\ &= \begin{pmatrix} L(v_1) & L(v_2) & L(v_3) \end{pmatrix} \begin{pmatrix}b^1\\b^2\\b^3\end{pmatrix}. \end{aligned}$$

(f) From the notation of the previous part, we see that we can list linear transformations $L : B^3 \to B$ by writing out all possible bit-valued row vectors

$$\begin{gathered} \begin{pmatrix}0 & 0 & 0\end{pmatrix}, \\ \begin{pmatrix}1 & 0 & 0\end{pmatrix}, \\ \begin{pmatrix}0 & 1 & 0\end{pmatrix}, \\ \begin{pmatrix}0 & 0 & 1\end{pmatrix}, \\ \begin{pmatrix}1 & 1 & 0\end{pmatrix}, \\ \begin{pmatrix}1 & 0 & 1\end{pmatrix}, \\ \begin{pmatrix}0 & 1 & 1\end{pmatrix}, \\ \begin{pmatrix}1 & 1 & 1\end{pmatrix}. \end{gathered}$$

There are $2^3 = 8$ different linear transformations $L : B^3 \to B$, exactly the same as the number of elements in B^3.

(g) Yes, essentially just because L_1 and L_2 are linear transformations. In detail for any bits (a, b) and vectors (u, v) in B^3 it is easy to check the linearity property for $(\alpha L_1 + \beta L_2)$

$$(\alpha L_1 + \beta L_2)(au + bv) = \alpha L_1(au + bv) + \beta L_2(au + bv)$$

$$= \alpha a L_1(u) + \alpha b L_1(v) + \beta a L_1(u) + \beta b L_1(v)$$
$$= a(\alpha L_1(u) + \beta L_2(v)) + b(\alpha L_1(u) + \beta L_2(v))$$
$$= a(\alpha L_1 + \beta L_2)(u) + b(\alpha L_1 + \beta L_2)(v)\,.$$

Here the first line used the definition of $(\alpha L_1 + \beta L_2)$, the second line depended on the linearity of L_1 and L_2, the third line was just algebra and the fourth used the definition of $(\alpha L_1 + \beta L_2)$ again.

(h) Yes. The easiest way to see this is the identification above of these maps with bit-valued column vectors. In that notation, a basis is

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \right\}.$$

Since this (spanning) set has three (linearly independent) elements, the vector space of linear maps $B^3 \to B$ has dimension 3. This is an example of a general notion called the *dual vector space*.

11. ...

12. (a) If we call $M = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$, then $X^T M X = ax^2 + 2bxy + dy^2$. Similarly putting $C = \begin{pmatrix} c \\ e \end{pmatrix}$ yields $X^T C + C^T X = 2X \cdot C = 2cx + 2ey$. Thus

$$0 = ax^2 + 2bxy + dy^2 + 2cx + 2ey + f$$
$$= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} c \\ e \end{pmatrix} + \begin{pmatrix} c & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + f\,.$$

(b) Yes, the matrix M is symmetric, so it will have a basis of eigenvectors and is similar to a diagonal matrix of real eigenvalues.

To find the eigenvalues notice that $\det \begin{pmatrix} a-\lambda & b \\ b & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - b^2 = \left(\lambda - \frac{a+d}{2}\right)^2 - b^2 - \left(\frac{a-d}{2}\right)^2$. So the eigenvalues are

$$\lambda = \frac{a+d}{2} + \sqrt{b^2 + \left(\frac{a-d}{2}\right)^2} \text{ and } \mu = \frac{a+d}{2} - \sqrt{b^2 + \left(\frac{a-d}{2}\right)^2}\,.$$

(c) The trick is to write

$$X^T MX+C^T X+X^T C = (X^T+C^T M^{-1})M(X+M^{-1}C)-C^T M^{-1}C\,,$$

so that

$$(X^T + C^T M^{-1})M(X + M^{-1}C) = C^T MC - f\,.$$

Hence $Y = X + M^{-1}C$ and $g = C^T MC - f$.

(d) The cosine of the angle between vectors V and W is given by

$$\frac{V \cdot W}{\sqrt{V \cdot V\, W \cdot W}} = \frac{V^T W}{\sqrt{V^T V\, W^T W}}\,.$$

So replacing $V \to PV$ and $W \to PW$ will always give a factor $P^T P$ inside all the products, but $P^T P = I$ for orthogonal matrices. Hence none of the dot products in the above formula changes, so neither does the angle between V and W.

(e) If we take the eigenvectors of M, normalize them (*i.e.* divide them by their lengths), and put them in a matrix P (as columns) then P will be an orthogonal matrix. (If it happens that $\lambda = \mu$, then we also need to make sure the eigenvectors spanning the two dimensional eigenspace corresponding to λ are orthogonal.) Then, since M times the eigenvectors yields just the eigenvectors back again multiplied by their eigenvalues, it follows that $MP = PD$ where D is the diagonal matrix made from eigenvalues.

(f) If $Y = PZ$, then $Y^T MY = Z^T P^T MPZ = Z^T P^T PDZ = Z^T DZ$ where $D = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$.

(g) Using part (f) and (c) we have

$$\lambda z^2 + \mu w^2 = g\,.$$

(h) When $\lambda = \mu$ and $g/\lambda = R^2$, we get the equation for a circle radius R in the (z, w)-plane. When λ, μ and g are postive, we have the equation for an ellipse. Vanishing g along with λ and μ of opposite signs gives a pair of straight lines. When g is non-vanishing, but λ and μ have opposite signs, the result is a pair of hyperbolæ. These shapes all come from cutting a cone with a plane, and are therefore called conic sections.

13. We show that L is bijective if and only if M is invertible.

    (a) We suppose that L is bijective.

    i. Since L is injective, its kernel consists of the zero vector alone. Hence
    $$L = \dim \ker L = 0.$$
    So by the Dimension Formula,
    $$\dim V = L + \operatorname{rank} L = \operatorname{rank} L.$$
    Since L is surjective, $L(V) = W$. Thus
    $$\operatorname{rank} L = \dim L(V) = \dim W.$$
    Thereby
    $$\dim V = \operatorname{rank} L = \dim W.$$

    ii. Since $\dim V = \dim W$, the matrix M is square so we can talk about its eigenvalues. Since L is injective, its kernel is the zero vector alone. That is, the only solution to $LX = 0$ is $X = 0_V$. But LX is the same as MX, so the only solution to $MX = 0$ is $X = 0_V$. So M does not have zero as an eigenvalue.

    iii. Since $MX = 0$ has no non-zero solutions, the matrix M is invertible.

    (b) Now we suppose that M is an invertible matrix.

    i. Since M is invertible, the system $MX = 0$ has no non-zero solutions. But LX is the same as MX, so the only solution to $LX = 0$ is $X = 0_V$. So L does not have zero as an eigenvalue.

    ii. Since $LX = 0$ has no non-zero solutions, the kernel of L is the zero vector alone. So L is injective.

    iii. Since M is invertible, we must have that $\dim V = \dim W$. By the Dimension Formula, we have
    $$\dim V = L + \operatorname{rank} L$$
    and since $\ker L = \{0_V\}$ we have $L = \dim \ker L = 0$, so
    $$\dim W = \dim V = \operatorname{rank} L = \dim L(V).$$

Since $L(V)$ is a subspace of W with the same dimension as W, it must be equal to W. To see why, pick a basis B of $L(V)$. Each element of B is a vector in W, so the elements of B form a linearly independent set in W. Therefore B is a basis of W, since the size of B is equal to $\dim W$. So $L(V) = \operatorname{span} B = W$. So L is surjective.

14. (a) $F_4 = F_2 + F_3 = 2 + 3 = 5$.

(b) The number of pairs of doves in any given year equals the number of the previous years plus those that hatch and there are as many of them as pairs of doves in the year before the previous year.

(c) $X_1 = \begin{pmatrix} F_1 \\ F_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $X_2 = \begin{pmatrix} F_2 \\ F_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$$MX_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = X_2 \,.$$

(d) We just need to use the recursion relationship of part (b) in the top slot of X_{n+1}:

$$X_{n+1} = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} F_n + F_{n-1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = MX_n \,.$$

(e) Notice M is symmetric so this is guaranteed to work.

$$\det \begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda(\lambda - 1) - 1 = \big(\lambda - \frac{1}{2}\big)^2 - \frac{5}{4} \,,$$

so the eigenvalues are $\frac{1\pm\sqrt{5}}{2}$. Hence the eigenvectors are $\begin{pmatrix} \frac{1\pm\sqrt{5}}{2} \\ 1 \end{pmatrix}$, respectively (notice that $\frac{1+\sqrt{5}}{2} + 1 = \frac{1+\sqrt{5}}{2} . \frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2} + 1 = \frac{1-\sqrt{5}}{2} . \frac{1-\sqrt{5}}{2}$). Thus $M = PDP^{-1}$ with

$$D = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \text{ and } P = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} .$$

(f) $M^n = (PDP^{-1})^n = PDP^{-1}PDP^{-1} \ldots PDP^{-1} = PD^nP^{-1}$.

(g) Just use the matrix recursion relation of part (d) repeatedly:

$$X_{n+1} = MX_n = M^2 X_{n-1} = \cdots = M^n X_1\,.$$

(h) The eigenvalues are $\varphi = \frac{1+\sqrt{5}}{2}$ and $1-\varphi = \frac{1-\sqrt{5}}{2}$.

(i)

$$X_{n+1} = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = M^n X_n = PD^nP^{-1}X_1$$

$$= P\begin{pmatrix} \varphi & 0 \\ 0 & 1-\varphi \end{pmatrix}^n \begin{pmatrix} \frac{1}{\sqrt{5}} & \star \\ -\frac{1}{\sqrt{5}} & \star \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = P\begin{pmatrix} \varphi^n & 0 \\ 0 & (1-\varphi)^n \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{\varphi^n}{\sqrt{5}} \\ -\frac{(1-\varphi)^n}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \star \\ \frac{\varphi^n-(1-\varphi)^n}{\sqrt{5}} \end{pmatrix}.$$

Hence

$$F_n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}\,.$$

These are the famous Fibonacci numbers.

15. Call the three vectors u, v and w, respectively. Then

$$v^\perp = v - \frac{u \cdot v}{u \cdot u}u = v - \frac{3}{4}u = \begin{pmatrix} \frac{1}{4} \\ -\frac{3}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix},$$

and

$$w^\perp = w - \frac{u \cdot w}{u \cdot u}u - \frac{v^\perp \cdot w}{v^\perp \cdot v^\perp}v^\perp = w - \frac{3}{4}u - \frac{\frac{3}{4}}{\frac{3}{4}}v^\perp = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Dividing by lengths, an orthonormal basis for span$\{u, v, w\}$ is

$$\left\{ \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix} \right\}.$$

16. ...

17. ...

18. We show that L is bijective if and only if M is invertible.

   (a) We suppose that L is bijective.

      i. Since L is injective, its kernel consists of the zero vector alone. So
      $$L = \dim \ker L = 0.$$
      So by the Dimension Formula,
      $$\dim V = L + \operatorname{rank} L = \operatorname{rank} L.$$
      Since L is surjective, $L(V) = W$. So
      $$\operatorname{rank} L = \dim L(V) = \dim W.$$
      So
      $$\dim V = \operatorname{rank} L = \dim W.$$

      ii. Since $\dim V = \dim W$, the matrix M is square so we can talk about its eigenvalues. Since L is injective, its kernel is the zero vector alone. That is, the only solution to $LX = 0$ is $X = 0_V$. But LX is the same as MX, so the only solution to $MX = 0$ is $X = 0_V$. So M does not have zero as an eigenvalue.

      iii. Since $MX = 0$ has no non-zero solutions, the matrix M is invertible.

   (b) Now we suppose that M is an invertible matrix.

      i. Since M is invertible, the system $MX = 0$ has no non-zero solutions. But LX is the same as MX, so the only solution to $LX = 0$ is $X = 0_V$. So L does not have zero as an eigenvalue.

      ii. Since $LX = 0$ has no non-zero solutions, the kernel of L is the zero vector alone. So L is injective.

      iii. Since M is invertible, we must have that $\dim V = \dim W$. By the Dimension Formula, we have
      $$\dim V = L + \operatorname{rank} L$$

and since $\ker L = \{0_V\}$ we have $L = \dim \ker L = 0$, so

$$\dim W = \dim V = \operatorname{rank} L = \dim L(V).$$

Since $L(V)$ is a subspace of W with the same dimension as W, it must be equal to W. To see why, pick a basis B of $L(V)$. Each element of B is a vector in W, so the elements of B form a linearly independent set in W. Therefore B is a basis of W, since the size of B is equal to $\dim W$. So $L(V) = \operatorname{span} B = W$. So L is surjective.

19. ...

D Points Vs. Vectors

This is an expanded explanation of this remark. People might interchangeably use the term *point* and *vector* in $\mathbb{R}^n$, however these are not quite the same concept. There is a notion of a point in $\mathbb{R}^n$ representing a vector, and while we can do this in a purely formal (mathematical) sense, we really cannot add two points together (there is the related notion of this using convex combinations, but that is for a different course) or scale a point. We can "subtract" two points which gives us the vector between as done which describing choosing the origin, thus if we take any point P, we can represent it as a vector (based at the origin O) by taking $v = P - O$. Naturally (as we should be able to) we can add vectors to points and get a point back.

To make all of this mathematically (and computationally) rigorous, we "lift" $\mathbb{R}^n$ up to $\mathbb{R}^{n+1}$ (sometimes written as $\widetilde{\mathbb{R}}^n$) by stating that all tuples $\widetilde{p} = (p^1, p^2, \ldots, p^n, 1) \in \mathbb{R}^{n+1}$ correspond to a point $p \in \mathbb{R}^n$ and $\widetilde{v} = (v^1, \ldots, v^n, 0) \in \mathbb{R}^{n+1}$ correspond to a vector $v \in \mathbb{R}^n$. Note that if the last coordinate w is not 0 or 1, then it does not carry meaning in terms of $\mathbb{R}^n$ but just exists in a formal sense. However we can project it down to a point by scaling by $\frac{1}{w}$, and this concept is highly used in rendering computer graphics.

We also do a similar procedure for all matrices acting on $\mathbb{R}^n$ by the following. Let A be a $k \times n$ matrix, then when we lift, we get the following $(k+1) \times (n+1)$ matrix

$$\widetilde{A} = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array} \right).$$

Note that we keep the last coordinate fixed, so we move points to points and vectors to vectors. We can also act on $\mathbb{R}^n$ in a somewhat non-linear fashion by taking matrices of the form

$$\left(\begin{array}{c|c} * & * \\ \hline 0 & 1 \end{array} \right)$$

and this still fixes the last coordinate. For example we can also represent a translation, which is non-linear since it moves the origin, in the direction $v = (v^1, v^2, \ldots, v^n)$ by the following matrix

$$T_v = \left(\begin{array}{c|c} I_n & v \\ \hline 0 & 1 \end{array} \right).$$

where I_n is the $n \times n$ identity matrix. Note that this is an invertible matrix with determinant 1, and it is stronger, a translation is what is known an *isometry* on $\mathbb{R}^n$ (note it is not an isometry on $\mathbb{R}^{n+1}$), an operator where $\|T_v x\| = \|x\|$ for all vectors $x \in \mathbb{R}^n$.

A good exercises to try are to check that lifting $\mathbb{R}^2$ to $\mathbb{R}^3$ allows us to add, subtract, and scale points and vectors as described and generates nonsense when we can't (i.e. adding two points gives us a 2 in the last coordinate, so it is neither a point nor a vector). Another good exercise is to describe all isometries of $\mathbb{R}^2$. As hint, you can get all of them by rotation about the origin, reflection about a single line, and translation.

E Abstract Concepts

Here we will introduce some abstract concepts which are mentioned or used in this book. This is material is more advanced but will be interesting to anybody wanting a deeper understanding of the underlying mathematical structures behind linear algebra. In all cases below, we assume that the given set is closed under the operation(s) introduced.

E.1 Dual Spaces

Definition A *bounded operator* is a linear operator $\phi \colon V \to W$ such that $\|\phi v\|_W \leq C \|v\|_V$ where $C > 0$ is a fixed constant.

Let V be a vector space over $\mathbb{F}$, and a *functional* is a function $\phi \colon V \to \mathbb{F}$.

Definition The *dual space* V^* of a vector space V is the vector space of all bounded linear functionals on V.

There is a natural basis $\{\Lambda_i\}$ for V^* by $\Lambda_i(e_j) = \delta_{ij}$ where $\{e_j\}$ is the canonical (standard) basis for V and δ_{ij} is the *Kronecker delta*, which is 1 if $i = j$ and 0 otherwise. Concretely for a finite dimensional vector space V, we can associate V^* with row vectors w^T as a functional by the matrix multiplication $w^T v$ for vectors $v \in V$. Alternatively we can associated V^* with vectors in V as a functional by taking the usual dot product. So the basis for V^* is e_i^T or $\langle e_i, v \rangle$ for vectors $v \in V$.

E.2 Groups

Definition A *group* is a set G with a single operation $\cdot$ which satisfies the axioms:

- Associativity $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in G$.
- There exists an identity $1 \in G$.
- There exists an inverse $g^{-1} \in G$ for all $g \in G$.

Groups can be finite or infinite, and notice that not alls element in a group must commute (*i.e.*, the order of multiplication can matter). Here are some examples of groups:

- Non-zero real numbers under multiplication.
- All real numbers under addition.
- All invertible $n \times n$ real matrices.
- All $n \times n$ real matrices of determinant 1.
- All permutations of $[1, 2, \ldots, n]$ under compositions.
- Any vector space under addition.

Note that all real numbers under multiplication is not a group since 0 does not have an inverse.

E.3 Fields

Definition A *field* $\mathbb{F}$ is a set with two operations $+$ and $\cdot$ that for all $a, b, c \in \mathbb{F}$ the following axioms are satisfied:

A1. Addition is associative $(a + b) + c = a + (b + c)$.

A2. There exists an additive identity 0.

A3. Addition is commutative $a + b = b + a$.

A4. There exists an additive inverse $-a$.

M1. Multiplication is associative $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

M2. There exists a multiplicative identity 1.

M3. Multiplication is commutative $a \cdot b = b \cdot a$.

M4. There exists a multiplicative inverse a^{-1} if $a \neq 0$.

D. The distributive law holds $a \cdot (b + c) = ab + ac$.

Roughly, all of the above mean that you have notions of $+$, $-$, $\times$ and $\div$ just as for regular real numbers.

Fields are a very beautiful structure; some examples are $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$. We note that every one of the these examples are infinite, however this does not necessarily have to be the case. Let $q \geq 0$ and let $\mathbb{Z}_q$ be the set of remainders of $\mathbb{Z}$ (the set of integers) by dividing by q. We say $\mathbb{Z}_q$ is the set of all a modulo q or $a \mod q$ for short or $a \equiv q$ where $a \in \mathbb{Z}$, and we define addition and multiplication to be their usual counterparts in $\mathbb{Z}$ except we take the result mod q. So for example we have $\mathbb{Z}_2 = \{0, 1\}$ where $1+1 = 2 \equiv 0$ (these are exactly the bits used in bit matrices) and $\mathbb{Z}_3 = \{0, 1, 2\}$ with $1 + 1 = 2$, $2 \cdot 2 = 4 \equiv 1$. Now if p is a prime number, then $\mathbb{Z}_p$ is a field (often written as $\mathbb{Z}_p$). Clearly $\mathbb{Z}_2$ is a field, and from above for $\mathbb{Z}_3$ we have $2^{-1} = 2$, so $\mathbb{Z}_3$ is also a field. For $\mathbb{Z}_5$ we have $2^{-1} = 3$ since $2 \cdot 3 = 6 \equiv 1$ and $4^{-1} = 4$ since $4 \cdot 4 = 16 \equiv 1$. Often when $q = p^n$ where p is a prime, then people will write $\mathbb{F}_q$ to reinforce that it is a field.

E.4 Rings

However $\mathbb{Z}_4$ is not a field since $2 \cdot 2 = 4 \equiv 0$ and $2 \cdot 3 = 6 \equiv 2$. Similarly $\mathbb{Z}$ is not a field since 2 does not have a multiplicative inverse. These are known as *rings*. For rings all of the addition axioms hold, but none of the multiplicative ones must.

Definition A *ring* R is a set with two operations $+$ and $\cdot$ that for all $a, b, c \in R$ the following axioms are satisfied:

A1. Addition is associative $(a + b) + c = a + (b + c)$.

A2. There exists an additive identity 0.

A3. Addition is commutative $a + b = b + a$.

A4. There exists an additive inverse $-a$.

D. The distributive law holds $a \cdot (b+c) = a \cdot b + a \cdot c$ and $(a+b) \cdot c = a \cdot c + b \cdot c$.

Note that when we have axiom M3, then the two equations in axiom D are equivalent.

Clearly all fields are rings, but rings in general are not nearly as nice (for example, in $\mathbb{Z}_4$ two things can be multiplied together to give you 0). An important example of a ring is $\mathbb{F}[x]$, which is the ring of all polynomials in one variable x with coefficients in a field $\mathbb{F}$. Recall that you can do everything you want in a field except divide polynomials, but if you take the modulus with respect to a polynomial which is not a product of two smaller polynomials, you can get a field. We call such polynomials *irreducible*. In other words, you take a polynomial p and you set $p \equiv 0$, thus this is just making sure you don't have $ab \equiv 0$. For example, the polynomial $p(x) = x^2 + 1$ cannot be factored over $\mathbb{R}$ (i.e. with real coefficients), so what you get is actually the same field as $\mathbb{C}$ since we have $x^2 + 1 = 0$ or perhaps more suggestively $x^2 = -1$. This is what is known as a *field extension*; these are the central objects in Galois theory and are denoted $\mathbb{F}(\alpha)$ where α is a root of p.

One final definition: We say that a field $\mathbb{F}$ has characteristic p if $\sum_{i=1}^{p} 1 \equiv 0$ (*i.e.* we sum 1 together p times and return to 0). For example $\mathbb{Z}_3$ has characteristic 3 since $1 + 1 + 1 \equiv 0$, and in general $\mathbb{Z}_p$ has characteristic p.

A good exercise is to find an irreducible degree 2 polynomial p in $\mathbb{Z}_2[x]$, and check that the field extension $\mathbb{Z}_2(\alpha)$ has 4 elements and has characteristic 2 (hence it is not actually $\mathbb{Z}_4$).

E.5 Algebras

Definition An *algebra* A is a vector space over $\mathbb{F}$ with the operation $\cdot$ such that for all $u, v, w \in A$ and $\alpha, \beta \in \mathbb{F}$, we have

D. The distributive law holds $u \cdot (v + w) = u \cdot v + u \cdot w$ and $(u + v) \cdot w = u \cdot w + v \cdot w$.

S. We have $(\alpha v) \cdot (\beta w) = (\alpha\beta)(v \cdot w)$.

Essentially an algebra is a ring that is also a vector space over some field. Or in simpler words, an algebra is a vector space where you can multiply vectors.

For example, all $n \times n$ real matrices $M_n(\mathbb{R})$ is a ring but we can let scalars in $\mathbb{R}$ act on these matrices in their usual way. Another algebra is we can take $M_n(\mathbb{R})$ but take scalars in $\mathbb{C}$ and just formally say iM is another element in this algebra. Another example is $\mathbb{R}^3$ where multiplication is the cross-product $\times$. We note that this is not associative nor commutative under $\times$ and that $v \times v = 0$ (so there are in fact no multiplicative inverses), and there is no multiplicative identity. Lastly, recall that $\mathfrak{sl}_n$ defined here is an algebra under $[,]$.

F Sine and Cosine as an Orthonormal Basis

Definition Let $\Omega \subseteq \mathbb{R}^n$ for some n. Let $L_0^p(\Omega)$ denote the space of all continuous functions $f\colon \Omega \to \mathbb{R}$ (or $\mathbb{C}$) such that if $p < \infty$, then

$$\left(\int_\Omega |f(x)|^p \, dx \right)^{1/p} < \infty,$$

otherwise $|f(x)| < M$ for some fixed M and all $x \in \Omega$.

Note that this is a vector space over $\mathbb{R}$ (or $\mathbb{C}$) under addition (in fact it is an algebra under pointwise multiplication) with norm (the length of the vector)

$$\|f\|_p = \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p}.$$

For example, the space $L_0^1(\mathbb{R})$ is all absolutely integrable functions. However note that not every differentiable function is contained in $L_0^p(\Omega)$; for example we have

$$\int_{\mathbb{R}_+} |1|^p \, dx = \int_0^\infty dx = \lim_{x \to \infty} x = \infty.$$

In particular, we can take S^1, the unit circle in $\mathbb{R}^2$, and to turn this into a valid integral, take $\Omega = [0, 2\pi)$ and take functions $f\colon [0, 2\pi] \to \mathbb{R}$ such that $f(0) = f(2\pi)$ (or more generally for a *periodic* function $f\colon \mathbb{R} \to \mathbb{R}$ where $f(x) = f(x + 2\pi n)$ for all $n \in \mathbb{Z}$). Additionally we can define an inner product on $\mathcal{H} = L_0^2(S^1)$ by taking

$$\langle f, g \rangle = \int_0^{2\pi} f(x) g(x) \, dx,$$

and note that $\langle f, f \rangle = \|f\|_2^2$. So the natural question to ask is what is a good basis for $\mathcal{H}$? The answer is $\sin(nx)$ and $\cos(nx)$ for all $n \in \mathbb{Z}_{\geq 0}$, and in fact, they are orthogonal. First note that

$$\begin{aligned} \langle \sin(mx), \sin(nx) \rangle &= \int_0^{2\pi} \sin(mx) \sin(nx) \, dx \\ &= \int_0^{2\pi} \frac{\cos((m-n)x) - \cos((m+n)x)}{2} \, dx \end{aligned}$$

and if $m \neq n$, then we have

$$\langle \sin(mx), \sin(nx) \rangle = \frac{\sin((m-n)x)}{2(m-n)}\bigg|_0^{2\pi} - \frac{\sin((m+n)x)}{2(m+n)}\bigg|_0^{2\pi} = 0.$$

However if $m = n$, then we have

$$\langle \sin(mx), \sin(mx) \rangle = \int_0^{2\pi} 1 - \frac{\cos(2mx)}{2}\, dx = 2\pi,$$

so $\|\sin(mx)\|_2 = \sqrt{2\pi}$, and similarly we have $\|\cos(mx)\|_2 = \sqrt{2\pi}$. Finally we have

$$\begin{aligned}
\langle \sin(mx), \cos(nx) \rangle &= \int_0^{2\pi} \sin(mx)\cos(nx)\, dx \\
&= \int_0^{2\pi} \frac{\sin((m+n)x) + \sin((m-n)x)}{2}\, dx \\
&= \frac{\cos((m+n)x)}{2(m+n)}\bigg|_0^{2\pi} + \frac{\cos((m-n)x)}{2(m-n)}\bigg|_0^{2\pi} = 0.
\end{aligned}$$

Now it is not immediately apparent that we haven't missed some basis vector, but this is a consequence of the Stone-Weierstrauss theorem. Now only appealing to linear algebra, we have that e^{inx} is a also basis for $L^2(S^1)$ (only over $\mathbb{C}$ though) since

$$\sin(nx) = \frac{e^{inx} - e^{-inx}}{2i}, \quad \cos(nx) = \frac{e^{inx} + e^{inx}}{2}, \quad e^{inx} = \cos(nx) + i\sin(nx)$$

is a linear change of basis.

G Movie Scripts

The authors welcome your feedback on how useful these movies are for helping you learn. We also welcome suggestions for other movie themes. You might even like to try your hand at making your own!

G.1 Introductory Video

```
Three bears go into a cave, two come out.
Would you go in?
```

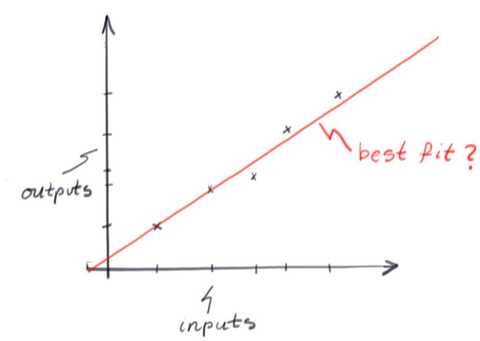
G.2 What is Linear Algebra: Overview

In this course, we start with linear systems

$$\begin{cases} f_1(x_1, \dots, x_m) & = & a_1 \\ \vdots & & \vdots \\ f_n(x_1, \dots, x_m) & = & a_n \,, \end{cases} \tag{3}$$

and discuss how to solve them.

We end with the problem of finding a least squares fit---find the line that best fits a given data set:



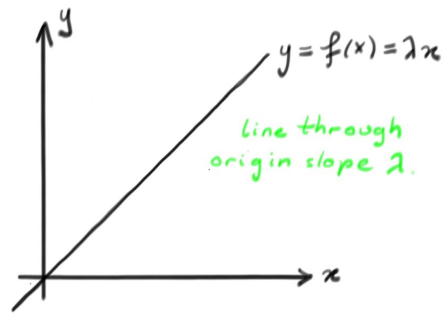
In equation (3) we have n *linear* functions called $f_1, \dots, f_n$, m unknowns $x_1, \dots, x_m$ and n given constants $a_1, \dots, a_n$. We need to say what it means for a function to be linear. In one variable, a linear function obeys the *linearity property*

$$f(a+b) = f(a) + f(b)\,.$$

The solution to this is

$$f(x) = \lambda x\,,$$

for some constant λ. The plot of this is just a straight line through the origin with slope λ



We should also check that our solution obeys the linearity property. The logic is to start with the left hand side $f(a+b)$ and try to turn it into the right hand side $f(a)+f(b)$ using correct manipulations:

$$f(a+b) = \lambda(a+b) = \lambda a + \lambda b = f(a) + f(b)\,.$$

The first step here just plugs $a+b$ into $f(x)$, the second is the distributive property, and in the third we recognize that $\lambda a = f(a)$ and $\lambda b = f(b)$. This proves our claim.

For functions of many variables, linearity must hold for every slot. For a linear function of two variables $f(x,y)$ this means

$$f(a+b, c+d) = f(a,c) + f(b,d)\,.$$

We finish with a question. The plot of $f(x) = \lambda x + \beta$ is a straight line, but does it obey the linearity property?

G.3 What is Linear Algebra: 3×3 Matrix Example

Your friend places a jar on a table and tells you that there is 65 cents in this jar with 7 coins consisting of quarters, nickels, and dimes, and that there are twice as many dimes as quarters. Your friend wants to know how many nickels, dimes, and quarters are in the jar.

We can translate this into a system of the following linear equations:

$$\begin{aligned} 5n+10d+25q &= 65 \\ n+d+q &= 7 \\ d &= 2q \end{aligned}$$

Now we can rewrite the last equation in the form of $-d+2q=0$, and thus express this problem as the matrix equation

$$\begin{pmatrix} 5 & 10 & 25 \\ 1 & 1 & 1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} n \\ d \\ q \end{pmatrix} = \begin{pmatrix} 65 \\ 7 \\ 0 \end{pmatrix}.$$

or as an augmented matrix (see also this script on the notation).

$$\left(\begin{array}{ccc|c} 5 & 10 & 25 & 65 \\ 1 & 1 & 1 & 7 \\ 0 & -1 & 2 & 0 \end{array}\right)$$

Now to solve it, using our original set of equations and by substitution, we have

$$\begin{aligned} 5n+20q+25q = 5n+45q &= 65 \\ n+2q+q = n+3q &= 7 \end{aligned}$$

and by subtracting 5 times the bottom equation from the top, we get

$$45q-15q = 30q = 65-35 = 30$$

and hence $q=1$. Clearly $d=2$, and hence $n=7-2-1=4$. Therefore there are four nickels, two dimes, and one quarter.

G.4 What is Linear Algebra: Hint

Looking at the problem statement we find some important information, first that oranges always have twice as much sugar as apples, and second that the information about the barrel is recorded as (s, f), where $s =$ units of sugar in the barrel and $f =$ number of pieces of fruit in the barrel.

We are asked to find a linear transformation relating this new representation to the one in the lecture, where in the lecture $x =$ the number of apples and $y =$ the number of oranges. This means we must create a system of equations relating the variable x and y to the variables s and f in matrix form. Your answer should be the matrix that transforms one set of variables into the other.

Hint: Let λ represent the amount of sugar in each apple.

1. To find the first equation find a way to relate f to the variables x and y.

2. To find the second equation, use the hint to figure out how much sugar is in x apples, and y oranges in terms of λ. Then write an equation for s using x, y and λ.

G.5 Gaussian Elimination: Augmented Matrix Notation

Why is the augmented matrix

$$\left(\begin{array}{cc|c} 1 & 1 & 27 \\ 2 & -1 & 0 \end{array}\right),$$

equivalent to the system of equations

$$\begin{aligned} x + y &= 27 \\ 2x - y &= 0\,? \end{aligned}$$

Well the augmented matrix is just a new notation for the matrix equation

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 27 \\ 0 \end{pmatrix}$$

and if you review your matrix multiplication remember that

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ 2x - y \end{pmatrix}$$

This means that

$$\begin{pmatrix} x + y \\ 2x - y \end{pmatrix} = \begin{pmatrix} 27 \\ 0 \end{pmatrix}$$

Which is our original equation.

G.6 Gaussian Elimination: Equivalence of Augmented Matrices

Lets think about what it means for the two augmented matrices

$$\left(\begin{array}{cc|c} 1 & 1 & 27 \\ 2 & -1 & 0 \end{array}\right),$$

and

$$\left(\begin{array}{cc|c} 1 & 0 & 9 \\ 0 & 1 & 18 \end{array}\right),$$

to be equivalent?

They are certainly not equal, because they don't match in each component, but since these augmented matrices represent a system, we might want to introduce a new kind of equivalence relation.

Well we could look at the system of linear equations this represents

$$\begin{array}{rcl} x + y & = & 27 \\ 2x - y & = & 0\,? \end{array}$$

and notice that the solution is $x = 9$ and $y = 18$. The other augmented matrix represents the system

$$\begin{array}{rcl} x + 0 \cdot y & = & 9 \\ 0 \cdot x + y & = & 18\,? \end{array}$$

This which clearly has the same solution. The first and second system are related in the sense that their solutions are the same. Notice that it is really nice to have the augmented matrix in the second form, because the matrix multiplication can be done in your head.

G.7 Gaussian Elimination: Hints for Review Questions 4 and 5

The hint for Review Question 4 is simple--just read the lecture on Elementary Row Operations.

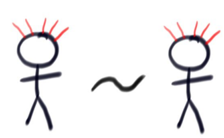
Question 5 looks harder than it actually is:

> Row equivalence of matrices is an example of an *equivalence relation*. Recall that a relation $\sim$ on a set of objects U is an equivalence relation if the following three properties are satisfied:
>
> - Reflexive: For any $x \in U$, we have $x \sim x$.
> - Symmetric: For any $x, y \in U$, if $x \sim y$ then $y \sim x$.
> - Transitive: For any x, y and $z \in U$, if $x \sim y$ and $y \sim z$ then $x \sim z$.
>
> (For a more complete discussion of equivalence relations, see Webwork Homework 0, Problem 4)
>
> Show that row equivalence of augmented matrices is an equivalence relation.

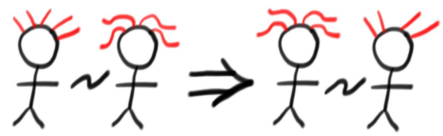
Firstly remember that an equivalence relation is just a more general version of ''equals''. Here we defined row equivalence for augmented matrices whose linear systems have solutions by the property that their solutions are the same.

So this question is really about the word *same*. Lets do a silly example: Lets replace the set of augmented matrices by the set of people who have hair. We will call two people equivalent if they have the same hair color. There are three properties to check:

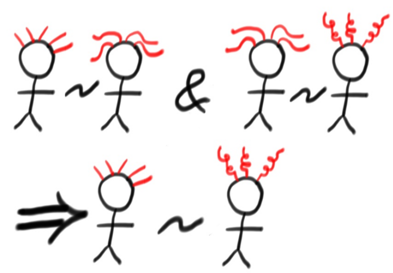
- Reflexive: This just requires that you have the same hair color as yourself so obviously holds.



- Symmetric: If the first person, Bob (say) has the same hair color as a second person Betty(say), then Bob has the same hair color as Betty, so this holds too.



- Transitive: If Bob has the same hair color as Betty (say) and Betty has the same color as Brenda (say), then it follows that Bob and Brenda have the same hair color, so the transitive property holds too and we are done.



G.8 Gaussian Elimination: 3×3 Example

We'll start with the matrix from the What is Linear Algebra: 3×3 Matrix Example which was

$$\left(\begin{array}{ccc|c} 5 & 10 & 25 & 65 \\ 1 & 1 & 1 & 7 \\ 0 & -1 & 2 & 0 \end{array}\right),$$

and recall the solution to the problem was $n = 4$, $d = 2$, and $q = 1$. So as a matrix equation we have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} n \\ d \\ q \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

or as an augmented matrix

$$\left(\begin{array}{ccc|c} 1 & & & 4 \\ & 1 & & 2 \\ & & 1 & 1 \end{array}\right)$$

Note that often in diagonal matrices people will either omit the zeros or write in a single large zero. Now the first matrix is equivalent to the second matrix and is written as

$$\left(\begin{array}{ccc|c} 5 & 10 & 25 & 65 \\ 1 & 1 & 1 & 7 \\ 0 & -1 & 2 & 0 \end{array}\right), \sim \left(\begin{array}{ccc|c} 1 & & & 4 \\ & 1 & & 2 \\ & & 1 & 1 \end{array}\right)$$

since they have the same solutions.

G.9 Elementary Row Operations: Example

We have three basic rules

1. Row Swap
2. Scalar Multiplication
3. Row Sum

Lets look at an example. The system

$$\begin{aligned} 3x + y &= 7 \\ x + 2y &= 4 \end{aligned}$$

is something we learned to solve in high school algebra. Now we can write it in augmented matrix for this way

$$\left(\begin{array}{cc|c} 3 & 1 & 7 \\ 1 & 2 & 4 \end{array}\right).$$

We can see what these operations allow us to do:

1. Row swap allows us to switch the order of rows. In this example there are only two equations, so I will switch them. This will work with a larger system as well, but you have to decide which equations to switch. So we get

   $$\begin{aligned} x + 2y &= 4 \\ 3x + y &= 7 \end{aligned}$$

   The augmented matrix looks like

   $$\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 3 & 1 & 7 \end{array}\right).$$

   Notice that this won't change the solution of the system, but the augmented matrix will look different. This is where we can say that the original augmented matrix is equivalent to the one with the rows swapped. This will work with a larger system as well, but you have to decide which equations, or rows to switch. Make sure that you don't forget to switch the entries in the right-most column.

2. Scalar multiplication allows us to multiply both sides of an equation by a non-zero constant. So if we are starting with

$$\begin{aligned} x + 2y &= 4 \\ 3x + y &= 7 \end{aligned}$$

Then we can multiply the first equation by -3 which is a non-zero scalar. This operation will give us

$$\begin{aligned} -3x + -6y &= -12 \\ 3x + y &= 7 \end{aligned}$$

which has a corresponding augmented matrix

$$\left(\begin{array}{cc|c} -3 & -6 & -12 \\ 3 & 1 & 7 \end{array}\right).$$

Notice that we have multiplied the entire first row by -3, and this changes the augmented matrix, but not the solution of the system. We are not allowed to multiply by zero because it would be like replacing one of the equations with $0 = 0$, effectively destroying the information contained in the equation.

3. Row summing allows us to add one equation to another. In our example we could start with

$$\begin{aligned} -3x + -6y &= -12 \\ 3x + y &= 7 \end{aligned}$$

and replace the first equation with the sum of both equations. So we get

$$\begin{aligned} -3x + 3x + -6y + y &= -12 + 7 \\ 3x + y &= 7, \end{aligned}$$

which after some simplification is translates to

$$\left(\begin{array}{cc|c} 0 & -5 & -5 \\ 3 & 1 & 7 \end{array}\right).$$

When using this row operation make sure that you end up with as many equations as you started with. Here we replaced the first equation with a sum, but the second equation remained untouched.

In the example, notice that the x-terms in the first equation disappeared, which makes it much easier to solve for y. Think about what the next steps for solving this system would be using the language of elementary row operations.

G.10 Elementary Row Operations: Worked Examples

Let us consider that we are given two systems of equations that give rise to the following two (augmented) matrices:

$$\left(\begin{array}{cccc|c} 2 & 5 & 2 & 0 & 2 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 4 & 1 & 0 & 1 \end{array}\right) \qquad \left(\begin{array}{cc|c} 5 & 2 & 9 \\ 0 & 5 & 10 \\ 0 & 3 & 6 \end{array}\right)$$

and we want to find the solution to those systems. We will do so by doing Gaussian elimination.

For the first matrix we have

$$\begin{aligned}
\left(\begin{array}{cccc|c} 2 & 5 & 2 & 0 & 2 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 4 & 1 & 0 & 1 \end{array}\right) \stackrel{R_1 \leftrightarrow R_2}{\sim} & \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 2 & 5 & 2 & 0 & 2 \\ 1 & 4 & 1 & 0 & 1 \end{array}\right) \\
\stackrel{R_2 - 2R_1; R_3 - R_1}{\sim} & \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \end{array}\right) \\
\stackrel{\frac{1}{3}R_2}{\sim} & \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \end{array}\right) \\
\stackrel{R_1 - R_2; R_3 - 3R_2}{\sim} & \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)
\end{aligned}$$

1. We begin by interchanging the first two rows in order to get a 1 in the upper-left hand corner and avoiding dealing with fractions.

2. Next we subtract row 1 from row 3 and twice from row 2 to get zeros in the left-most column.

3. Then we scale row 2 to have a 1 in the eventual pivot.

4. Finally we subtract row 2 from row 1 and three times from row 2 to get it into Row-Reduced Echelon Form.

Therefore we can write $x = 1 - \lambda$, $y = 0$, $z = \lambda$ and $w = \mu$, or in vector form

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Now for the second system we have

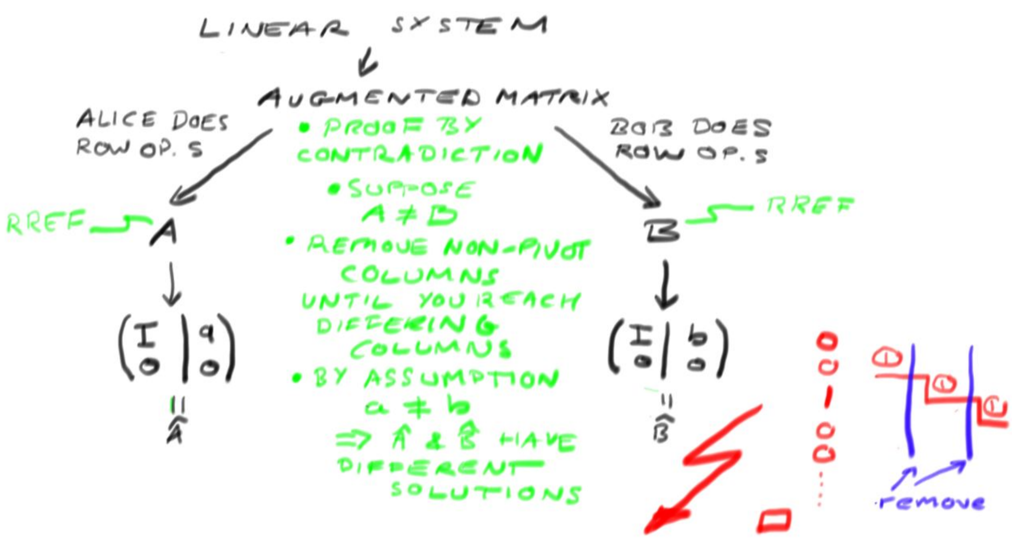
$$\left(\begin{array}{cc|c} 5 & 2 & 9 \\ 0 & 5 & 10 \\ 0 & 3 & 6 \end{array}\right) \overset{\frac{1}{5}R_2}{\sim} \left(\begin{array}{cc|c} 5 & 2 & 9 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \end{array}\right)$$
$$\overset{R_3-3R_2}{\sim} \left(\begin{array}{cc|c} 5 & 2 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$$
$$\overset{R_1-2R_2}{\sim} \left(\begin{array}{cc|c} 5 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$$
$$\overset{\frac{1}{5}R_1}{\sim} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$$

We scale the second and third rows appropriately in order to avoid fractions, then subtract the corresponding rows as before. Finally scale the first row and hence we have $x = 1$ and $y = 2$ as a unique solution.

G.11 Elementary Row Operations: Explanation of Proof for Theorem 3.1

The first thing to realize is that there are choices in the Gaussian elimination recipe, so maybe that could lead to two different RREF's and in turn two different solution sets for the same linear system. But that would be weird, in fact this Theorem says that this can never happen!

Because this proof comes at the end of the section it is often glossed over, but it is a very important result. Here's a sketch of what happens in the video:



In words: we start with a linear system and convert it to an augmented matrix. Then, because we are studying a uniqueness statement, we try a proof by contradiction. That is the method where to show that a statement is true, you try to demonstrate that the opposite of the statement leads to a contradiction. Here, the opposite statement to the theorem would be to find two different RREFs for the same system.

Suppose, therefore, that Alice and Bob do find different RREF augmented matrices called A and B. Then remove all the non-pivot columns from A and B until you hit the first column that differs. Record that in the last column and call the results $\widehat{A}$ and $\widehat{B}$. Removing columns does change the solution sets, but it does not ruin row equivalence, so $\widehat{A}$ and $\widehat{B}$ have the same solution sets.

Now, because we left only the pivot columns (plus the first column that differs) we have

$$\hat{A} = \begin{pmatrix} I_N & a \\ 0 & 0 \end{pmatrix} \text{ and } \hat{B} = \begin{pmatrix} I_N & b \\ 0 & 0 \end{pmatrix},$$

where I_N is an identity matrix and a and b are column vectors. Importantly, by assumption,

$$a \neq b .$$

So if we try to wrote down the solution sets for $\widehat{A}$ and $\widehat{B}$ they would be different. But at all stages, we only performed operations that kept Alice's solution set the same as Bob's. This is a contradiction so the proof is complete.

G.12 Elementary Row Operations: Hint for Review Question 3

The first part for Review Question 3 is simple--just write out the associated linear system and you will find the equation $0 = 6$ which is inconsistent. Therefore we learn that we must avoid a row of zeros preceding a non-vanishing entry after the vertical bar.

Turning to the system of equations, we first write out the augmented matrix and then perform two row operations

$$\left(\begin{array}{ccc|c} 1 & -3 & 0 & 6 \\ 1 & 0 & 3 & -3 \\ 2 & k & 3-k & 1 \end{array}\right)$$

$$\stackrel{R_2-R_1;R_3-2R_1}{\sim} \quad \left(\begin{array}{ccc|c} 1 & -3 & 0 & 6 \\ 0 & 3 & 3 & -9 \\ 0 & k+6 & 3-k & -11 \end{array}\right).$$

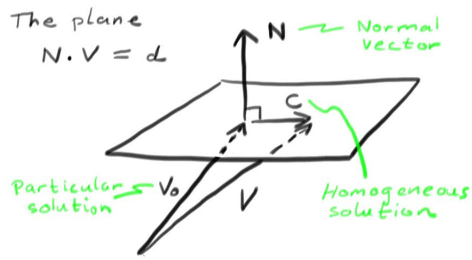
Next we would like to subtract some amount of R_2 from R_3 to achieve a zero in the third entry of the second column. But if

$$k + 6 = 3 - k \Rightarrow k = -\frac{3}{2},$$

this would produce zeros in the third row before the vertical line. You should also check that this does not make the whole third line zero. You now have enough information to write a complete solution.

G.13 Solution Sets for Systems of Linear Equations: Planes

Here we want to describe the mathematics of planes in space. The video is summarised by the following picture:



A plane is often called $\mathbb{R}^2$ because it is spanned by two coordinates, and space is called $\mathbb{R}^3$ and has three coordinates, usually called (x, y, z). The equation for a plane is

$$ax + by + cz = d\,.$$

Lets simplify this by calling $V = (x, y, z)$ the vector of unknowns and $N = (a, b, c)$. Using the dot product in $\mathbb{R}^3$ we have

$$N \cdot V = d\,.$$

Remember that when vectors are perpendicular their dot products vanish. *I.e.* $U \cdot V = 0 \Leftrightarrow U \perp V$. This means that if a vector V_0 solves our equation $N \cdot V = d$, then so too does $V_0 + C$ whenever C is perpendicular to N. This is because

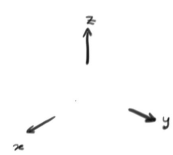
$$N \cdot (V_0 + C) = N \cdot V_0 + N \cdot C = d + 0 = d\,.$$

But C is ANY vector perpendicular to N, so all the possibilities for C span a plane whose normal vector is N. Hence we have shown that solutions to the equation $ax + by + cz = 0$ are a plane with normal vector $N = (a, b, c)$.

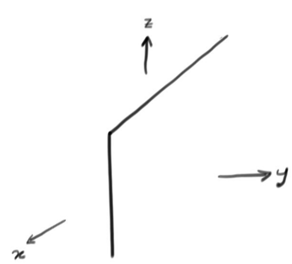
G.14 Solution Sets for Systems of Linear Equations: Pictures and Explanation

This video considers solutions sets for linear systems with three unknowns. These are often called (x, y, z) and label points in $\mathbb{R}^3$. Lets work case by case:

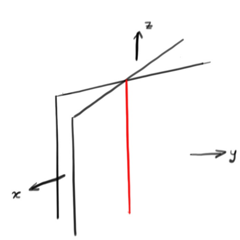
- If you have no equations at all, then any (x, y, z) is a solution, so the solution set is all of $\mathbb{R}^3$. The picture looks a little silly:



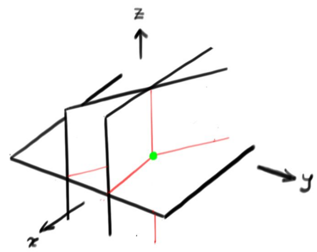
- For a single equation, the solution is a plane. This is explained in this video or the accompanying script. The picture looks like this:



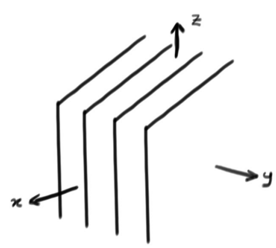
- For two equations, we must look at two planes. These usually intersect along a line, so the solution set will also (usually) be a line:



- For three equations, most often their intersection will be a single point so the solution will then be unique:



- Of course stuff can go wrong. Two different looking equations could determine the same plane, or worse equations could be inconsistent. If the equations are inconsistent, there will be no solutions at all. For example, if you had four equations determining four parallel planes the solution set would be empty. This looks like this:



G.15 Solution Sets for Systems of Linear Equations: Example

Here is an augmented matrix, let's think about what the solution set looks like

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 1 \end{array}\right)$$

This looks like the system

$$\begin{aligned} x_1 + 3x_3 &= 2 \\ x_2 &= 1 \end{aligned}$$

Notice that when the system is written this way the copy of the 2×2 identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ makes it easy to write a solution in terms of the variables x_1 and x_2. We will call x_1 and x_2 the ***pivot*** variables. The third column $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ does not look like part of an identity matrix, and there is no 3×3 identity in the augmented matrix. Notice there are more variables than equations and that this means we will have to write the solutions for the system in terms of the variable x_3. We'll call x_3 the ***free*** variable.

Let $x_3 = \mu$. Then we can rewrite the first equation in our system

$$\begin{aligned} x_1 + 3x_3 &= 2 \\ x_1 + 3\mu &= 2 \\ x_1 &= 2 - 3\mu. \end{aligned}$$

Then since the second equation doesn't depend on μ we can keep the equation

$$x_2 = 1,$$

and for a third equation we can write

$$x_3 = \mu$$

so that we get system

$$\begin{aligned}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 2-3\mu \\ 1 \\ \mu \end{pmatrix} \\
&= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -3\mu \\ 0 \\ \mu \end{pmatrix} \\
&= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}.
\end{aligned}$$

So for any value of μ will give a solution of the system, and any system can be written in this form for some value of μ. Since there are multiple solutions, we can also express them as a set:

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \middle| \mu \in \mathbb{R} \right\}.$$

G.16 Solution Sets for Systems of Linear Equations: Hint

For the first part of this problem, the key is to consider the vector as a $n \times 1$ matrix. For the second part, all you need to show is that

$$M(\alpha \cdot X + \beta \cdot Y) = \alpha \cdot (MX) + \beta \cdot (MY)$$

where $\alpha, \beta \in \mathbb{R}$ (or whatever field we are using) and

$$Y = \begin{pmatrix} y^1 \\ y^2 \\ \vdots \\ y^k \end{pmatrix}.$$

Note that this will be somewhat tedious, and many people use summation notation or Einstein's summation convention with the added notation of M_j denoting the j-th row of the matrix. For example, for any j we have

$$(MX)_j = \sum_{i=1}^{k} a_i^j x^i = a_i^j x^i.$$

You can see a concrete example after the definition of the linearity property.

G.17 Vectors in Space, n-Vectors: Overview

What is the space $\mathbb{R}^n$. In short it is the usual vectors we are used to. For example, if $n = 1$, then it is just the number line where we either move in the positive or negative directions, and we clearly have a notion of distance. This is something you should understand well, but ultimately there is nothing really interesting that goes on here.

Luckily when $n = 2$, things begin to get interesting. It is lucky for us because we can represent this by drawing arrows on paper. However what is interesting is that we no longer just have two directions, but an infinite number which we typically encapsulate as 0 to 2π radians (i.e. 0 to 360 degrees). Recall that the length of the vector is also known as its magnitude. We can add vectors by putting them head to toe and we can scale our vectors, and this concept is useful in physics such as Force Vector Diagrams. So why is this system $\mathbb{R}^2$? The answer comes from trigonometry, and what I have described is polar coordinates which you should be able to translate back to the usual Cartesian coordinates of (x, y). You still should be familiar with what things look like here.

Now for $\mathbb{R}^3$, if we look at this in Cartesian coordinates (x, y, z), this is exactly the same as $\mathbb{R}^2$, just we can now move around in our usual ''3D'' space by basically being able to draw in the air. Now our notion of direction in somewhat more complicated using azimuth and altitude (see Figure G.17 below), but it is secretly still there. So we will just use the tuple to encapsulate the data of it's direction and magnitude. Also we can equivalently write our tuple (x, y, z) as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

so our notation is consistent with matrix multiplication. Thus for all $n \geq 3$, we just use the tuple $(x^1, x^2, \ldots, x^n)$ to encapsulate our direction and magnitude and you can just treat vectors in $\mathbb{R}^n$ the same way as you would for vectors in $\mathbb{R}^2$.

Just one final closing remark; I have been somewhat sloppy through here on points and vectors, so make sure you read the note: Points Versus Vectors or Appendix D.

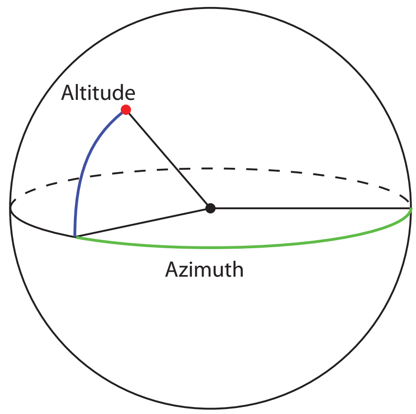


Figure 2: The azimuth and altitude in spherical coordinates.

G.18 Vectors in Space, n-Vectors: Review of Parametric Notation

The equation for a plane in three variables x, y and z looks like

$$ax + by + cz = d$$

where a, b, c, and d are constants. Lets look at the example

$$x + 2y + 5z = 3\,.$$

In fact this is a system of linear equations whose solutions form a plane with normal vector $(1, 2, 5)$. As an augmented matrix the system is simply

$$\left(\begin{array}{ccc|c} 1 & 2 & 5 & 3 \end{array}\right).$$

This is actually RREF! So we can let x be our pivot variable and y, z be represented by free parameters λ_1 and λ_2:

$$x = \lambda_1\,, \qquad y = \lambda_2\,.$$

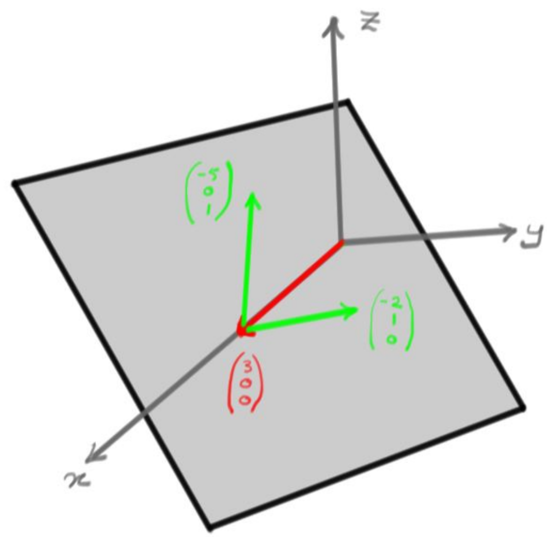
Thus we write the solution as

$$\begin{array}{rcrrr} x & = & -2\lambda_1 & -5\lambda_2 & +3 \\ y & = & \lambda_1 & & \\ z & = & & \lambda_2 & \end{array}$$

or in vector notation

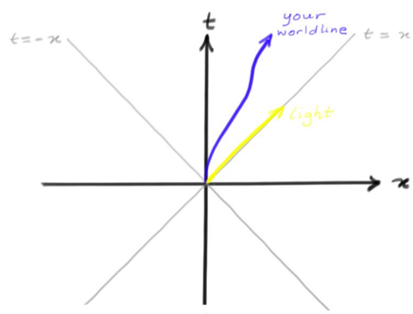
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} .$$

This describes a plane parametric equation. Planes are ‘‘two-dimensional’’ because they are described by two free variables. Here’s a picture of the resulting plane:



G.19 Vectors in Space, n-Vectors: The Story of Your Life

This video talks about the weird notion of a ''length-squared'' for a vector $v = (x, t)$ given by $||v||^2 = x^2 - t^2$ used in Einstein's theory of relativity. The idea is to plot the story of your life on a plane with coordinates (x, t). The coordinate x encodes *where* an event happened (for real life situations, we must replace $x \to (x, y, z) \in \mathbb{R}^3$). The coordinate t says *when* events happened. Therefore you can plot your life history as a worldline as shown:



Each point on the worldline corresponds to a place and time of an event in your life. The slope of the worldline has to do with your speed. Or to be precise, the inverse slope is your velocity. Einstein realized that the maximum speed possible was that of light, often called c. In the diagram above $c = 1$ and corresponds to the lines $x = \pm t \Rightarrow x^2 - t^2 = 0$. This should get you started in your search for vectors with zero length.

G.20 Vector Spaces: Examples of Each Rule

Lets show that $\mathbb{R}^2$ is a vector space. To do this (unless we invent some clever tricks) we will have to check all parts of the definition. Its worth doing this once, so here we go:

Before we start, remember that for $\mathbb{R}^2$ we define vector addition and scalar multiplication component-wise.

(+i) Additive closure: We need to make sure that when we add $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ that we do not get something outside the original vector space $\mathbb{R}^2$. This just relies on the underlying structure of real numbers whose sums are again real numbers so, using our component-wise addition law we have

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} := \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} \in \mathbb{R}^2 .$$

(+ii) Additive commutativity: We want to check that when we add any two vectors we can do so in either order, *i.e.*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .$$

This again relies on the underlying real numbers which for any $x, y \in \mathbb{R}$ obey

$$x + y = y + x .$$

This fact underlies the middle step of the following computation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} = \begin{pmatrix} y_1 + x_1 \\ y_2 + x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} ,$$

which demonstrates what we wished to show.

(+iii) Additive Associativity: This shows that we needn't specify with parentheses which order we intend to add triples of vectors because their sums will agree for either choice. What we have to check is

$$\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) + \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \left(\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right) .$$

Again this relies on the underlying associativity of real numbers:

$$(x + y) + z = x + (y + z)\,.$$

The computation required is

$$\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) + \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} (x_1 + y_1) + z_1 \\ (x_2 + y_2) + z_2 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 + (y_1 + z_1) \\ x_2 + (y_2 + z_2) \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} y_1 + z_1 \\ y_2 + z_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \left(\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)\,.$$

(iv) Zero: There needs to exist a vector $\vec{0}$ that works the way we would expect zero to behave, *i.e.*

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \vec{0} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\,.$$

It is easy to find, the answer is

$$\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\,.$$

You can easily check that when this vector is added to any vector, the result is unchanged.

(+v) Additive Inverse: We need to check that when we have $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, there is another vector that can be added to it so the sum is $\vec{0}$. (Note that it is important to first figure out what $\vec{0}$ is here!) The answer for the additive inverse of $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is $\begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix}$ because

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_1 \\ x_2 - x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0}\,.$$

We are half-way done, now we need to consider the rules for scalar multiplication. Notice, that we multiply vectors by scalars (*i.e.* numbers) but do NOT multiply a vectors by vectors.

(·i) Multiplicative closure: Again, we are checking that an operation does not produce vectors outside the vector space. For a scalar $a \in \mathbb{R}$, we require that $a\begin{pmatrix}x_1\\x_2\end{pmatrix}$ lies in $\mathbb{R}^2$. First we compute using our component-wise rule for scalars times vectors:

$$a\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}ax_1\\ax_2\end{pmatrix} .$$

Since products of real numbers ax_1 and ax_2 are again real numbers we see this is indeed inside $\mathbb{R}^2$.

(·ii) Multiplicative distributivity: The equation we need to check is

$$(a+b)\begin{pmatrix}x_1\\x_2\end{pmatrix} \stackrel{?}{=} a\begin{pmatrix}x_1\\x_2\end{pmatrix} + b\begin{pmatrix}x_1\\x_2\end{pmatrix} .$$

Once again this is a simple LHS=RHS proof using properties of the real numbers. Starting on the left we have

$$\begin{aligned}(a+b)\begin{pmatrix}x_1\\x_2\end{pmatrix} &= \begin{pmatrix}(a+b)x_1\\(a+b)x_2\end{pmatrix} = \begin{pmatrix}ax_1+bx_1\\ax_2+bx_2\end{pmatrix}\\ &= \begin{pmatrix}ax_1\\ax_2\end{pmatrix} + \begin{pmatrix}bx_1\\bx_2\end{pmatrix} = a\begin{pmatrix}x_1\\x_2\end{pmatrix} + b\begin{pmatrix}x_1\\x_2\end{pmatrix} ,\end{aligned}$$

as required.

(·iii) Additive distributivity: This time we need to check the equation The equation we need to check is

$$a\left(\begin{pmatrix}x_1\\x_2\end{pmatrix} + \begin{pmatrix}y_1\\y_2\end{pmatrix}\right) \stackrel{?}{=} a\begin{pmatrix}x_1\\x_2\end{pmatrix} + a\begin{pmatrix}y_1\\y_2\end{pmatrix} ,$$

i.e., one scalar but two different vectors. The method is by now becoming familiar

$$\begin{aligned}a\left(\begin{pmatrix}x_1\\x_2\end{pmatrix} + \begin{pmatrix}y_1\\y_2\end{pmatrix}\right) &= a\left(\begin{pmatrix}x_1+y_1\\x_2+y_2\end{pmatrix}\right) = \begin{pmatrix}a(x_1+y_1)\\a(x_2+y_2)\end{pmatrix}\\ &= \begin{pmatrix}ax_1+ay_1\\ax_2+ay_2\end{pmatrix} = \begin{pmatrix}ax_1\\ax_2\end{pmatrix} + \begin{pmatrix}ay_1\\ay_2\end{pmatrix} = a\begin{pmatrix}x_1\\x_2\end{pmatrix} + a\begin{pmatrix}y_1\\y_2\end{pmatrix} ,\end{aligned}$$

again as required.

(·iv) Multiplicative associativity. Just as for addition, this is the requirement that the order of bracketing does not matter. We need to establish whether

$$(a.b)\cdot\begin{pmatrix}x_1\\x_2\end{pmatrix} \stackrel{?}{=} a\cdot\left(b\cdot\begin{pmatrix}x_1\\x_2\end{pmatrix}\right) .$$

This clearly holds for real numbers $a.(b.x) = (a.b).x$. The computation is

$$(a.b)\cdot\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}(a.b).x_1\\(a.b).x_2\end{pmatrix} = \begin{pmatrix}a.(b.x_1)\\a.(b.x_2)\end{pmatrix} = a.\begin{pmatrix}(b.x_1)\\(b.x_2)\end{pmatrix} = a\cdot\left(b\cdot\begin{pmatrix}x_1\\x_2\end{pmatrix}\right) ,$$

which is what we want.

(·v) Unity: We need to find a special scalar acts the way we would expect ``1'' to behave. *I.e.*

$$\text{``1''}\cdot\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1\\x_2\end{pmatrix} .$$

There is an obvious choice for this special scalar---just the real number 1 itself. Indeed, to be pedantic lets calculate

$$1\cdot\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1.x_1\\1.x_2\end{pmatrix} = \begin{pmatrix}x_1\\x_2\end{pmatrix} .$$

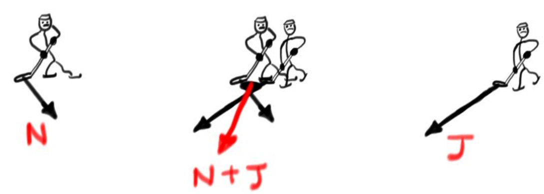
Now we are done---we have really proven the $\mathbb{R}^2$ is a vector space so lets write a little square $\square$ to celebrate.

G.21 Vector Spaces: Example of a Vector Space

This video talks about the definition of a vector space. Even though the defintion looks long, complicated and abstract, it is actually designed to model a very wide range of real life situations. As an example, consider the vector space

$$V = \{\text{all possible ways to hit a hockey puck}\}.$$

The different ways of hitting a hockey puck can all be considered as vectors. You can think about adding vectors by having two players hitting the puck at the same time. This picture shows vectors N and J corresponding to the ways Nicole Darwitz and Jenny Potter hit a hockey puck, plus the vector obtained when they hit the puck together.



You can also model the new vector $2J$ obtained by scalar multiplication by 2 by thinking about Jenny hitting the puck twice (or a world with two Jenny Potters....). Now ask yourself questions like whether the multiplicative distributive law

$$2J + 2N = 2(J + N)$$

make sense in this context.

G.22 Vector Spaces: Hint

I will only really worry about the last part of the problem. The problem can be solved by considering a non-zero simple polynomial, such as a degree 0 polynomial, and multiplying by $i \in \mathbb{C}$. That is to say we take a vector $p \in P_3^{\mathbb{R}}$ and then considering $i \cdot p$. This will violate one of the vector space rules about scalars, and you should take from this that the scalar field matters.

As a second hint, consider $\mathbb{Q}$ (the field of rational numbers). This is not a vector space over $\mathbb{R}$ since $\sqrt{2} \cdot 1 = \sqrt{2} \notin \mathbb{Q}$, so it is not closed under scalar multiplication, but it is clearly a vector space over $\mathbb{Q}$.

G.23 Linear Transformations: A Linear and A Non-Linear Example

This video gives an example of a linear transformation as well as a transformation that is not linear. In what happens below remember the properties that make a transformation linear:

$$L(u+v) = L(u) + L(v) \qquad \text{and} \qquad L(cu) = cL(u)\,.$$

The first example is the map

$$L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2\,,$$

via

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\,.$$

Here we focus on the scalar multiplication property $L(cu) = cL(u)$ which needs to hold for *any* scalar $c \in \mathbb{R}$ and *any* vector u. The additive property $L(u+v) = L(u) + L(v)$ is left as a fun exercise. The calculation looks like this:

$$L(cu) = L\left(c\begin{pmatrix} x \\ y \end{pmatrix}\right) = L\left(\begin{pmatrix} cx \\ cy \end{pmatrix}\right) = \begin{pmatrix} 2cx - 3cy \\ cx + cy \end{pmatrix}$$

$$= c\begin{pmatrix} 2x - 3y \\ x + y \end{pmatrix} = cL\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = cL(u)\,.$$

The first equality uses the fact that u is a vector in $\mathbb{R}^2$, next comes the rule for multiplying a vector by a number, then the rule for the given linear transformation L is used. The c is then factored out and we recognize that the vector next to c is just our linear transformation again. This verifies the scalar multiplication property $L(cu) = cL(u)$.

For a non-linear example lets take the vector space $\mathbb{R}^1 = \mathbb{R}$ with

$$L : \mathbb{R} \longrightarrow \mathbb{R}$$

via

$$x \mapsto x + 1\,.$$

This looks linear because the variable x appears once, but the constant term will be our downfall! Computing $L(cx)$ we get:

$$L(cx) = cx + 1\,,$$

but on the other hand

$$cL(x) = c(x + 1) = cx + c\,.$$

Now we see the problem, unless we are lucky and $c = 1$ the two expressions above are not linear. Since we need $L(cu) = cL(u)$ for *any* c, the game is up! $x \mapsto x + 1$ is *not* a linear transformation.

G.24 Linear Transformations: Derivative and Integral of (Real) Polynomials of Degree at Most 3

For this, we consider the vector space $\mathbb{P}_3^{\mathbb{R}}$ of real coefficient polynomials p such that the degree of $\deg p$ is at most 3. Let D denote the usual derivative operator and we note that it is linear, and we can write this as the matrix

$$D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Similarly now consider the map I where $I(f) = \int f(x)\,dx$ is the ***indefinite*** integral on any integrable function f. Now we first note that for any $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned} I(\alpha \cdot p + \beta \cdot q) &= \int \big(\alpha \cdot p(x) + \beta \cdot q(x)\big)\,dx \\ &= \alpha \int p(x)\,dx + \beta \int q(x)\,dx = \alpha I(p) + \beta I(q), \end{aligned}$$

so I is a linear map on functions. However we note that this is not a well-defined map on vector spaces since the additive constant states the image is not unique. For example $I(3x^2) = x^3 + c$ where c can be ***any*** constant. Therefore we have to perform a ***definite*** integral instead, so we define $I(f) := \int_0^x f(y)\,dy$. The other thing we could do is explicitly choose our constant, and we note that this does not necessarily give the same map (ex. take the constant to be non-zero with polynomials which in-fact will make it non-linear).

Now going to our vector space $\mathbb{P}_3^{\mathbb{R}}$, if we take any $p(x) = \alpha x^3 + \beta x^2 + \gamma x + \delta$, then we have

$$I(p) = \frac{\alpha}{4}x^4 + \frac{\beta}{3}x^3 + \frac{\gamma}{2}x^2 + \delta x,$$

and we note that this is outside of $\mathbb{P}_3^{\mathbb{R}}$. So to make our image in $\mathbb{P}_3^{\mathbb{R}}$, we formally set $I(x^3) = 0$. Thus we can now (finally) write this as the linear map $I\colon \mathbb{P}_3^{\mathbb{R}} \to \mathbb{P}_3^{\mathbb{R}}$ as the matrix:

$$I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \end{pmatrix}.$$

Finally we have

$$ID = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$DI = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and note the subspaces that are preserved under these compositions.

G.25 Linear Transformations: Linear Transformations Hint

The first thing we see in the problem is a definition of this new space P_n. Elements of P_n are polynomials that look like

$$a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$$

where the a_i's are constants. So this means if L is a linear transformation from $P_2 \to P_3$ that the inputs of L are degree two polynomials which look like

$$a_0 + a_1 t + a_2 t^2$$

and the output will have degree three and look like

$$b_0 + b_1 t + b_2 t^2 + b_3 t^3$$

We also know that L is a linear transformation, so what does that mean in this case? Well, by linearity we know that we can separate out the sum, and pull out the constants so we get

$$L(a_0 + a_1 t + a_2 t^2) = a_0 L(1) + a_1 L(t) + a_2 L(t^2)$$

Just this should be really helpful for the first two parts of the problem. The third part of the problem is asking us to think about this as a linear algebra problem, so lets think about how we could write this in the vector notation we use in the class. We could write

$$a_0 + a_1 t + a_2 t^2 \text{ as } \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

And think for a second about how you add polynomials, you match up terms of the same degree and add the constants component-wise. So it makes some sense to think about polynomials this way, since vector addition is also component-wise.

We could also write the output

$$b_0 + b_1 t + b_2 t^2 + b_3 t^3 \text{ as } \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} b_3$$

Then lets look at the information given in the problem and think about it in terms of column vectors

- $L(1) = 4$ but we can think of the input $1 = 1 + 0t + 0t^2$ and the output $4 = 4 + 0t + 0t^2 0t^3$ and write this as $L(\begin{pmatrix}1\\0\\0\end{pmatrix}) = \begin{pmatrix}4\\0\\0\\0\end{pmatrix}$

- $L(t) = t^3$ This can be written as $L(\begin{pmatrix}0\\1\\0\end{pmatrix}) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

- $L(t^2) = t - 1$ It might be a little trickier to figure out how to write $t - 1$ but if we write the polynomial out with the terms in order and with zeroes next to the terms that do not appear, we can see that

$$t - 1 = -1 + t + 0t^2 + 0t^3 \text{ corresponds to } \begin{pmatrix}-1\\1\\0\\0\end{pmatrix}$$

So this can be written as $L(\begin{pmatrix}0\\0\\1\end{pmatrix}) = \begin{pmatrix}-1\\1\\0\\0\end{pmatrix}$

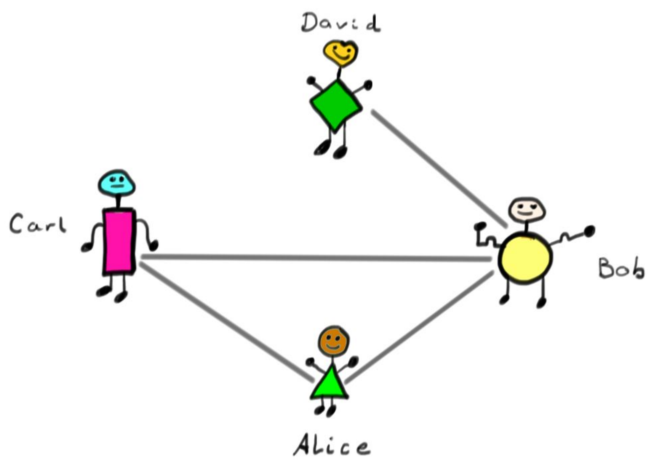
Now to think about how you would write the linear transformation L as a matrix, first think about what the dimensions of the matrix would be. Then look at the first two parts of this problem to help you figure out what the entries should be.

G.26 Matrices: Adjacency Matrix Example

Lets think about a graph as a mini-facebook. In this tiny facebook there are only four people, Alice, Bob, Carl, and David.

Suppose we have the following relationships

- Alice and Bob are friends.
- Alice and Carl are friends.
- Carl and Bob are friends.
- David and Bob are friends.



Now draw a picture where each person is a dot, and then draw a line between the dots of people who are friends. This is an example of a graph if you think of the people as nodes, and the friendships as edges.

Now lets make a 4×4 matrix, which is an adjacency matrix for the graph. Make a column and a row for each of the four people. It will look a lot like a table. When two people are friends put a 1 the the row of one and the column of the other. For example Alice and Carl are friends so we can label the table below.

	A	B	C	D
A			1	
B				
C	1			
D				

We can continue to label the entries for each friendship. Here lets assume that people are friends with themselves, so the diagonal will be all ones.

	A	B	C	D
A	1	1	1	0
B	1	1	1	1
C	1	1	1	0
D	0	1	0	1

Then take the entries of this table as a matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Notice that this table is symmetric across the diagonal, the same way a multiplication table would be symmetric. This is because on facebook friendship is symmetric in the sense that you can't be friends with someone if they aren't friends with you too. This is an example of a symmetric matrix.

You could think about what you would have to do differently to draw a graph for something like twitter where you don't have to follow everyone who follows you. The adjacency matrix might not be symmetric then.

G.27 Matrices: Do Matrices Commute?

This video shows you a funny property of matrices. Some matrix properties look just like those for numbers. For example numbers obey

$$a(bc) = (ab)c$$

and so do matrices:

$$A(BC) = (AB)C.$$

This says the order of bracketing does not matter and is called associativity. Now we ask ourselves whether the basic property of numbers

$$ab = ba\,,$$

holds for matrices

$$AB \stackrel{?}{=} BA\,.$$

For this, firstly note that we need to work with square matrices even for both orderings to even make sense. Lets take a simple 2×2 example, let

$$A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\,, \qquad B = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\,, \qquad C = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\,.$$

In fact, computing AB and BA we get the same result

$$AB = BA = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix}\,,$$

so this pair of matrices do commute. Lets try A and C:

$$AC = \begin{pmatrix} 1+a^2 & a \\ a & 1 \end{pmatrix}\,, \qquad \text{and} \qquad CA = \begin{pmatrix} 1 & a \\ a & 1+a^2 \end{pmatrix}$$

so

$$AC \neq CA$$

and this pair of matrices does *not* commute. Generally, matrices usually do not commute, and the problem of finding those that do is a very interesting one.

G.28 Matrices: Hint for Review Question 4

This problem just amounts to remembering that the dot product of $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ is

$$x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \,.$$

Then try multiplying the above row vector times y^T and compare.

G.29 Matrices: Hint for Review Question 5

The majority of the problem comes down to showing that matrices are right distributive. Let M_k is all $n \times k$ matrices for any n, and define the map $f_R\colon M_k \to M_m$ by $f_R(M) = MR$ where R is some $k \times m$ matrix. It should be clear that $f_R(\alpha \cdot M) = (\alpha M)R = \alpha(MR) = \alpha f_R(M)$ for any scalar α. Now all that needs to be proved is that

$$f_R(M + N) = (M + N)R = MR + NR = f_R(M) + f_R(N),$$

and you can show this by looking at each entry.

We can actually generalize the concept of this problem. Let V be some vector space and $\mathbb{M}$ be some collection of matrices, and we say that $\mathbb{M}$ is a ***left-action*** on V if

$$(M \cdot N) \circ v = M \circ (N \circ v)$$

for all $M, N \in \mathbb{N}$ and $v \in V$ where $\cdot$ denoted multiplication in $\mathbb{M}$ (i.e. standard matrix multiplication) and $\circ$ denotes the matrix is a linear map on a vector (i.e. $M(v)$). There is a corresponding notion of a right action where

$$v \circ (M \cdot N) = (v \circ M) \circ N$$

where we treat $v \circ M$ as $M(v)$ as before, and note the order in which the matrices are applied. People will often omit the left or right because they are essentially the same, and just say that $\mathbb{M}$ acts on V.

G.30 Properties of Matrices: Matrix Exponential Example

This video shows you how to compute

$$\exp \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} .$$

For this we need to remember that the matrix exponential is defined by its power series

$$\exp M := I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \cdots .$$

Now lets call

$$\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} = i\theta$$

where the *matrix*

$$i := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and by matrix multiplication is seen to obey

$$i^2 = -I \,, \qquad i^3 = -i \,, i^4 = I \,.$$

Using these facts we compute by organizing terms according to whether they have an i or not:

$$\begin{aligned}
\exp i\theta &= I + \frac{1}{2!}\,\theta^2(-I) + \frac{1}{4!}\,(+I) + \cdots \\
&+ i\theta + \frac{1}{3!}\,\theta^3(-i) + \frac{1}{5!}\,i + \cdots \\
&= I(1 - \frac{1}{2!}\,\theta^2 + \frac{1}{4!}\,\theta^4 + \cdots) \\
&+ i(\theta - \frac{1}{3!}\,\theta^3 + \frac{1}{5!}\,\theta^5 + \cdots) \\
&= I\cos\theta + i\sin\theta \\
&= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} .
\end{aligned}$$

Here we used the familiar Taylor series for the cosine and sine functions. A fun thing to think about is how the above matrix acts on vector in the plane.

G.31 Properties of Matrices: Explanation of the Proof

In this video we will talk through the steps required to prove

$$\operatorname{tr} MN = \operatorname{tr} NM \,.$$

There are some useful things to remember, first we can write

$$M = (m^i_j) \qquad \text{and} \qquad N = (n^i_j)$$

where the upper index labels rows and the lower one columns. Then

$$MN = \Big(\sum_l m^i_l n^l_j \Big) \,,$$

where the ''open'' indices i and j label rows and columns, but the index l is a ''dummy'' index because it is summed over. (We could have given it any name we liked!).

Finally the trace is the sum over diagonal entries for which the row and column numbers must coincide

$$\operatorname{tr} M = \sum_i m^i_i \,.$$

Hence starting from the left of the statement we want to prove, we have

$$\texttt{LHS} = \operatorname{tr} MN = \sum_i \sum_l m^i_l n^l_i \,.$$

Next we do something obvious, just change the order of the entries m^i_l and n^l_i (they are just numbers) so

$$\sum_i \sum_l m^i_l n^l_i = \sum_i \sum_l n^l_i m^i_l \,.$$

Equally obvious, we now rename $i \to l$ and $l \to i$ so

$$\sum_i \sum_l m^i_l n^l_i = \sum_l \sum_i n^i_l m^l_i \,.$$

Finally, since we have finite sums it is legal to change the order of summations

$$\sum_l \sum_i n^i_l m^l_i = \sum_i \sum_l n^i_l m^l_i \,.$$

This expression is the same as the one on the line above where we started except the m and n have been swapped so

$$\sum_i \sum_l m^i_l n^l_i = \operatorname{tr} NM = \mathtt{RHS}\,.$$

This completes the proof. $\square$

G.32 Properties of Matrices: A Closer Look at the Trace Function

This seemingly boring function which extracts a single real number does not seem immediately useful, however it is an example of an element in the *dual-space* of all $n \times n$ matrices since it is a bounded linear operator to the underlying field $\mathbb{F}$. By a bounded operator, I mean it will at most scale the length of the matrix (think of it as a vector in $\mathbb{F}^{n^2}$) by some fixed constant $C > 0$ (this can depend upon n), and for example if the length of a matrix M is d, then $\operatorname{tr}(M) \leq Cd$ (I believe $C = 1$ should work).

Some other useful properties is for block matrices, it should be clear that we have

$$\operatorname{tr}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \operatorname{tr} A + \operatorname{tr} D.$$

and that

$$\operatorname{tr}(PAP^{-1}) = \operatorname{tr}\big(P(AP^{-1})\big) = \operatorname{tr}\big((AP^{-1})P\big) = \operatorname{tr}(AP^{-1}P) = \operatorname{tr}(A)$$

so the trace function is conjugate (i.e. similarity) invariant. Using a concept from Chapter 17, it is basis invariant. Additionally in later chapters we will see that the trace function can be used to calculate the determinant (in a sense it is the derivative of the determinant, see Lecture 13 Problem 5) and eigenvalues.

Additionally we can define the set $\mathfrak{sl}_n$ as the set of all $n \times n$ matrices with trace equal to 0, and since the trace is linear and $a \cdot 0 = 0$, we note that $\mathfrak{sl}_n$ is a vector space. Additionally we can use the fact $\operatorname{tr}(MN) = \operatorname{tr}(NM)$ to define an operation called bracket

$$[M, N] = MN - NM,$$

and we note that $\mathfrak{sl}_n$ is closed under bracket since

$$\operatorname{tr}(MN - NM) = \operatorname{tr}(MN) - \operatorname{tr}(NM) = \operatorname{tr}(MN) - \operatorname{tr}(MN) = 0.$$

G.33 Properties of Matrices: Matrix Exponent Hint

This is a hint for computing exponents of matrices. So what is e^A if A is a matrix? We remember that the Taylor series for

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

So as matrices we can think about

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

This means we are going to have an idea of what A^n looks like for any n. Lets look at the example of one of the matrices in the problem. Let

$$A = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}.$$

Lets compute A^n for the first few n.

$$\begin{aligned} A^0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ A^1 &= \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \\ A^2 = A \cdot A &= \begin{pmatrix} 1 & 2\lambda \\ 0 & 1 \end{pmatrix} \\ A^3 = A^2 \cdot A &= \begin{pmatrix} 1 & 3\lambda \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

There is a pattern here which is that

$$A^n = \begin{pmatrix} 1 & n\lambda \\ 0 & 1 \end{pmatrix},$$

then we can think about the first few terms of the sequence

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = A^0 + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots.$$

Looking at the entries when we add this we get that the upper left-most entry looks like this:

$$1+1+\frac{1}{2}+\frac{1}{3!}+\ldots=\sum_{n=0}^{\infty}\frac{1}{n!}=e^{1}.$$

Continue this process with each of the entries using what you know about Taylor series expansions to find the sum of each entry.

G.34 Inverse Matrix: A 2×2 Example

Lets go though and show how this 2×2 example satisfies all of these properties. Lets look at

$$M = \begin{pmatrix} 7 & 3 \\ 11 & 5 \end{pmatrix}$$

We have a rule to compute the inverse

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

So this means that

$$M^{-1} = \frac{1}{35 - 33} \begin{pmatrix} 5 & -3 \\ -11 & 7 \end{pmatrix}$$

Lets check that $M^{-1}M = I = MM^{-1}$.

$$M^{-1}M = \frac{1}{35 - 33} \begin{pmatrix} 5 & -3 \\ -11 & 7 \end{pmatrix} \begin{pmatrix} 7 & 3 \\ 11 & 5 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = I$$

You can compute MM^{-1}, this should work the other way too.

Now lets think about products of matrices

$$\text{Let } A = \begin{pmatrix} 1 & 3 \\ 1 & 5 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

Notice that $M = AB$. We have a rule which says that $(AB)^{-1} = B^{-1}A^{-1}$. Lets check to see if this works

$$A^{-1} = \frac{1}{2} \begin{pmatrix} 5 & -3 \\ -1 & 1 \end{pmatrix} \text{ and } B^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

and

$$B^{-1}A^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 5 & -3 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

G.35 Inverse Matrix: Hints for Problem 3

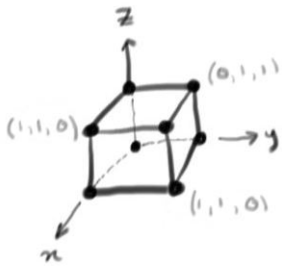
First I want to state that (b) implies (a) is the easy direction by just thinking about what it means for M to be non-singular and for a linear function to be well-defined. Therefore we assume that M is singular which implies that there exists a non-zero vector X_0 such that $MX_0 = 0$. Now assume there exists some vector X_V such that $MX_V = V$, and look at what happens to $X_V + c \cdot X_0$ for any c in your field. Lastly don't forget to address what happens if X_V does not exist.

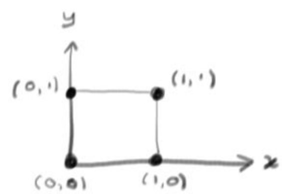
G.36 Inverse Matrix: Left and Right Inverses

This video is a hint for question 4 in the Inverse Matrixlecture 10. In the lecture, only inverses for square matrices were discussed, but there is a notion of left and right inverses for matrices that are not square. It helps to look at an example with bits to see why. To start with we look at vector spaces

$$\mathbb{Z}_2^3 = \{(x,y,z)|x,y,z=0,1\} \qquad \text{and} \qquad \mathbb{Z}_{2}{}_{2}^{2}\,.$$

These have 8 and 4 vectors, respectively, that can be depicted as corners of a cube or square:

$\mathbb{Z}_2^3$ 

or $\mathbb{Z}_2^2$ 

Now lets consider a linear transformation

$$L : \mathbb{Z}_2^3 \longrightarrow \mathbb{Z}_2^2\,.$$

This must be represented by a matrix, and lets take the example

$$L\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0&1&1\\1&1&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} := AX\,.$$

Since we have bits, we can work out what L does to every vector, this is listed below

$$
\begin{array}{rcl}
(0,0,0) & \stackrel{L}{\mapsto} & (0,0) \\[6pt]
(0,0,1) & \stackrel{L}{\mapsto} & (1,0) \\
(1,1,0) & \stackrel{L}{\mapsto} & (1,0) \\[6pt]
(1,0,0) & \stackrel{L}{\mapsto} & (0,1) \\
(0,1,1) & \stackrel{L}{\mapsto} & (0,1) \\[6pt]
(0,1,0) & \stackrel{L}{\mapsto} & (1,1) \\
(1,0,1) & \stackrel{L}{\mapsto} & (1,1) \\
(1,1,1) & \stackrel{L}{\mapsto} & (1,1)
\end{array}
$$

Now lets think about left and right inverses. A left inverse B to the matrix A would obey

$$BA = I$$

and since the identity matrix is square, B must be 2×3. It would have to undo the action of A and return vectors in $\mathbb{Z}_2^3$ to where they started from. But above, we see that different vectors in $\mathbb{Z}_2^3$ are mapped to the same vector in $\mathbb{Z}_2^2$ by the linear transformation L with matrix A. So B cannot exist. However a right inverse C obeying

$$AC = I$$

can. It would be 2×2. Its job is to take a vector in $\mathbb{Z}_2^2$ back to one in $\mathbb{Z}_2^3$ in a way that gets undone by the action of A. This can be done, but not uniquely.

G.37 LU Decomposition: Example: How to Use LU Decomposition

Lets go through how to use a LU decomposition to speed up solving a system of equations. Suppose you want to solve for x in the equation $Mx = b$

$$\begin{pmatrix} 1 & 0 & -5 \\ 3 & -1 & -14 \\ 1 & 0 & -3 \end{pmatrix} x = \begin{pmatrix} 6 \\ 19 \\ 4 \end{pmatrix}$$

where you are given the decomposition of M into the product of L and U which are lower and upper and lower triangular matrices respectively.

$$M = \begin{pmatrix} 1 & 0 & -5 \\ 3 & -1 & -14 \\ 1 & 0 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -5 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = LU$$

First you should solve $L(Ux) = b$ for Ux. The augmented matrix you would use looks like this

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 3 & 1 & 0 & 19 \\ 1 & 0 & 2 & 4 \end{array}\right)$$

This is an easy augmented matrix to solve because it is upper triangular. If you were to write out the three equations using variables, you would find that the first equation has already been solved, and is ready to be plugged into the second equation. This backward substitution makes solving the system much faster. Try it and in a few steps you should be able to get

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

This tells us that $Ux = \begin{pmatrix} 6 \\ 1 \\ -1 \end{pmatrix}$. Now the second part of the problem is to solve for x. The augmented matrix you get is

$$\left(\begin{array}{ccc|c} 1 & 0 & -5 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

It should take only a few step to transform it into

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right),$$

which gives us the answer $x = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$.

G.38 LU Decomposition: Worked Example

Here we will perform an LU decomposition on the matrix

$$M = \begin{pmatrix} 1 & 7 & 2 \\ -3 & -21 & 4 \\ 1 & 6 & 3 \end{pmatrix}$$

following the procedure outlined in Section 11.2. So initially we have $L_1 = I_3$ and $U_1 = M$, and hence

$$L_2 = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad U_2 = \begin{pmatrix} 1 & 7 & 2 \\ 0 & 0 & 10 \\ 0 & -1 & -1 \end{pmatrix}.$$

However we now have a problem since $0 \cdot c = 0$ for any value of c since we are working over a field, but we can quickly remedy this by swapping the second and third rows of U_2 to get U_2' and note that we just interchange the corresponding rows all columns left of and including the column we added values to in L_2 to get L_2'. Yet this gives us a small problem as $L_2'U_2' \neq M$; in fact it gives us the similar matrix M' with the second and third rows swapped. In our original problem $MX = V$, we also need to make the corresponding swap on our vector V to get a V' since all of this amounts to changing the order of our two equations, and note that this clearly does not change the solution. Back to our example, we have

$$L_2' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \qquad U_2' = \begin{pmatrix} 1 & 7 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 10 \end{pmatrix},$$

and note that U_2' is upper triangular. Finally you can easily see that

$$L_2'U_2' = \begin{pmatrix} 1 & 7 & 2 \\ 1 & 6 & 3 \\ -3 & -21 & 4 \end{pmatrix} = M'$$

which solves the problem of $L_2'U_2'X = M'X = V'$. (We note that as augmented matrices $(M'|V') \sim (M|V)$.)

G.39 LU Decomposition: Block LDU Explanation

This video explains how to do a block LDU decomposition. Firstly remember some key facts about block matrices: It is important that the blocks fit together properly. For example, if we have matrices

matrix	shape
X	$r \times r$
Y	$r \times t$
Z	$t \times r$
W	$t \times t$

we could fit these together as a $(r+t) \times (r+t)$ square block matrix

$$M = \left(\begin{array}{c|c} X & Y \\ \hline Z & W \end{array}\right) .$$

Matrix multiplication works for blocks just as for matrix entries:

$$M^2 = \left(\begin{array}{c|c} X & Y \\ \hline Z & W \end{array}\right) \left(\begin{array}{c|c} X & Y \\ \hline Z & W \end{array}\right) = \left(\begin{array}{c|c} X^2 + YZ & XY + YW \\ \hline ZX + WZ & ZY + W^2 \end{array}\right) .$$

Now lets specialize to the case where the square matrix X has an inverse. Then we can multiply out the following triple product of a lower triangular, a block diagonal and an upper triangular matrix:

$$\left(\begin{array}{c|c} I & 0 \\ \hline ZX^{-1} & I \end{array}\right) \left(\begin{array}{c|c} X & 0 \\ \hline 0 & W - ZX^{-1}Y \end{array}\right) \left(\begin{array}{c|c} I & X^{-1}Y \\ \hline 0 & I \end{array}\right)$$

$$= \left(\begin{array}{c|c} X & 0 \\ \hline Z & W - ZX^{-1}Y \end{array}\right) \left(\begin{array}{c|c} I & X^{-1}Y \\ \hline 0 & I \end{array}\right)$$

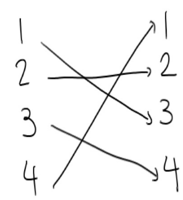
$$= \left(\begin{array}{c|c} X & Y \\ \hline ZX^{-1}Y + Z & W - ZX^{-1}Y \end{array}\right)$$

$$= \left(\begin{array}{c|c} X & Y \\ \hline Z & W \end{array}\right) = M .$$

This shows that the LDU decomposition given in Section 11 is correct.

G.40 Elementary Matrices and Determinants: Permutations

Lets try to get the hang of permutations. A permutation is a function which scrambles things. Suppose we had



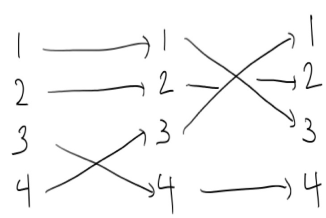
This looks like a function σ that has values

$$\sigma(1) = 3, \sigma(2) = 2, \sigma(3) = 4, \sigma(4) = 1$$

Then we could write this as

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{bmatrix}$$

We could write this permutation in two steps by saying that first we swap 3 and 4, and then we swap 1 and 3. The order here is important.



This is an even permutation, since the number of swaps we used is two (an even number).

G.41 Elementary Matrices and Determinants: Some Ideas Explained

This video will explain some of the ideas behind elementary matrices. First think back to linear systems, for example n equations in n unknowns:

$$\left\{ \begin{array}{rcl} a^1_1x^1 + a^1_2x^2 + \cdots + a^1_nx^n &=& v^1 \\ a^2_1x^1 + a^2_2x^2 + \cdots + a^2_nx^n &=& v^2 \\ & \vdots & \\ a^n_1x^1 + a^n_2x^2 + \cdots + a^n_nx^n &=& v^n \,. \end{array} \right.$$

We know it is helpful to store the above information with matrices and vectors

$$M := \begin{pmatrix} a^1_1 & a^1_2 & \cdots & a^1_n \\ a^2_1 & a^2_2 & \cdots & a^2_n \\ \vdots & \vdots & & \vdots \\ a^n_1 & a^n_2 & \cdots & a^n_n \end{pmatrix}, \qquad X := \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix}, \qquad V := \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{pmatrix}.$$

Here we will focus on the case the M is square because we are interested in its inverse M^{-1} (if it exists) and its determinant (whose job it will be to determine the existence of M^{-1}).

We know at least three ways of handling this linear system problem:

1. As an augmented matrix
$$\left(\begin{array}{c|c} M & V \end{array} \right).$$
Here our plan would be to perform row operations until the system looks like
$$\left(\begin{array}{c|c} I & M^{-1}V \end{array} \right),$$
(assuming that M^{-1} exists).

2. As a matrix equation
$$MX = V\,,$$
which we would solve by finding M^{-1} (again, if it exists), so that
$$X = M^{-1}V\,.$$

3. As a linear transformation
$$L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
via
$$\mathbb{R}^n \ni X \longmapsto MX \in \mathbb{R}^n \,.$$
In this case we have to study the equation $L(X) = V$ because $V \in \mathbb{R}^n$.

Lets focus on the first two methods. In particular we want to think about how the augmented matrix method can give information about finding M^{-1}. In particular, how it can be used for handling determinants.

The main idea is that the row operations changed the augmented matrices, but we also know how to change a matrix M by multiplying it by some other matrix E, so that $M \to EM$. In particular can we find ''elementary matrices'' the perform row operations?

Once we find these elementary matrices is is *very important* to ask how they effect the determinant, but you can think about that for your own self right now.

Lets tabulate our names for the matrices that perform the various row operations:

Row operation	Elementary Matrix
$R_i \leftrightarrow R_j$	E^i_j
$R_i \to \lambda R_i$	$R^i(\lambda)$
$R_i \to R_i + \lambda R_j$	$S^i_j(\lambda)$

To finish off the video, here is how all these elementary matrices work for a 2×2 example. Lets take
$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,.$$
A good thing to think about is what happens to $\det M = ad - bc$ under the operations below.

- Row swap:
$$E^1_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,, \qquad E^1_2 M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix} \,.$$

- `Scalar multiplying`:

$$R^1(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \qquad E_2^1 M = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ c & d \end{pmatrix}.$$

- `Row sum`:

$$S_2^1(\lambda) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \quad S_2^1(\lambda) M = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + \lambda c & b + \lambda d \\ c & d \end{pmatrix}.$$

G.42 Elementary Matrices and Determinants: Hints for Problem 4

Here we will examine the inversion number and the effect of the transposition $\tau_{1,2}$ and $\tau_{2,4}$ on the permutation $\nu = [3, 4, 1, 2]$. Recall that the inversion number is basically the number of items out of order. So the inversion number of ν is 4 since $3 > 1$ and $4 > 1$ and $3 > 2$ and $4 > 2$. Now we have $\tau_{1,2}\nu = [4, 3, 1, 2]$ by interchanging the first and second entries, and the inversion number is now 5 since we now also have $4 > 3$. Next we have $\tau_{2,4}\nu = [3, 2, 1, 4]$ whose inversion number is 3 since $3 > 2 > 1$. Finally we have $\tau_{1,2}\tau_{2,4}\nu = [2, 3, 1, 4]$ and the resulting inversion number is 2 since $2 > 1$ and $3 > 1$. Notice how when we are applying $\tau_{i,j}$ the parity of the inversion number changes.

G.43 Elementary Matrices and Determinants II: Elementary Determinants

This video will show you how to calculate determinants of elementary matrices. First remember that the job of an elementary row matrix is to perform row operations, so that if E is an elementary row matrix and M some given matrix,

$$EM$$

is the matrix M with a row operation performed on it.

The next thing to remember is that the determinant of the identity is 1. Moreover, we also know what row operations do to determinants:

- Row swap E^i_j: flips the sign of the determinant.
- Scalar multiplication $R^i(\lambda)$: multiplying a row by λ multiplies the determinant by λ.
- Row addition $S^i_j(\lambda)$: adding some amount of one row to another does not change the determinant.

The corresponding elementary matrices are obtained by performing exactly these operations on the identity:

$$E^i_j = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix},$$

$$R^i(\lambda) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix},$$

$$S^i_j(\lambda) = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & \lambda & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

So to calculate their determinants, we just have to apply the above list of what happens to the determinant of a matrix under row operations to the determinant of the identity. This yields

$$\det E^i_j = -1\,, \qquad \det R^i(\lambda) = \lambda\,, \qquad \det S^i_j(\lambda) = 1\,.$$

G.44 Elementary Matrices and Determinants II: Determinants and Inverses

Lets figure out the relationship between determinants and invertibility. If we have a system of equations $Mx = b$ and we have the inverse M^{-1} then if we multiply on both sides we get $x = M^{-1}Mx = M^{-1}b$. If the inverse exists we can solve for x and get a solution that looks like a point.

So what could go wrong when we want solve a system of equations and get a solution that looks like a point? Something would go wrong if we didn't have enough equations for example if we were just given

$$x + y = 1$$

or maybe, to make this a square matrix M we could write this as

$$\begin{aligned} x + y &= 1 \\ 0 &= 0 \end{aligned}$$

The matrix for this would be $M = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $\det(M) = 0$. When we compute the determinant, this row of all zeros gets multiplied in every term. If instead we were given redundant equations

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= 2 \end{aligned}$$

The matrix for this would be $M = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ and $\det(M) = 0$. But we know that with an elementary row operation, we could replace the second row with a row of all zeros. Somehow the determinant is able to detect that there is only one equation here. Even if we had a set of contradictory set of equations such as

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= 0, \end{aligned}$$

where it is not possible for both of these equations to be true, the matrix M is still the same, and still has a determinant zero.

Lets look at a three by three example, where the third equation is the sum of the first two equations.

$$\begin{aligned} x + y + z &= 1 \\ y + z &= 1 \\ x + 2y + 2z &= 2 \end{aligned}$$

and the matrix for this is

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

If we were trying to find the inverse to this matrix using elementary matrices

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right)$$

And we would be stuck here. The last row of all zeros cannot be converted into the bottom row of a 3×3 identity matrix. this matrix has no inverse, and the row of all zeros ensures that the determinant will be zero. It can be difficult to see when one of the rows of a matrix is a linear combination of the others, and what makes the determinant a useful tool is that with this reasonably simple computation we can find out if the matrix is invertible, and if the system will have a solution of a single point or column vector.

G.45 Elementary Matrices and Determinants II: Product of Determinants

Here we will prove more directly that the determinant of a product of matrices is the product of their determinants. First we reference that for a matrix M with rows r_i, if M' is the matrix with rows $r'_j = r_j + \lambda r_i$ for $j \neq i$ and $r'_i = r_i$, then $\det(M) = \det(M')$ Essentially we have M' as M multiplied by the elementary row sum matrices $S^i_j(\lambda)$. Hence we can create an upper-triangular matrix U such that $\det(M) = \det(U)$ by first using the first row to set $m^1_i \mapsto 0$ for all $i > 1$, then iteratively (increasing k by 1 each time) for fixed k using the k-th row to set $m^k_i \mapsto 0$ for all $i > k$.

Now note that for two upper-triangular matrices $U = (u^j_i)$ and $U' = (u'^j_i)$, by matrix multiplication we have $X = UU' = (x^j_i)$ is upper-triangular and $x^i_i = u^i_i u'^i_i$. Also since every permutation would contain a lower diagonal entry (which is 0) have $\det(U) = \prod_i u^i_i$. Let A and A' have corresponding upper-triangular matrices U and U' respectively (i.e. $\det(A) = \det(U)$), we note that AA' has a corresponding upper-triangular matrix UU', and hence we have

$$\begin{aligned}
\det(AA') = \det(UU') &= \prod_i u^i_i u'^i_i \\
&= \left(\prod_i u^i_i\right)\left(\prod_i u'^i_i\right) \\
&= \det(U)\det(U') = \det(A)\det(A').
\end{aligned}$$

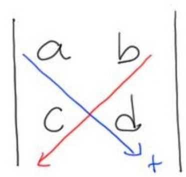
G.46 Properties of the Determinant: Practice taking Determinants

Lets practice taking determinants of 2×2 and 3×3 matrices.

For 2×2 matrices we have a formula

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

This formula can be easier to remember when you think about this picture.

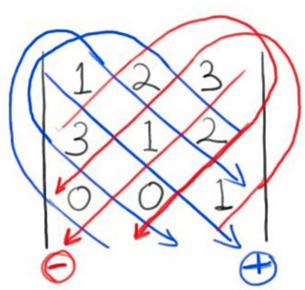


Now we can look at three by three matrices and see a few ways to compute the determinant. We have a similar pattern for 3×3 matrices.

Consider the example

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = ((1{\cdot}1{\cdot}1)+(2{\cdot}2{\cdot}0)+(3{\cdot}3{\cdot}0))-((3{\cdot}1{\cdot}0)+(1{\cdot}2{\cdot}0)+(3{\cdot}2{\cdot}1)) = -5$$

We can draw a picture with similar diagonals to find the terms that will be positive and the terms that will be negative.



Another way to compute the determinant of a matrix is to use this recursive formula. Here I take the coefficients of the first row and multiply them by the determinant of the minors and the cofactor. Then we can use the formula for a two by two determinant to compute the determinant of the minors

$$\texttt{det}\begin{bmatrix}1 & 2 & 3\\ 3 & 1 & 2\\ 0 & 0 & 1\end{bmatrix} = 1\begin{vmatrix}1 & 2\\ 0 & 1\end{vmatrix} - 2\begin{vmatrix}3 & 2\\ 0 & 1\end{vmatrix} + 3\begin{vmatrix}3 & 1\\ 0 & 0\end{vmatrix} = 1(1-0)-2(3-0)+3(0-0) = -5$$

Decide which way you prefer and get good at taking determinants, you'll need to compute them in a lot of problems.

G.47 Properties of the Determinant: The Adjoint Matrix

In this video we show how the adjoint matrix works in detail for the 3x3 case. Recall, that for a 2×2 matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the matrix

$$N = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

had the marvelous property

$$MN = (\det M)\, I$$

(you can easily check this for yourself). We call

$$N := \mathrm{adj} M\,,$$

the adjoint matrix of M. When the determinant $\det M \neq 0$, we can use it to immediately compute the inverse

$$M^{-1} = \frac{1}{\det M}\, \mathrm{adj} M\,.$$

Lets now think about a 3×3 matrix

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

The first thing to remember is that we can compute the determinant by expanding in a row and computing determinants of minors, so

$$\det M = a \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} - b \det \begin{pmatrix} d & g \\ f & i \end{pmatrix} + c \det \begin{pmatrix} e & f \\ h & i \end{pmatrix}.$$

We can think of this as the product of a row and column vector

$$\det M = \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ -\det \begin{pmatrix} d & g \\ f & i \end{pmatrix} \\ \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} \end{pmatrix}.$$

Now, we try a little experiment. Lets multiply the same column vector by the other two rows of M

$$\begin{pmatrix} d & e & f \end{pmatrix} \begin{pmatrix} \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ -\det\begin{pmatrix} d & g \\ f & i \end{pmatrix} \\ \det\begin{pmatrix} e & f \\ h & i \end{pmatrix} \end{pmatrix} = 0 = \begin{pmatrix} g & h & i \end{pmatrix} \begin{pmatrix} \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ -\det\begin{pmatrix} d & g \\ f & i \end{pmatrix} \\ \det\begin{pmatrix} e & f \\ h & i \end{pmatrix} \end{pmatrix}$$

The answer, ZERO, for both these computations, has been written in already because it is obvious. This is because these two computations are really computing

$$\det\begin{pmatrix} d & e & f \\ d & e & f \\ g & h & i \end{pmatrix} \quad \texttt{and} \quad \det\begin{pmatrix} g & h & i \\ d & e & f \\ g & h & i \end{pmatrix} .$$

These vanish because the determinant of an matrix with a pair of equal rows is zero. Thus we have found a nice result

$$M \begin{pmatrix} \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ -\det\begin{pmatrix} d & g \\ f & i \end{pmatrix} \\ \det\begin{pmatrix} e & f \\ h & i \end{pmatrix} \end{pmatrix} = \det M \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} .$$

Notice the answer is the number $\det M$ times the first column of the identity matrix. In fact, the column vector above is exactly the first column of the adjoint matrix $\mathrm{adj}M$. The rule how to get the rest of the adjoint matrix is not hard. You first compute the cofactor matrix obtained by replacing the entries of M with the signed determinants of the corresponding minors got by deleting the row and column of the particular entry. For the 3×3 case this is

$$\mathrm{cofactor}M = \begin{pmatrix} \det\begin{pmatrix} e & f \\ h & i \end{pmatrix} & -\det\begin{pmatrix} d & f \\ g & i \end{pmatrix} & \det\begin{pmatrix} d & e \\ g & h \end{pmatrix} \\ -\det\begin{pmatrix} b & c \\ h & i \end{pmatrix} & \det\begin{pmatrix} a & c \\ g & i \end{pmatrix} & -\det\begin{pmatrix} a & b \\ g & h \end{pmatrix} \\ \det\begin{pmatrix} b & c \\ e & f \end{pmatrix} & -\det\begin{pmatrix} a & c \\ d & f \end{pmatrix} & \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{pmatrix} .$$

Then the adjoint is just the transpose

$$\mathrm{adj}M = \big(\mathrm{cofactor}M\big)^T \, .$$

Computing all this is a little tedious, but always works, even for any $n \times n$ matrix. Moreover, when $\det M \neq 0$, we thus obtain the inverse $M^{-1} = \frac{1}{\det M}\,\mathrm{adj}M$.

G.48 Properties of the Determinant: Hint for Problem 3

For an arbitrary 3×3 matrix $A = (a^i_j)$, we have

$$\det(A) = a^1_1 a^2_2 a^3_3 + a^1_2 a^2_3 a^3_1 + a^1_3 a^2_1 a^3_2 - a^1_1 a^2_3 a^3_2 - a^1_2 a^2_1 a^3_3 - a^1_3 a^2_2 a^3_1$$

and so the complexity is $5a + 12m$. Now note that in general, the complexity c_n of the expansion minors formula of an arbitrary $n \times n$ matrix should be

$$c_n = (n-1)a + nc_{n-1}m$$

since $\det(A) = \sum_{i=1}^n (-1)^i a^1_i \operatorname{cofactor}(a^1_i)$ and $\operatorname{cofactor}(a^1_i)$ is an $(n-1) \times (n-1)$ matrix. This is one way to prove part (c).

G.49 Subspaces and Spanning Sets: Worked Example

Suppose that we were given a set of linear equations $l^j(x^1, x^2, \dots, x^n)$ and we want to find out if $l^j(X) = v^j$ for all j for some vector $V = (v^j)$. We know that we can express this as the matrix equation

$$\sum_i l_i^j x^i = v^j$$

where l_i^j is the coefficient of the variable x^i in the equation l^j. However, this is also stating that V is in the span of the vectors $\{L_i\}_i$ where $L_i = (l_i^j)_j$. For example, consider the set of equations

$$\begin{aligned} 2x + 3y - z &= 5 \\ -x + 3y + z &= 1 \\ x + y - 2z &= 3 \end{aligned}$$

which corresponds to the matrix equation

$$\begin{pmatrix} 2 & 3 & -1 \\ -1 & 3 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}.$$

We can thus express this problem as determining if the vector

$$V = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}$$

lies in the span of

$$\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \right\}.$$

G.50 Subspaces and Spanning Sets: Hint for Problem 2

We want to check whether

$$x - x^3 \in \texttt{span}\{x^2, 2x + x^2, x + x^3\}$$

If you are wondering what it means to be in the span of these polynomials here is an example

$$2(x^2) + 5(2x + x^2) \in \texttt{span}\{x^2, 2x + x^2, x + x^3\}$$

Linear combinations where the polynomials are multiplied by scalars in $\mathbb{R}$ is fine. We are not allowed to multiply the polynomials together, since in a vector space there is not necessarily a notion of multiplication for two vectors.

Lets put this problem in the language of matrices. Since we can write $x^2 = 0 + 0x + 1x^2 + 0x^3$ we can write it as a column vector, where the coefficient of each of the terms is an entry.

$$x^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \ , \ 2x + x^2 = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} \text{ and } x + x^3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Since we want to find out if $x - x^3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$ is in the span of these polynomials above we can ask, do there exist r_1, r_2 and r_3 such that

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

There are two ways to do this, one is by finding a r_1, r_2 and r_3 that work, another is to notice that there are no constant terms in any of the equations and to simplify the system so that it becomes

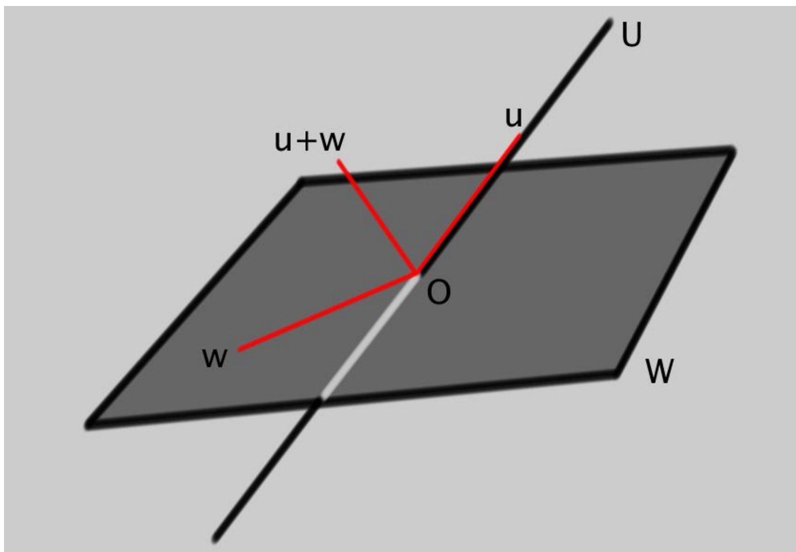
$$\begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

From here you can determine if the now square matrix has an inverse. If the matrix has an inverse you can say that there are r_1, r_2 and r_3 that satisfy this equation, without actually finding them.

G.51 Subspaces and Spanning Sets: Hint

This is a hint for the problem on intersections and unions of subspaces.

For the first part, try drawing an example in $\mathbb{R}^3$:



Here we have taken the subspace W to be a plane through the origin and U to be a line through the origin. The hint now is to think about what happens when you add a vector $u \in U$ to a vector $w \in W$. Does this live in the union $U \cup W$?

For the second part, we take a more theoretical approach. Lets suppose that $v \in U \cap W$ and $v' \in U \cap W$. This implies

$$v \in U \quad \text{and} \quad v' \in U\,.$$

So, since U is a subspace and all subspaces are vector spaces, we know that the linear combination

$$\alpha v + \beta v' \in U\,.$$

Now repeat the same logic for W and you will be nearly done.

G.52 Linear Independence: Worked Example

This video gives some more details behind the example for the following four vectors in $\mathbb{R}^3$ Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} -3 \\ 7 \\ 4 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 5 \\ 12 \\ 17 \end{pmatrix}, \qquad v_4 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

The example asks whether they are linearly independent, and the answer is immediate: NO, four vectors can never be linearly independent in $\mathbb{R}^3$. This vector space is simply not big enough for that, but you need to understand the notion of the dimension of a vector space to see why. So we think the vectors v_1, v_2, v_3 and v_4 are linearly dependent, which means we need to show that there is a solution to

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = 0$$

for the numbers α_1, α_2, α_3 and α_4 not all vanishing.

To find this solution we need to set up a linear system. Writing out the above linear combination gives

$$\begin{array}{rrrrcl} 4\alpha_1 & -3\alpha_2 & +5\alpha_3 & -\alpha_4 & = & 0\,, \\ -\alpha_1 & +7\alpha_2 & +12\alpha_3 & +\alpha_4 & = & 0\,, \\ 3\alpha_1 & +4\alpha_2 & +17\alpha_3 & & = & 0\,. \end{array}$$

This can be easily handled using an augmented matrix whose columns are just the vectors we started with

$$\left(\begin{array}{rrrr|l} 4 & -3 & 5 & -1 & 0\,, \\ -1 & 7 & 12 & 1 & 0\,, \\ 3 & 4 & 17 & 0 & 0\,. \end{array}\right).$$

Since there are only zeros on the right hand column, we can drop it. Now we perform row operations to achieve RREF

$$\begin{pmatrix} 4 & -3 & 5 & -1 \\ -1 & 7 & 12 & 1 \\ 3 & 4 & 17 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{71}{25} & -\frac{4}{25} \\ 0 & 1 & \frac{53}{25} & \frac{3}{25} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This says that α_3 and α_4 are not pivot variable so are arbitrary, we set them to μ and ν, respectively. Thus

$$\alpha_1 = \Big(-\frac{71}{25}\,\mu+\frac{4}{25}\,\nu\Big)\,, \qquad \alpha_2 = \Big(-\frac{53}{25}\,\mu-\frac{3}{25}\,\nu\Big)\,, \qquad \alpha_3 = \mu\,, \qquad \alpha_4 = \nu\,.$$

Thus we have found a relationship among our four vectors

$$\Big(-\frac{71}{25}\,\mu+\frac{4}{25}\,\nu\Big)\,v_1 + \Big(-\frac{53}{25}\,\mu-\frac{3}{25}\,\nu\Big)\,v_2 + \mu\,v_3 + \mu_4\,v_4 = 0\,.$$

In fact this is not just one relation, but infinitely many, for any choice of μ, ν. The relationship quoted in the notes is just one of those choices.

Finally, since the vectors v_1, v_2, v_3 and v_4 are linearly dependent, we can try to eliminate some of them. The pattern here is to keep the vectors that correspond to columns with pivots. For example, setting $\mu = -1$ (say) and $\nu = 0$ in the above allows us to solve for v_3 while $\mu = 0$ and $\nu = -1$ (say) gives v_4, explicitly we get

$$v_3 = \frac{71}{25}\,v_1 + \frac{53}{25}\,v_2\,, \qquad v_4 = -\frac{4}{25}\,v_3 + \frac{3}{25}\,v_4\,.$$

This eliminates v_3 and v_4 and leaves a pair of linearly independent vectors v_1 and v_2.

G.53 Linear Independence: Proof of Theorem 16.1

Here we will work through a quick version of the proof. Let $\{v_i\}$ denote a set of linearly dependent vectors, so $\sum_i c^i v_i = 0$ where there exists some $c^k \neq 0$. Now without loss of generality we order our vectors such that $c^1 \neq 0$, and we can do so since addition is commutative (i.e. $a + b = b + a$). Therefore we have

$$
\begin{aligned}
c^1 v_1 &= -\sum_{i=2}^{n} c^i v_i \\
v_1 &= -\sum_{i=2}^{n} \frac{c^i}{c^1} v_i
\end{aligned}
$$

and we note that this argument is completely reversible since every $c^i \neq 0$ is invertible and $0/c^i = 0$.

G.54 Linear Independence: Hint for Problem 1

Lets first remember how $\mathbb{Z}_2$ works. The only two elements are 1 and 0. Which means when you add $1+1$ you get 0. It also means when you have a vector $\vec{v} \in B^n$ and you want to multiply it by a scalar, your only choices are 1 and 0. This is kind of neat because it means that the possibilities are finite, so we can look at an entire vector space.

Now lets think about B^3 there is choice you have to make for each coordinate, you can either put a 1 or a 0, there are three places where you have to make a decision between two things. This means that you have $2^3 = 8$ possibilities for vectors in B^3.

When you want to think about finding a set S that will span B^3 and is linearly independent, you want to think about how many vectors you need. You will need you have enough so that you can make every vector in B^3 using linear combinations of elements in S but you don't want too many so that some of them are linear combinations of each other. I suggest trying something really simple perhaps something that looks like the columns of the identity matrix

For part (c) you have to show that you can write every one of the elements as a linear combination of the elements in S, this will check to make sure S actually spans B^3.

For part (d) if you have two vectors that you think will span the space, you can prove that they do by repeating what you did in part (c), check that every vector can be written using only copies of of these two vectors. If you don't think it will work you should show why, perhaps using an argument that counts the number of possible vectors in the span of two vectors.

G.55 Basis and Dimension: Proof of Theorem

Lets walk through the proof of this theorem. We want to show that for $S = \{v_1, \dots, v_n\}$ a basis for a vector space V, then every vector $w \in V$ can be written ***uniquely*** as a linear combination of vectors in the basis S:

$$w = c^1 v_1 + \cdots + c^n v_n.$$

We should remember that since S is a basis for V, we know two things

- $V = \operatorname{span} S$
- $v_1, \dots, v_n$ are linearly independent, which means that whenever we have $a^1 v_1 + \ldots + a^n v_n = 0$ this implies that $a^i = 0$ for all $i = 1, \dots, n$.

This first fact makes it easy to say that there exist constants c^i such that $w = c^1 v_1 + \cdots + c^n v_n$. What we don't yet know is that these $c^1, \dots c^n$ are unique.

In order to show that these are unique, we will suppose that they are not, and show that this causes a contradiction. So suppose there exists a second set of constants d^i such that

$$w = d^1 v_1 + \cdots + d^n v_n \, .$$

For this to be a contradiction we need to have $c^i \neq d^i$ for some i. Then look what happens when we take the difference of these two versions of w:

$$\begin{aligned} 0_V &= w - w \\ &= (c^1 v_1 + \cdots + c^n v_n) - (d^1 v_1 + \cdots + d^n v_n) \\ &= (c^1 - d^1) v_1 + \cdots + (c^n - d^n) v_n. \end{aligned}$$

Since the v_i's are linearly independent this implies that $c^i - d^i = 0$ for all i, this means that we cannot have $c^i \neq d^i$, which is a contradiction.

G.56 Basis and Dimension: Worked Example

In this video we will work through an example of how to extend a set of linearly independent vectors to a basis. For fun, we will take the vector space

$$V = \{(x,y,z,w)|x,y,z,w \in \mathbb{Z}^5\}.$$

This is like four dimensional space $\mathbb{R}^4$ except that the numbers can only be $\{0,1,2,3,4\}$. This is like bits, but now the rule is

$$0 = 5.$$

Thus, for example, $\frac{1}{4} = 4$ because $4 = 16 = 1 + 3 \times 5 = 1$. Don't get too caught up on this aspect, its a choice of base field designed to make computations go quicker!

Now, here's the problem we will solve:

Find a basis for V that includes the vectors $\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$ and $\begin{pmatrix}0\\3\\2\\1\end{pmatrix}$.

The way to proceed is to add a known (and preferably simple) basis to the vectors given, thus we consider

$$v_1 = \begin{pmatrix}1\\2\\3\\4\end{pmatrix},\ v_2 = \begin{pmatrix}0\\3\\2\\1\end{pmatrix},\ e_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ e_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ e_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix},\ e_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

The last four vectors are clearly a basis (make sure you understand this....) and are called the *canonical basis*. We want to keep v_1 and v_2 but find a way to turf out two of the vectors in the canonical basis leaving us a basis of four vectors. To do that, we have to study linear independence, or in other words a linear system problem defined by

$$0 = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 v_1 + \alpha_4 v_2 + \alpha_5 e_3 + \alpha_6 e_4.$$

We want to find solutions for the $\alpha's$ which allow us to determine two of the $e's$. For that we use an augmented matrix

$$\left(\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

Next comes a bunch of row operations. Note that we have dropped the last column of zeros since it has no information--you can fill in the row operations used above the $\sim$'s as an exercise:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 & 1 & 0 \\ 4 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 3 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}$$

$$\sim \begin{pmatrix} \underline{1} & 0 & 1 & 0 & 0 & 0 \\ 0 & \underline{1} & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & \underline{1} & 0 & 2 \\ 0 & 0 & 0 & 0 & \underline{1} & 3 \end{pmatrix}$$

The pivots are underlined. The columns corresponding to non-pivot variables are the ones that can be eliminated--their coefficients (the α's) will be arbitrary, so set them all to zero save for the one next to the vector you are solving for which can be taken to be unity. Thus that vector can certainly be expressed in terms of previous ones. Hence, altogether, our basis is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Finally, as a check, note that $e_1 = v_1 + v_2$ which explains why we had to throw it away.

G.57 Basis and Dimension: Hint for Problem 2

Since there are two possible values for each entry, we have $|B^n| = 2^n$. We note that $\dim B^n = n$ as well. Explicitly we have $B^1 = \{(0), (1)\}$ so there is only 1 basis for B^1. Similarly we have

$$B^2 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

and so choosing any two non-zero vectors will form a basis. Now in general we note that we can build up a basis $\{e_i\}$ by arbitrarily (independently) choosing the first $i-1$ entries, then setting the i-th entry to 1 and all higher entries to 0.

G.58 Eigenvalues and Eigenvectors: Worked Example

Lets consider a linear transformation

$$L : V \longrightarrow W$$

where a basis for V is the pair of vectors $\{\rightarrow, \uparrow\}$ and a basis for W is given by some other pair of vectors $\{\nearrow, \nwarrow\}$. (Don't be afraid that we are using arrows instead of latin letters to denote vectors!) To test your understanding, see if you know what $\dim V$ and $\dim W$ are. Now suppose that L does the following to the basis vectors in V

$$\rightarrow \stackrel{L}{\mapsto} \; a \nearrow + \, c \nwarrow =: L(\rightarrow)\,, \qquad \uparrow \stackrel{L}{\mapsto} \; b \nearrow + \, d \nwarrow =: L(\uparrow)\,.$$

Now arrange L acting on the basis vectors in a row vector (this will be a row vector whose entries are vectors).

$$\begin{pmatrix} L(\rightarrow) & L(\uparrow) \end{pmatrix} = \begin{pmatrix} a \nearrow + \, c \nwarrow & b \nearrow + \, d \nwarrow \end{pmatrix}.$$

Now we rewrite the right hand side as a matrix acting from the right on the basis vectors in W:

$$\begin{pmatrix} L(\rightarrow) & L(\uparrow) \end{pmatrix} = \begin{pmatrix} \nearrow & \nwarrow \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The matrix on the right is the matrix of L with respect to this pair of bases.

We can also write what happens when L acts on a general vector $v \in V$. Such a v can be written

$$v = x \rightarrow \; + \, y \uparrow \; .$$

First we compute L acting on this using linearity of L

$$L(v) = L(x \rightarrow \; + \, y \uparrow)$$

and then arrange this as a row vector (whose entries are vectors) times a column vector of numbers

$$L(v) = \begin{pmatrix} L(\rightarrow) & L(\uparrow) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Now we use our result above for the row vector$(L(\rightarrow)\ L(\uparrow))$ and obtain

$$L(v) = \begin{pmatrix} \nearrow & \nwarrow \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} .$$

Finally, as a fun exercise, suppose that you want to make a change of basis in W via

$$\nearrow = \rightarrow + \uparrow \quad \texttt{and} \quad \nwarrow = -\rightarrow + \uparrow .$$

Can you compute what happens to the matrix of L?

G.59 Eigenvalues and Eigenvectors: 2×2 Example

Here is an example of how to find the eigenvalues and eigenvectors of a 2×2 matrix.

$$M = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}.$$

Remember that an eigenvector v with eigenvalue λ for M will be a vector such that $Mv = \lambda v$ i.e. $M(v) - \lambda I(v) = \vec{0}$. When we are talking about a nonzero v then this means that $\det(M - \lambda I) = 0$. We will start by finding the eigenvalues that make this statement true. First we compute

$$\det(M - \lambda I) = \det\left(\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = \det\begin{pmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix}$$

so $\det(M - \lambda I) = (4-\lambda)(3-\lambda) - 2 \cdot 1$. We set this equal to zero to find values of λ that make this true:

$$(4-\lambda)(3-\lambda) - 2 \cdot 1 = 10 - 7\lambda + \lambda^2 = (2-\lambda)(5-\lambda) = 0\,.$$

This means that $\lambda = 2$ and $\lambda = 5$ are solutions. Now if we want to find the eigenvectors that correspond to these values we look at vectors v such that

$$\begin{pmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix} v = \vec{0}\,.$$

For $\lambda = 5$

$$\begin{pmatrix} 4-5 & 2 \\ 1 & 3-5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}\,.$$

This gives us the equalities $-x + 2y = 0$ and $x - 2y = 0$ which both give the line $y = \frac{1}{2}x$. Any point on this line, so for example $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, is an eigenvector with eigenvalue $\lambda = 5$.

Now lets find the eigenvector for $\lambda = 2$

$$\begin{pmatrix} 4-2 & 2 \\ 1 & 3-2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0},$$

which gives the equalities $2x + 2y = 0$ and $x + y = 0$. (Notice that these equations are not independent of one another, so our eigenvalue must be correct.) This means any vector $v = \begin{pmatrix} x \\ y \end{pmatrix}$ where $y = -x$, such as $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, or any scalar multiple of this vector , *i.e.* any vector on the line $y = -x$ is an eigenvector with eigenvalue 2. This solution could be written neatly as

$$\lambda_1 = 5,\, v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } \lambda_2 = 2,\, v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

G.60 Eigenvalues and Eigenvectors: Jordan Cells

Consider the matrix

$$J_2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

and we note that we can just read off the eigenvector e_1 with eigenvalue λ. However the characteristic polynomial of J_2 is $P_{J_2}(\mu) = (\mu - \lambda)^2$ so the only possible eigenvalue is λ, but we claim it does not have a second eigenvector v. To see this, we require that

$$\begin{aligned} \lambda v^1 + v^2 &= \lambda v^1 \\ \lambda v^2 &= \lambda v^2 \end{aligned}$$

which clearly implies that $v^2 = 0$. This is known as a Jordan 2-cell, and in general, a Jordan n-cell with eigenvalue λ is (similar to) the $n \times n$ matrix

$$J_n = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda & 1 \\ 0 & \cdots & 0 & 0 & \lambda \end{pmatrix}$$

which has a single eigenvector e_1.

Now consider the following matrix

$$M = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

and we see that $P_M(\lambda) = (\lambda - 3)^2(\lambda - 2)$. Therefore for $\lambda = 3$ we need to find the solutions to $(M - 3I_3)v = 0$ or in equation form:

$$\begin{aligned} v^2 &= 0 \\ v^3 &= 0 \\ -v^3 &= 0, \end{aligned}$$

and we immediately see that we must have $V = e_1$. Next for $\lambda = 2$, we need to solve $(M - 2I_3)v = 0$ or

$$\begin{aligned} v^1 + v^2 &= 0 \\ v^2 + v^3 &= 0 \\ 0 &= 0, \end{aligned}$$

and thus we choose $v^1 = 1$, which implies $v^2 = -1$ and $v^3 = 1$. Hence this is the only other eigenvector for M.

This is a specific case of Problem 20.5.

G.61 Eigenvalues and Eigenvectors II: Eigenvalues

Eigenvalues and eigenvectors are extremely important. In this video we review the theory of eigenvalues. Consider a linear transformation

$$L : V \longrightarrow V$$

where $\dim V = n < \infty$. Since V is finite dimensional, we can represent L by a square matrix M by choosing a basis for V.

So the eigenvalue equation

$$\boxed{Lv = \lambda v}$$

becomes

$$\boxed{Mv = \lambda v\,,}$$

where v is a column vector and M is an $n \times n$ matrix (both expressed in whatever basis we chose for V). The scalar λ is called an eigenvalue of M and the job of this video is to show you how to find all the eigenvalues of M.

The first step is to put all terms on the left hand side of the equation, this gives

$$(M - \lambda I)v = 0\,.$$

Notice how we used the identity matrix I in order to get a matrix times v equaling zero. Now here comes a VERY important fact

$$\boxed{Nu = 0 \text{ and } u \neq 0 \Longleftrightarrow \det N = 0\,.}$$

I.e., a square matrix can have an eigenvector with vanishing eigenvalue if and only if its determinant vanishes! Hence

$$\boxed{\det(M - \lambda I) = 0\,.}$$

The quantity on the left (up to a possible minus sign) equals the so-called characteristic polynomial

$$P_M(\lambda) := \det(\lambda I - M)\,.$$

It is a polynomial of degree n in the variable λ. To see why, try a simple 2×2 example

$$\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - bc\,,$$

which is clearly a polynomial of order 2 in λ. For the $n \times n$ case, the order n term comes from the product of diagonal matrix elements also.

There is an amazing fact about polynomials called the *fundamental theorem of algebra*: they can always be factored over complex numbers. This means that degree n polynomials have n complex roots (counted with multiplicity). The word can does not mean that explicit formulas for this are known (in fact explicit formulas can only be give for degree four or less). The necessity for complex numbers is easily seems from a polynomial like

$$z^2 + 1$$

whose roots would require us to solve $z^2 = -1$ which is impossible for real number z. However, introducing the imaginary unit i with

$$i^2 = -1\,,$$

we have

$$z^2 + 1 = (z-i)(z+i)\,.$$

Returning to our characteristic polynomial, we call on the fundamental theorem of algebra to write

$$P_M(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)\,.$$

The roots λ_1, λ_2,...,λ_n are the eigenvalues of M (or its underlying linear transformation L).

G.62 Eigenvalues and Eigenvectors II: Eigenspaces

Consider the linear map

$$L = \begin{pmatrix} -4 & 6 & 6 \\ 0 & 2 & 0 \\ -3 & 3 & 5 \end{pmatrix}.$$

Direct computation will show that we have

$$L = Q \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} Q^{-1}$$

where

$$Q = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Therefore the vectors

$$v_1^{(2)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad v_2^{(2)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

span the eigenspace $E^{(2)}$ of the eigenvalue 2, and for an explicit example, if we take

$$v = 2v_1^{(2)} - v_2^{(2)} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

we have

$$Lv = \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix} = 2v$$

so $v \in E^{(2)}$. In general, we note the linearly independent vectors $v_i^{(\lambda)}$ with the same eigenvalue λ span an eigenspace since for any $v = \sum_i c^i v_i^{(\lambda)}$, we have

$$Lv = \sum_i c^i L v_i^{(\lambda)} = \sum_i c^i \lambda v_i^{(\lambda)} = \lambda \sum_i c^i v_i^{(\lambda)} = \lambda v.$$

G.63 Eigenvalues and Eigenvectors II: Hint

We are looking at the matrix M, and a sequence of vectors starting with $v(0) = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$ and defined recursively so that

$$v(1) = \begin{pmatrix} x(1) \\ y(1) \end{pmatrix} = M \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}.$$

We first examine the eigenvectors and eigenvalues of

$$M = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}.$$

We can find the eigenvalues and vectors by solving

$$\det(M - \lambda I) = 0$$

for λ.

$$\det \begin{pmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{pmatrix} = 0$$

By computing the determinant and solving for λ we can find the eigenvalues $\lambda = 1$ and 5, and the corresponding eigenvectors. You should do the computations to find these for yourself.

When we think about the question in part (b) which asks to find a vector $v(0)$ such that $v(0) = v(1) = v(2) \ldots$, we must look for a vector that satisfies $v = Mv$. What eigenvalue does this correspond to? If you found a $v(0)$ with this property would $cv(0)$ for a scalar c also work? Remember that eigenvectors have to be nonzero, so what if $c = 0$?

For part (c) if we tried an eigenvector would we have restrictions on what the eigenvalue should be? Think about what it means to be pointed in the same direction.

G.64 Diagonalization: Derivative Is Not Diagonalizable

First recall that the derivative operator is linear and that we can write it as the matrix

$$\frac{d}{dx} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We note that this transforms into an infinite Jordan cell with eigenvalue 0 or

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which is in the basis $\{n^{-1}x^n\}_n$ (where for $n = 0$, we just have 1). Therefore we note that 1 (constant polynomials) is the only eigenvector with eigenvalue 0 for polynomials since they have finite degree, and so the derivative is not diagonalizable. Note that we are ignoring infinite cases for simplicity, but if you want to consider infinite terms such as convergent series or all formal power series where there is no conditions on convergence, there are many eigenvectors. Can you find some? This is an example of how things can change in infinite dimensional spaces.

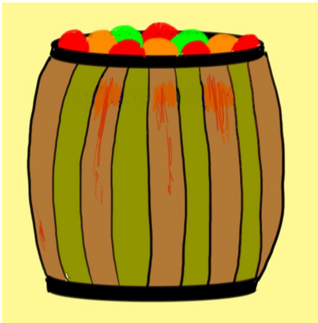
For a more finite example, consider the space $\mathbb{P}_3^{\mathbb{C}}$ of complex polynomials of degree at most 3, and recall that the derivative D can be written as

$$D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

You can easily check that the only eigenvector is 1 with eigenvalue 0 since D always lowers the degree of a polynomial by 1 each time it is applied. Note that this is a nilpotent matrix since $D^4 = 0$, but the only nilpotent matrix that is ''diagonalizable'' is the 0 matrix.

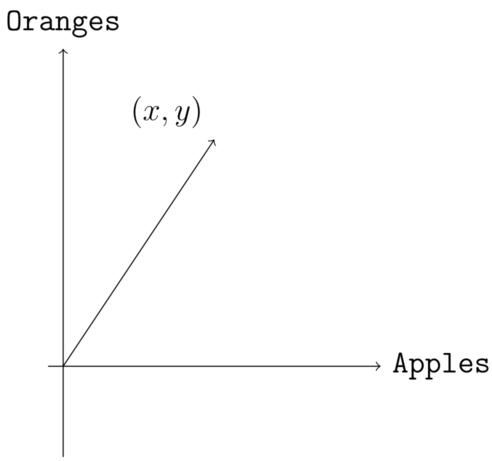
G.65 Diagonalization: Change of Basis Example

This video returns to our first example of a barrel filled with fruit



as a demonstration of changing basis.

Since this was a linear systems problem, we can try to represent what's in the barrel using a vector space. The first representation was the one where $(x, y) = (\texttt{apples}, \texttt{oranges})$:

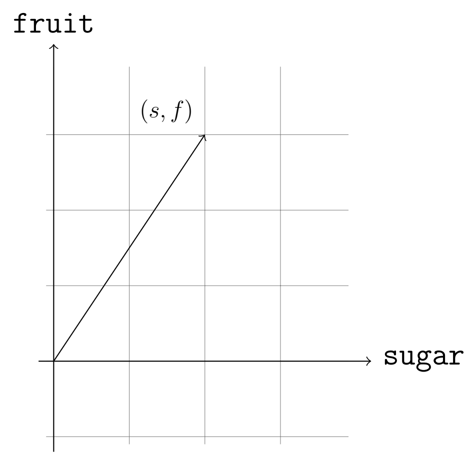


Calling the basis vectors $\vec{e}_1 := (1, 0)$ and $\vec{e}_2 := (0, 1)$, this representation would label what's in the barrel by a vector

$$\vec{x} := x\vec{e}_1 + y\vec{e}_2 = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} .$$

Since this is the method ordinary people would use, we will call this the ''engineer's'' method!

But this is not the approach nutritionists would use. They would note the amount of sugar and total number of fruit (s, f):

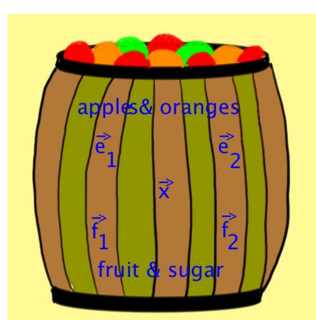


WARNING: To make sense of what comes next you need to allow for the possibity of a negative amount of fruit or sugar. This would be just like a bank, where if money is owed to somebody else, we can use a minus sign.

The vector $\vec{x}$ says what is in the barrel and does not depend which mathematical description is employed. The way nutritionists label $\vec{x}$ is in terms of a pair of basis vectors $\vec{f}_1$ and $\vec{f}_2$:

$$\vec{x} = s\vec{f}_1 + f\vec{f}_2 = \begin{pmatrix} \vec{f}_1 & \vec{f}_2 \end{pmatrix} \begin{pmatrix} s \\ f \end{pmatrix} .$$

Thus our vector space now has a bunch of interesting vectors:



The vector $\vec{x}$ labels generally the contents of the barrel. The vector $\vec{e}_1$ corresponds to one apple and one orange. The vector $\vec{e}_2$ is one orange and no apples. The vector $\vec{f}_1$ means one unit of sugar and zero total fruit (to achieve this you could lend out some apples and keep a few oranges). Finally the vector $\vec{f}_2$ represents a total of one piece of fruit and no sugar.

You might remember that the amount of sugar in an apple is called λ while oranges have twice as much sugar as apples. Thus

$$\left\{\begin{array}{l} s=\lambda\,(x+2y) \\ f=x+y\,. \end{array}\right.$$

Essentially, this is already our change of basis formula, but lets play around and put it in our notations. First we can write this as a matrix

$$\begin{pmatrix} s\\ f\end{pmatrix}=\begin{pmatrix}\lambda & 2\lambda\\ 1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\,.$$

We can easily invert this to get

$$\begin{pmatrix} x\\ y\end{pmatrix}=\begin{pmatrix}-\frac{1}{\lambda} & 2\\ \frac{1}{\lambda} & -1\end{pmatrix}\begin{pmatrix}s\\ f\end{pmatrix}\,.$$

Putting this in the engineer's formula for $\vec{x}$ gives

$$\vec{x}=\begin{pmatrix}\vec{e}_1 & \vec{e}_2\end{pmatrix}\begin{pmatrix}-\frac{1}{\lambda} & 2\\ \frac{1}{\lambda} & -1\end{pmatrix}\begin{pmatrix}s\\ f\end{pmatrix}=\begin{pmatrix}-\frac{1}{\lambda}\big(\vec{e}_1-\vec{e}_2\big) & 2\vec{e}_1-2\vec{e}_2\end{pmatrix}\begin{pmatrix}s\\ f\end{pmatrix}\,.$$

Comparing to the nutritionist's formula for the same object $\vec{x}$ we learn that

$$\vec{f}_1=-\frac{1}{\lambda}\big(\vec{e}_1-\vec{e}_2\big)\quad\text{and}\quad\vec{f}_2=2\vec{e}_1-2\vec{e}_2\,.$$

Rearranging these equation we find the change of base matrix P from the engineer's basis to the nutritionist's basis:

$$\begin{pmatrix}\vec{f}_1 & \vec{f}_2\end{pmatrix}=\begin{pmatrix}\vec{e}_1 & \vec{e}_2\end{pmatrix}\begin{pmatrix}-\frac{1}{\lambda} & 2\\ \frac{1}{\lambda} & -1\end{pmatrix}=:\begin{pmatrix}\vec{e}_1 & \vec{e}_2\end{pmatrix}P\,.$$

We can also go the other direction, changing from the nutritionist's basis to the engineer's basis

$$\begin{pmatrix}\vec{e}_1 & \vec{e}_2\end{pmatrix}=\begin{pmatrix}\vec{f}_1 & \vec{f}_2\end{pmatrix}\begin{pmatrix}\lambda & 2\lambda\\ 1 & 1\end{pmatrix}=:\begin{pmatrix}\vec{f}_1 & \vec{f}_2\end{pmatrix}Q\,.$$

Of course, we must have

$$Q = P^{-1} ,$$

(which is in fact how we constructed P in the first place).

Finally, lets consider the very first linear systems problem, where you were given that there were 27 pieces of fruit in total and twice as many oranges as apples. In equations this says just

$$x + y = 27 \quad \text{and} \quad 2x - y = 0 .$$

But we can also write this as a matrix system

$$MX = V$$

where

$$M := \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} , \qquad X := \begin{pmatrix} x \\ y \end{pmatrix} \qquad V := \begin{pmatrix} 0 \\ 27 \end{pmatrix} .$$

Note that

$$\vec{x} = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 \end{pmatrix} X .$$

Also lets call

$$\vec{v} := \begin{pmatrix} \vec{e}_1 & \vec{e}_2 \end{pmatrix} V .$$

Now the matrix M is the matrix of some linear transformation L in the basis of the engineers. Lets convert it to the basis of the nutritionists:

$$L\vec{x} = L \begin{pmatrix} \vec{f}_1 & \vec{f}_2 \end{pmatrix} \begin{pmatrix} s \\ f \end{pmatrix} = L \begin{pmatrix} \vec{e}_1 & \vec{e}_2 \end{pmatrix} P \begin{pmatrix} s \\ f \end{pmatrix} = \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix} MP \begin{pmatrix} s \\ f \end{pmatrix} .$$

Note here that the linear transformation on acts on *vectors* -- these are the objects we have written with a $\vec{}$ sign on top of them. It does not act on columns of numbers!

We can easily compute MP and find

$$MP = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -\frac{1}{\lambda} & 2 \\ \frac{1}{\lambda} & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{3}{\lambda} & 5 \end{pmatrix} .$$

Note that $P^{-1}MP$ is the matrix of L in the nutritionists basis, but we don't need this quantity right now.

Thus the last task is to solve the system, lets solve for sugar and fruit. We need to solve

$$MP\begin{pmatrix} s \\ f \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{3}{\lambda} & 5 \end{pmatrix}\begin{pmatrix} s \\ f \end{pmatrix} = \begin{pmatrix} 27 \\ 0 \end{pmatrix}.$$

This is solved immediately by forward substitution (the nutritionists basis is nice since it directly gives f):

$$f = 27 \quad \text{and} \quad s = 45\lambda\,.$$

G.66 Diagonalization: Diagionalizing Example

Lets diagonalize the matrix M from a previous example

Eigenvalues and Vectors: Example

$$M = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$$

We found the eigenvalues and eigenvectors of M, our solution was

$$\lambda_1 = 5,\ \mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } \lambda_2 = 2,\ \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

So we can diagonalize this matrix using the formula $D = P^{-1}MP$ where $P = (\mathbf{v}_1, \mathbf{v}_2)$. This means

$$P = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } P^{-1} = -\frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$$

The inverse comes from the formula for inverses of 2×2 matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \text{ so long as } ad - bc \neq 0.$$

So we get:

$$D = -\frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}$$

But this doesn't really give any intuition into why this happens. Let look at what happens when we apply this matrix $D = P^{-1}MP$ to a vector $v = \begin{pmatrix} x \\ y \end{pmatrix}$. Notice that applying P translates $v = \begin{pmatrix} x \\ y \end{pmatrix}$ into $x\mathbf{v}_1 + y\mathbf{v}_2$.

$$\begin{aligned}
P^{-1}MP\begin{pmatrix}x\\y\end{pmatrix} &= P^{-1}M\begin{pmatrix}2x+y\\x-y\end{pmatrix}\\
&= P^{-1}M[\begin{pmatrix}2x\\x\end{pmatrix}+\begin{pmatrix}y\\-y\end{pmatrix}]\\
&= P^{-1}[(x)M\begin{pmatrix}2\\1\end{pmatrix}+(y)M\begin{pmatrix}1\\-1\end{pmatrix}]\\
&= P^{-1}[(x)M\mathbf{v}_1+(y)\cdot M\mathbf{v}_2]
\end{aligned}$$

Remember that we know what M does to $\mathbf{v}_1$ and $\mathbf{v}_2$, so we get

$$\begin{aligned}
P^{-1}[(x)M\mathbf{v}_1+(y)M\mathbf{v}_2] &= P^{-1}[(x\lambda_1)\mathbf{v}_1+(y\lambda_2)\mathbf{v}_2]\\
&= (5x)P^{-1}\mathbf{v}_1+(2y)P^{-1}\mathbf{v}_2\\
&= (5x)\begin{pmatrix}1\\0\end{pmatrix}+(2y)\begin{pmatrix}0\\1\end{pmatrix}\\
&= \begin{pmatrix}5x\\2y\end{pmatrix}
\end{aligned}$$

Notice that multiplying by P^{-1} converts $\mathbf{v}_1$ and $\mathbf{v}_2$ back in to $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ respectively. This shows us why $D=P^{-1}MP$ should be the diagonal matrix:

$$D=\begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix}=\begin{pmatrix}5 & 0\\0 & 2\end{pmatrix}$$

G.67 Orthonormal Bases: Sine and Cosine Form All Orthonormal Bases for $\mathbb{R}^2$

We wish to find all orthonormal bases for the space $\mathbb{R}^2$, and they are $\{e_1^\theta, e_2^\theta\}$ up to reordering where

$$e_1^\theta = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad e_2^\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix},$$

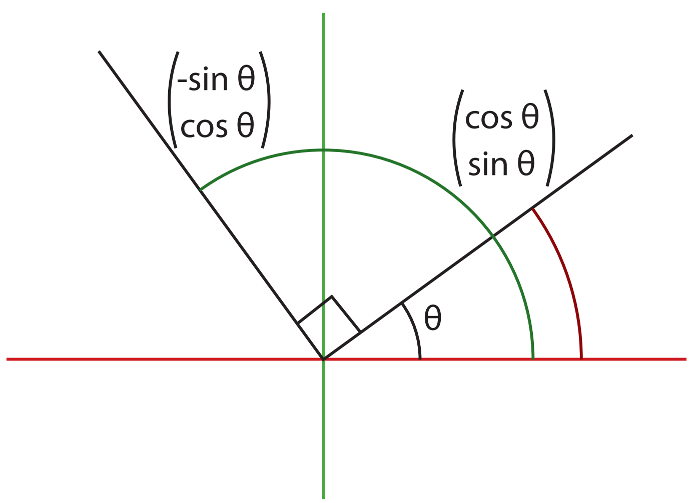
for some $\theta \in [0, 2\pi)$. Now first we need to show that for a fixed θ that the pair is orthogonal:

$$e_1^\theta \cdot e_2^\theta = -\sin\theta\cos\theta + \cos\theta\sin\theta = 0.$$

Also we have

$$\|e_1^\theta\|^2 = \|e_2^\theta\|^2 = \sin^2\theta + \cos^2\theta = 1,$$

and hence $\{e_1^\theta, e_2^\theta\}$ is an orthonormal basis. To show that every orthonormal basis of $\mathbb{R}^2$ is $\{e_1^\theta, e_2^\theta\}$ for some θ, consider an orthonormal basis $\{b_1, b_2\}$ and note that b_1 forms an angle ϕ with the vector e_1 (which is e_1^0). Thus $b_1 = e_1^\phi$ and if $b_2 = e_2^\phi$, we are done, otherwise $b_2 = -e_2^\phi$ and it is the reflected version. However we can do the same thing except starting with b_2 and get $b_2 = e_1^\psi$ and $b_1 = e_2^\psi$ since we have just interchanged two basis vectors which corresponds to a reflection which picks up a minus sign as in the determinant.



G.68 Orthonormal Bases: Hint for Question 2, Lecture 21

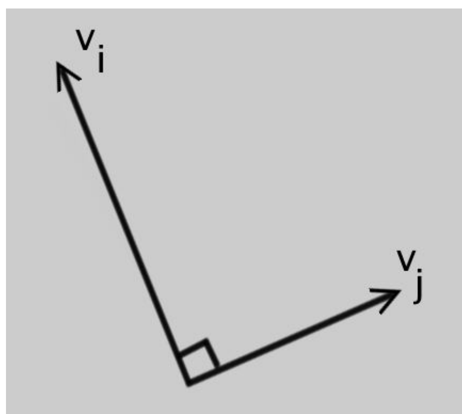
This video gives a hint for problem 2 in lecture 21. You are asked to consider an orthogonal basis $\{v_1, v_2, \ldots v_n\}$. Because this is a basis any $v \in V$ can be uniquely expressed as

$$v = c^1 v_1 + c^2 v_2 + \cdots + v^n c_n \,,$$

and the number $n = \dim V$. Since this is an orthogonal basis

$$v_i \cdot v_j = 0 \,, \qquad i \neq j \,.$$

So different vectors in the basis are orthogonal:



However, the basis is *not* orthonormal so we know nothing about the lengths of the basis vectors (save that they cannot vanish).

To complete the hint, lets use the dot product to compute a formula for c^1 in terms of the basis vectors and v. Consider

$$v_1 \cdot v = c^1 v_1 \cdot v_1 + c^2 v_1 \cdot v^2 + \cdots + c^n v_1 \cdot v_n = c^1 v_1 \cdot v_1 \,.$$

Solving for c^1 (remembering that $v_1 \cdot v_1 \neq 0$) gives

$$c^1 = \frac{v_1 \cdot v}{v_1 \cdot v_1} \,.$$

This should get you started on this problem.

G.69 Orthonormal Bases: Hint

This video gives a hint for problem 3 in lecture 21.

(a) Is the vector $v^{\perp} = v - \frac{u\cdot v}{u\cdot u}u$ in the plane P?

Remember that the dot product gives you a scalar not a vector, so if you think about this formula $\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}$ is a scalar, so this is a linear combination of $\mathbf{v}$ and $\mathbf{u}$. Do you think it is in the span?

(b) What is the angle between $v^{\perp}$ and u?

This part will make more sense if you think back to the dot product formulas you probably first saw in multivariable calculus. Remember that

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos(\theta),$$

and in particular if they are perpendicular $\theta = \frac{\pi}{2}$ and $\cos(\frac{\pi}{2}) = 0$ you will get $\mathbf{u}\cdot\mathbf{v} = 0$.

Now try to compute the dot product of u and $v^{\perp}$ to find $\|\mathbf{u}\|\|\mathbf{v}^{\perp}\|\cos(\theta)$

$$\begin{aligned} u\cdot v^{\perp} &= u\cdot\left(v - \frac{u\cdot v}{u\cdot u}u\right) \\ &= u\cdot v - u\cdot\left(\frac{u\cdot v}{u\cdot u}\right)u \\ &= u\cdot v - \left(\frac{u\cdot v}{u\cdot u}\right)u\cdot u \end{aligned}$$

Now you finish simplifying and see if you can figure out what θ has to be.

(c) Given your solution to the above, how can you find a third vector perpendicular to both u and $v^{\perp}$?

Remember what other things you learned in multivariable calculus? This might be a good time to remind your self what the cross product does.

(d) Construct an orthonormal basis for $\mathbb{R}^3$ from u and v.

If you did part (c) you can probably find 3 orthogonal vectors to make a orthogonal basis. All you need to do to turn this into an orthonormal basis is make these into unit vectors.

(e) Test your abstract formulae starting with

$$u = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix} \text{ and } v = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}.$$

Try it out, and if you get stuck try drawing a sketch of the vectors you have.

G.70 Gram-Schmidt and Orthogonal Complements: 4×4 Gram Schmidt Example

Lets do an example of how to "Gram-Schmidt" some vectors in $\mathbb{R}^4$. Given the following vectors

$$v_1 = \begin{pmatrix} o \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } v_4 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix},$$

we start with v_1

$$v_1^{\perp} = v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Now the work begins

$$\begin{aligned} v_2^{\perp} &= v_2 - \frac{(v_1^{\perp} \cdot v_2)}{\|v_1^{\perp}\|^2} v_1^{\perp} \\ &= \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{1} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \end{aligned}$$

This gets a little longer with every step.

$$\begin{aligned} v_3^{\perp} &= v_3 - \frac{(v_1^{\perp} \cdot v_3)}{\|v_1^{\perp}\|^2} v_1^{\perp} - \frac{(v_2^{\perp} \cdot v_3)}{\|v_2^{\perp}\|^2} v_2^{\perp} \\ &= \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{0}{1} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{1} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

This last step requires subtracting off the term of the form $\frac{u \cdot v}{u \cdot u}\mathbf{u}$ for each of the previously defined basis vectors.

$$\begin{aligned}
v_4^\perp &= v_4 - \frac{(v_1^\perp \cdot v_4)}{\|v_1^\perp\|^2} v_1^\perp - \frac{(v_2^\perp \cdot v_4)}{\|v_2^\perp\|^2} v_2^\perp - \frac{(v_3^\perp \cdot v_4)}{\|v_3^\perp\|^2} v_3^\perp \\
&= \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix} - \frac{1}{1} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{0}{1} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{3}{9} \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}
\end{aligned}$$

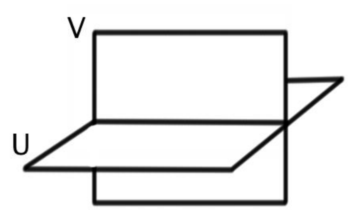
Now $v_1^\perp$, $v_2^\perp$, $v_3^\perp$, and $v_4^\perp$ are an orthogonal basis. Notice that even with very, very nice looking vectors we end up having to do quite a bit of arithmetic. This a good reason to use programs like matlab to check your work.

G.71 Gram-Schmidt and Orthogonal Complements: Overview

This video depicts the ideas of a subspace sum, a direct sum and an orthogonal complement in $\mathbb{R}^3$. Firstly, lets start with the subspace sum. Remember that even if U and V are subspaces, their union $U \cup V$ is usually not a subspace. However, the span of their union certainly is and is called the subspace sum

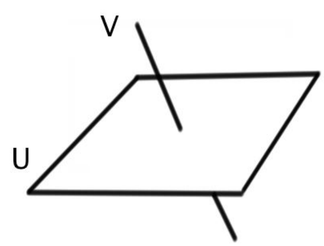
$$U + V = \text{span}(U \cup V)\,.$$

You need to be aware that this is a sum of vector spaces (not vectors). A picture of this is a pair of planes in $\mathbb{R}^3$:



Here $U + V = \mathbb{R}^3$.

Next lets consider a direct sum. This is just the subspace sum for the case when $U \cap V = \{0\}$. For that we can keep the plane U but must replace V by a line:

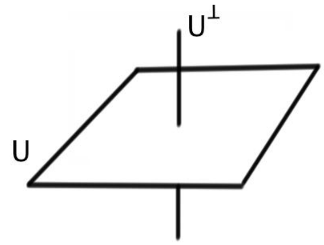


Taking a direct sum we again get the whole space, $U \oplus V = \mathbb{R}^3$.

Now we come to an orthogonal complement. There is not really a notion of subtraction for subspaces but the orthogonal complement comes close. Given U it provides a space $U^{\perp}$ such that the direct sum returns the whole space:

$$U \oplus U^{\perp} = \mathbb{R}^3\,.$$

The orthogonal complement $U^{\perp}$ is the subspace made from all vectors perpendicular to any vector in U. Here, we need to just tilt the line V above until it hits U at a right angle:



Notice, we can apply the same operation to $U^{\perp}$ and just get U back again, *i.e.*

$$\left(U^{\perp}\right)^{\perp} = U\,.$$

G.72 Gram-Schmidt and Orthogonal Complements: QR Decomposition Example

We can alternatively think of the QR decomposition as performing the Gram-Schmidt procedure on the *column space*, the vector space of the column vectors of the matrix, of the matrix M. The resulting orthonormal basis will be stored in Q and the negative of the coefficients will be recorded in R. Note that R is upper triangular by how Gram-Schmidt works. Here we will explicitly do an example with the matrix

$$M = \begin{pmatrix} | & | & | \\ m_1 & m_2 & m_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ -1 & 1 & 1 \end{pmatrix}.$$

First we normalize m_1 to get $m_1' = \frac{m_1}{\|m_1\|}$ where $\|m_1\| = r_1^1 = \sqrt{2}$ which gives the decomposition

$$Q_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 1 & -1 \\ 0 & 1 & 2 \\ -\frac{1}{\sqrt{2}} & 1 & 1 \end{pmatrix}, \qquad R_1 = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Next we find

$$t_2 = m_2 - (m_1' \cdot m_2)m_1' = m_2 - r_2^1 m_1' = m_2 - 0m_1'$$

noting that

$$m_1' \cdot m_1' = \|m_1'\|^2 = 1$$

and $\|t_2\| = r_2^2 = \sqrt{3}$, and so we get $m_2' = \frac{t_2}{\|t_2\|}$ with the decomposition

$$Q_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -1 \\ 0 & \frac{1}{\sqrt{3}} & 2 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & 1 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Finally we calculate

$$\begin{aligned} t_3 &= m_3 - (m_1' \cdot m_3)m_1' - (m_2' \cdot m_3)m_2' \\ &= m_3 - r_3^1 m_1' - r_3^2 m_2' = m_3 + \sqrt{2}m_1' - \frac{2}{\sqrt{3}}m_2', \end{aligned}$$

again noting $m_2' \cdot m_2' = \|m_2'\| = 1$, and let $m_3' = \frac{t_3}{\|t_3\|}$ where $\|t_3\| = r_3^3 = 2\sqrt{\frac{2}{3}}$. Thus we get our final $M = QR$ decomposition as

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{3} & -\frac{1}{\sqrt{6}} \end{pmatrix}, \qquad R = \begin{pmatrix} \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{3} & \frac{2}{\sqrt{3}} \\ 0 & 0 & 2\sqrt{\frac{2}{3}} \end{pmatrix}.$$

G.73 Gram-Schmidt and Orthogonal Complements: Hint for Problem 1

This video shows you a way to solve problem 1 that's different to the method described in the Lecture. The first thing is to think of

$$M = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ -1 & 2 & 2 \end{pmatrix}$$

as a set of 3 vectors

$$v_1 = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}.$$

Then you need to remember that we are searching for a decomposition

$$M = QR$$

where Q is an orthogonal matrix. Thus the upper triangular matrix $R = Q^T M$ and $Q^T Q = I$. Moreover, orthogonal matrices perform rotations. To see this compare the inner product $u \cdot v = u^T v$ of vectors u and v with that of Qu and Qv:

$$(Qu) \cdot (Qv) = (Qu)^T (Qv) = u^T Q^T Q v = u^T v = u \cdot v\,.$$

Since the dot product doesn't change, we learn that Q does not change angles or lengths of vectors.

Now, here's an interesting procedure: rotate v_1, v_2 and v_3 such that v_1 is along the x-axis, v_2 is in the xy-plane. Then if you put these in a matrix you get something of the form

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$

which is exactly what we want for R!

Moreover, the vector

$$\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$$

is the rotated v_1 so must have length $||v_1|| = \sqrt{3}$. Thus $a = \sqrt{3}$.

The rotated v_2 is

$$\begin{pmatrix} b \\ d \\ 0 \end{pmatrix}$$

and must have length $||v_2|| = 2\sqrt{2}$. Also the dot product between

$$\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} b \\ d \\ 0 \end{pmatrix}$$

is ab and must equal $v_1 \cdot v_2 = 0$. (That v_1 and v_2 were orthogonal is just a coincidence here... .) Thus $b = 0$. So now we know most of the matrix R

$$R = \begin{pmatrix} \sqrt{3} & 0 & c \\ 0 & 2\sqrt{2} & e \\ 0 & 0 & f \end{pmatrix}.$$

You can work out the last column using the same ideas. Thus it only remains to compute Q from

$$Q = MR^{-1}.$$

G.74 Diagonalizing Symmetric Matrices: 3×3 Example

Lets diagonalize the matrix

$$M = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

If we want to diagonalize this matrix, we should be happy to see that it is symmetric, since this means we will have real eigenvalues, which means factoring won't be too hard. As an added bonus if we have three distinct eigenvalues the eigenvectors we find will automatically be orthogonal, which means that the inverse of the matrix P will be easy to compute. We can start by finding the eigenvalues of this

$$\begin{aligned}
\det \begin{pmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{pmatrix} &= (1-\lambda) \begin{vmatrix} 1-\lambda & 0 \\ 0 & 5-\lambda \end{vmatrix} \\
&\quad - (2) \begin{vmatrix} 2 & 0 \\ 0 & 5-\lambda \end{vmatrix} + 0 \begin{vmatrix} 2 & 1-\lambda \\ 0 & 0 \end{vmatrix} \\
&= (1-\lambda)(1-\lambda)(5-\lambda) + (-2)(2)(5-\lambda) + 0 \\
&= (1-2\lambda+\lambda^2)(5-\lambda) + (-2)(2)(5-\lambda) \\
&= ((1-4) - 2\lambda + \lambda^2)(5-\lambda) \\
&= (-3 - 2\lambda + \lambda^2)(5-\lambda) \\
&= (1+\lambda)(3-\lambda)(5-\lambda)
\end{aligned}$$

So we get $\lambda = -1, 3, 5$ as eigenvectors. First find v_1 for $\lambda_1 = -1$

$$(M+I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

implies that $2x + 2y = 0$ and $6z = 0$,which means any multiple of $v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ is an eigenvector with eigenvalue $\lambda_1 = -1$. Now for v_2

with $\lambda_2 = 3$

$$(M-3I)\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-2 & 2 & 0\\ 2 & -2 & 0\\ 0 & 0 & 4\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix},$$

and we can find that that $v_2 = \begin{pmatrix}1\\1\\0\end{pmatrix}$ would satisfy $-2x+2y=0$, $2x-2y=0$ and $4z=0$.

Now for v_3 with $\lambda_3 = 5$

$$(M-5I)\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-4 & 2 & 0\\ 2 & -4 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix},$$

Now we want v_3 to satisfy $-4x+2y=0$ and $2x-4y=0$, which imply $x=y=0$, but since there are no restrictions on the z coordinate we have $v_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$.

Notice that the eigenvectors form an orthogonal basis. We can create an orthonormal basis by rescaling to make them unit vectors. This will help us because if $P = [v_1, v_2, v_3]$ is created from orthonormal vectors then $P^{-1} = P^T$, which means computing P^{-1} should be easy. So lets say

$$v_1 = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\\ 0\end{pmatrix},\ v_2 = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\\ 0\end{pmatrix},\ \text{and } v_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

so we get

$$P = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{pmatrix} \text{ and } P^{-1} = \begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{pmatrix}$$

So when we compute $D = P^{-1}MP$ we'll get

$$\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 0\\ 2 & 5 & 0\\ 0 & 0 & 5\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}-1 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 5\end{pmatrix}$$

G.75 Diagonalizing Symmetric Matrices: Hints for Problem 1

For part (a), we can consider any complex number z as being a vector in $\mathbb{R}^2$ where complex conjugation corresponds to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Can you describe $z\bar{z}$ in terms of $\|z\|$? For part (b), think about what values $a \in \mathbb{R}$ can take if $a = -a$? Part (c), just compute it and look back at part (a).

For part (d), note that $x^\dagger x$ is just a number, so we can divide by it. Parts (e) and (f) follow right from definitions. For part (g), first notice that every row vector is the (unique) transpose of a column vector, and also think about why $(AA^T)^T = AA^T$ for any matrix A. Additionally you should see that $\overline{x^T} = x^\dagger$ and mention this. Finally for part (h), show that

$$\frac{x^\dagger M x}{x^\dagger x} = \overline{\left(\frac{x^\dagger M x}{x^\dagger x}\right)^T}$$

and reduce each side separately to get $\lambda = \overline{\lambda}$.

G.76 Kernel, Range, Nullity, Rank: Invertibility Conditions

Here I am going to discuss some of the conditions on the invertibility of a matrix stated in Theorem 24.6. Condition 1 states that $X = M^{-1}V$ uniquely, which is clearly equivalent to 4. Similarly, every square matrix M uniquely corresponds to a linear transformation $L\colon \mathbb{R}^n \to \mathbb{R}^n$, so condition 3 is equivalent to condition 1.

Condition 6 implies 4 by the adjoint construct the inverse, but the converse is not so obvious. For the converse (4 implying 6), we refer back the proofs in Chapter 18 and 19. Note that if $\det M = 0$, there exists an eigenvalue of M equal to 0, which implies M is not invertible. Thus condition 8 is equivalent to conditions 4, 5, 9, and 10.

The map M is injective if it does not have a null space by definition, however eigenvectors with eigenvalue 0 form a basis for the null space. Hence conditions 8 and 14 are equivalent, and 14, 15, and 16 are equivalent by the Dimension Formula (also known as the Rank-Nullity Theorem).

Now conditions 11, 12, and 13 are all equivalent by the definition of a basis. Therefore condition 13 is equivalent to 2.

G.77 Kernel, Range, Nullity, Rank: Hint for 1

Lets work through this problem.

Let $L\colon V \to W$ be a linear transformation. Show that $\ker L = \{0_V\}$ if and only if L is one-to-one:

1. First, suppose that $\ker L = \{0_V\}$. Show that L is one-to-one.

   Remember what one-one means, it means whenever $L(x) = L(y)$ we can be certain that $x = y$. While this might seem like a weird thing to require this statement really means that each vector in the range gets mapped to a unique vector in the range.

   We know we have the one-one property, but we also don't want to forget some of the more basic properties of linear transformations namely that they are linear, which means $L(ax + by) = aL(x) + bL(y)$ for scalars a and b.

   What if we rephrase the one-one property to say whenever $L(x) - L(y) = 0$ implies that $x - y = 0$? Can we connect that to the statement that $\ker L = \{0_V\}$? Remember that if $L(v) = 0$ then $v \in \ker L = \{0_V\}$.

2. Now, suppose that L is one-to-one. Show that $\ker L = \{0_V\}$. That is, show that 0_V is in $\ker L$, and then show that there are no other vectors in $\ker L$.

   What would happen if we had a nonzero kernel? If we had some vector v with $L(v) = 0$ and $v \neq 0$, we could try to show that this would contradict the given that L is one-one. If we found x and y with $L(x) = L(y)$, then we know $x = y$. But if $L(v) = 0$ then $L(x) + L(v) = L(y)$. Does this cause a problem?

G.78 Least Squares: Hint for Problem 1

Lets work through this problem. Let $L : U \to V$ be a linear transformation. Suppose $v \in L(U)$ and you have found a vector $u_{\rm ps}$ that obeys $L(u_{\rm ps}) = v$.

Explain why you need to compute $\ker L$ to describe the solution space of the linear system $L(u) = v$.

Remember the property of linearity that comes along with any linear transformation: $L(ax + by) = aL(x) + bL(y)$ for scalars a and b. This allows us to break apart and recombine terms inside the transformation.

Now suppose we have a solution x where $L(x) = v$. If we have an vector $y \in \ker L$ then we know $L(y) = 0$. If we add the equations together $L(x) + L(y) = L(x + y) = v + 0$ we get another solution for free. Now we have two solutions, is that all?

G.79 Least Squares: Hint for Problem 2

For the first part, what is the transpose of a 1×1 matrix? For the other two parts, note that $v \cdot v = v^T v$. Can you express this in terms of $\|v\|$? Also you need the trivial kernel only for the last part and just think about the null space of M. It might help to substitute $w = Mx$.

H Student Contributions

Here is a collection of useful material created by students. The copyright to this work belongs to them as does responsibility for the correctness of any information therein.

- 4D TIC TAC TOE by Davis Shih.
- A hint for review problem 1, lecture 2 by Ashley Coates.
- A hint for review problem 1, lecture 12 by Philip Digiglio.
- Some cartoons depicting matrix multiplication by Asun Oka.
- An eigenvector example for lecture 18 by Ashley Coates.

I Other Resources

Here are some suggestions for other places to get help with Linear Algebra:

- Strang's MIT Linear Algebra Course. Videos of lectures and more:

  http://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/

- The Khan Academy has thousands of free videos on a multitude of topics including linear algebra:

  http://www.khanacademy.org/

- The Linear Algebra toolkit:

  http://www.math.odu.edu/∼bogacki/lat/

- Carter, Tapia and Papakonstantinou's online linear algebra resource

  http://ceee.rice.edu/Books/LA/index.html

- S.O.S. Mathematics Matrix Algebra primer:

  http://www.sosmath.com/matrix/matrix.html

- The numerical methods guy on youtube. Lots of worked examples:

  http://www.youtube.com/user/numericalmethodsguy

- Interactive Mathematics. Lots of useful math lessons on many topics:

  http://www.intmath.com/

- Stat Trek. A quick matrix tutorial for statistics students:

  http://stattrek.com/matrix-algebra/matrix.aspx

- Wolfram's Mathworld. An online mathematics encyclopædia:

http://mathworld.wolfram.com/

- Paul Dawkin's online math notes

http://tutorial.math.lamar.edu/

- Math Doctor Bob:

http://www.youtube.com/user/MathDoctorBob?feature=watch

- Some pictures of how to rotate objects with matrices:

http://people.cornellcollege.edu/dsherman/visualize-matrix.html

- xkcd. Geek jokes:

http://xkcd.com/184/

- See the bridge actually fall down:

http://anothermathgeek.hubpages.com/hub/What-the-Heck-are-Eigenvalues-and-Eigenvectors

J List of Symbols

$\in$	"Is an element of".
$\sim$	"Is equivalent to", see equivalence relations. Also, "is row equivalent to" for matrices.
$\mathbb{R}$	The real numbers.
I_n	The $n \times n$ identity matrix.
$P_n^{\mathbb{F}}$	The vector space of polynomials of degree at most n with coefficients in the field $\mathbb{F}$.

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