Written Exercise part III: Approximate Solutions

Example

To measure the strength of the gravitational field at the surface of mars, a mars rover picks up a rock, throws it, and video records the rock flying through the air. A team of volunteers watches the video and estimates the height of the rock in each frame of the video.

frame number	1	2	3	4	5
Estimated height	2	4	5	3	1

Estimate the acceleration due to gravity at the surface of mars based on this data. Describe ways that more data can be obtained to yield a better estimation.

Response:
If the acceleration of the rock due to gravity is constant a then the function $h : [0,5] \to \mathbb{R}$ that gives the height of the rock (in the units were used to describe the height of the rock) as a function of time (in units of time equal to the time between video frames) satisfies

$$\frac{d^2}{dt^2}h(t) = a \implies h(t) = \frac{1}{2}at^2 + bt + c \text{ for some } b, c \in \mathbb{R}.$$

If h is to fit the data exactly then (a,b,c) should be a solution to the system of equations

$$\left.\begin{array}{ll} \frac{1}{2}a1^2 + b1 + c & = 2 \\ \frac{1}{2}a2^2 + b2 + c & = 4 \\ \frac{1}{2}a3^2 + b3 + c & = 5 \\ \frac{1}{2}a4^2 + b4 + c & = 3 \\ \frac{1}{2}a5^2 + b5 + c & = 1 \end{array}\right\} \Leftrightarrow \begin{pmatrix} \frac{1}{2} & 1 & 1 \\ 2 & 2 & 1 \\ \frac{9}{2} & 3 & 1 \\ 8 & 4 & 1 \\ \frac{25}{2} & 5 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 5 \\ 3 \\ 1 \end{pmatrix}.$$

This system has no solution, as shown by row reduction of the associated augmented matrix;

$$\left(\begin{array}{ccc|c} \frac{1}{2} & 1 & 1 & 2 \\ 2 & 2 & 1 & 4 \\ \frac{9}{2} & 3 & 1 & 5 \\ 8 & 4 & 1 & 3 \\ \frac{25}{2} & 5 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 0 & -2 & -3 & -4 \\ 0 & -6 & -8 & -13 \\ 0 & 4 & 1 & 3 \\ 0 & -20 & -24 & -49 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & 6 & -9 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 7 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -10 \\ 0 & 0 & 0 & -3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 7 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The system encoded by the augmented matrix $(M|V)$ does not have a solution if V can not be expressed as a linear combination of the columns of M since this means that V is not in the range of M. Out of all the systems of equations $(M|U)$ that do have solutions the system with U closest to V is the system $(M|V_r)$ where V_r is the projection of V onto ran M. The matrix that projects a vector onto ran M is $M(M^TM)^{-}1M$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Applying this matrix to V gives

$$V_r := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 7 \\ -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

The equation $Mx = V_r$ then has augmented matrix

$$\left(\begin{array}{ccc|r} 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 7 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and unique solution

$$\begin{pmatrix} -1 \\ \frac{7}{2} \\ -1 \end{pmatrix}.$$

The quadratic function that best approximates the data set is then

$$h : [0, 5] \to \mathbb{R}$$

$$h(t) = \frac{1}{2}(-1)t^2 + \frac{7}{2}t + (-1).$$

In particular, the data indicate that acceleration at the surface of mars a is approximately -1 (in whatever units were used to described the height in the data set per the the time between video frames squared.) Mode data would yield a better approximation for this value. That data may come in the form of more video frames per second, data for another rock throw, throwing a rock harder so that its hang time is longer, or setting up a different experiment such as spherical balls rolling down inclined planes as Galileo used to measure earth's gravity.