LECTURE #10 INVERSE MATRIX

We say that a square matrix M is <u>invertible</u> (or <u>non-singular</u>) if there exist a matrix M^{-1} such that

$$\boxed{M^{-1}M = MM^{-1} = I}$$

<u>Example</u>

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$N = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$MN = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = (ad-bc)I$$

$\Rightarrow$ M has a inverse

$$M^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

whenever

$$ad \neq bc$$

PROPERTIES OF THE INVERSE

Notice we can read

$$A A^{-1} = I = A^{-1} A$$

as saying A^{-1} is ~~the~~ inverse of A, or as A is the inverse of A^{-1} (after all, to find ~~the~~ inverse of A^{-1}, we need to find B such that $BA^{-1} = I$, clearly $B = A$ works).

i.e. $\boxed{(A^{-1})^{-1} = A}$

Observe

$$B^{-1} A^{-1} A B = B^{-1} I B$$
$$= B^{-1} B = I = A B B^{-1} A^{-1}$$

which means $\boxed{(AB)^{-1} = B^{-1} A^{-1}}$

Finally, recall $(AB)^T = B^T A^T$ and $I^T = I \Rightarrow (A^{-1}A)^T = A^T (A^{-1})^T = I$, similarly $(AA^{-1})^T = (A^{-1})^T A^T = I$ so

$$\boxed{(A^{-1})^T = (A^T)^{-1} \equiv A^{-T}}$$

FINDING INVERSES

Suppose M is a square matrix and $MX = V$ has a unique solution X_0. Then

$$(M \mid V) \sim (I \mid M^{-1}V)$$

augmented matrix

reduced row echelon form

Identity matrix because <u>all</u> variables are pivot variables.

So consider a larger augmented matrix

$$(M \mid I) \sim (I \mid M^{-1}I)$$

Instead of a single constant vector put n constant vectors

Row ops.

$$= (I \mid M^{-1})$$

$$\underbrace{\left(\begin{array}{c:c:c:c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \\ \vdots & \vdots & \vdots & \end{array}\right)}_{\text{Block matrix}}$$

EX Find $\begin{pmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{pmatrix}^{-1}$

$$\left(\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & & \\ 2 & 1 & & & 1 & \\ 4 & -2 & 5 & & & 1 \end{array}\right)$$

$$\begin{array}{c} -R_1 \\ R_2+2R_1 \\ \sim \\ R_3+4R_1 \end{array} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & & \\ & 5 & -6 & 2 & 1 & \\ & 6 & -7 & 4 & & 1 \end{array}\right)$$

$$\begin{array}{c} R_2/5 \\ R_1+2R_2/5 \\ \sim \\ R_3-6R_2/5 \end{array} \left(\begin{array}{ccc|ccc} 1 & & 3/5 & -1/5 & 2/5 & \\ & 1 & -6/5 & 2/5 & 1/5 & \\ & & 1/5 & 8/5 & -6/5 & 1 \end{array}\right)$$

$$\begin{array}{c} 5R_3 \\ R_1-3R_3 \\ \sim \\ R_2+6R_3 \end{array} \left(\begin{array}{ccc|ccc} 1 & & & -5 & 4 & -3 \\ & 1 & & 10 & -7 & 6 \\ & & 1 & 8 & -6 & 5 \end{array}\right)$$

Check $\begin{pmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{pmatrix}\begin{pmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{pmatrix}$$

LINEAR SYSTEMS & INVERSES

Once we know M^{-1}, we can immediately solve the associated linear system.

<u>Ex</u>

$$\begin{aligned} -x + 2y - 3z &= 1 \\ 2x + y &= 2 \\ 4x - 2y + 5z &= 0 \end{aligned}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 2 & -3 \\ 2 & 1 & \\ 4 & -2 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \\ \ \end{pmatrix} = \begin{pmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ \ \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix}$$

$\Rightarrow x = 3,\ y = -4,\ z = -4$

$$\boxed{\text{i.e. when } M^{-1} \text{ exists, } MX = V \Rightarrow X = M^{-1}V}$$

Homogeneous systems

Consider a square matrix M.

Then

M^{-1} exists and $MX = 0$

$$\Rightarrow X = M^{-1}0 = 0$$

This means that $MX = 0$ has non-zero solutions $X \neq 0$ IF M is singular.

In fact this is an

iff = "If and only if"

statement because if M is singular it is not row equivalent to the identity. This means not all variables are pivots and the system has homogeneous solutions.

BIT MATRICES $M^r_k(\mathbb{Z}_2)$

Matrices over $\mathbb{Z}_2$

Instead of real numbers, consider bits $\{0, 1\}$ with multiplication and addition rules

+	0	1
0	0	1
1	1	0

x	0	1
0	0	0
1	0	1

Notice $-1 = 1$ because $1 + 1 = 0$

EX CLAIM

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

because

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Bit matrices often in computing & code theory. (c.f. "The Codebook" Singh)

Lecture 10 Review Questions

1. Let M be a square matrix. Explain why the following statements are equivalent

   (i) $MX = V$ has a unique solution for every column vector V

   (ii) M is non-singular.

[N.B. You must explain why (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i)]

2. Find a formula for

(i) $\begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix}^{-1}$ (ii) $\begin{pmatrix} a & b & c \\ & d & e \\ & & f \end{pmatrix}^{-1}$

when they are not singular.

Give conditions for them to be singular.

3. Write down all 2×2 bit matrices & decide which of them are singular.

(i) Suppose $\mathcal{U} = \mathbb{R}$ (real numbers).
Explain why $=$ is an equivalence relation but $\geq$ is not.

(ii) Explain why equivalence of augmented matrices is an equivalence relation.