LECTURE #11 LU Decomposition

Let M be a square matrix.

We will attempt to write

$$M = LU$$

where

(i) U is upper triangular:

$$U = (u^i_j) \quad \text{with} \quad u^{i>j}_j = 0$$

$$= \begin{pmatrix} u^1_1 & u^1_2 & u^1_3 & \cdots & \\ 0 & u^2_2 & u^2_3 & & \\ 0 & 0 & u^3_3 & \ddots & \\ \vdots & \vdots & & & u^n_n \end{pmatrix}$$

(ii) L is lower triangular:

$$L = (l^i_j) \quad \text{with} \quad l^i_{j>i} = 0$$

$$= \begin{pmatrix} l^1_1 & 0 & 0 & \cdots & \\ l^2_1 & l^2_2 & 0 & & \\ l^3_1 & l^3_2 & l^3_3 & \ddots & \\ \vdots & \vdots & & & l^n_n \end{pmatrix}$$

This a useful trick for many reasons including

- rapid (computer) solutions of linear systems
- inverses are easy
- determinants are easy

Triangular systems are rapidly solved by substitution

Ex
$$\left(\begin{array}{cc|c} a & b & d \\ & c & e \end{array}\right) \Rightarrow \begin{cases} y = e/c \\ x = \frac{1}{a}(d - be/c) \end{cases}$$

"back substitution"

$$\left(\begin{array}{ccc|c} 1 & & & d \\ a & 1 & & e \\ b & c & 1 & f \end{array}\right) \Rightarrow \begin{cases} x = d \\ y = e - ad \\ z = f - bd - c(e-ad) \end{cases}$$

SOLVING LINEAR SYSTEMS

Suppose we know $M = LU$ (an LU decomposition) & want to solve

$$MX = V$$

$$\Rightarrow LUX = V$$

Step ① Find (UX), easy because L is lower triangular

Step ② Suppose you found $UX = W$, now solve for X, easy because U is upper triangular.

EX
$$\begin{cases} 6x + 18y + 3z = 3 \\ 2x + 12y + z = 19 \\ 4x + 15y + 3z = 0 \end{cases}$$

Data

$$\begin{pmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{pmatrix} = \begin{pmatrix} 3 & & \\ 1 & 6 & \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 6 & 1 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$$

① Find $W = UX$ by solving

$$L(UX) = LW = V$$

$$\Rightarrow \begin{pmatrix} 3 & & \\ 1 & 6 & \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 3 \\ 19 \\ 0 \end{pmatrix}$$

FWD SUBSTITUTION

$\Rightarrow u = 1 \Rightarrow 6v = 19 - 1 \Rightarrow v = 3$

$\Rightarrow w = -2 - 9 = -11$

$$\Rightarrow W = \begin{pmatrix} 1 \\ 3 \\ -11 \end{pmatrix}$$

② Solve $UX = W$

$$\begin{pmatrix} 2 & 6 & 1 \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -11 \end{pmatrix}$$

BACK SUBSTITUTION

$\Rightarrow z = -11,\ y = 3 \Rightarrow 2x = 1 - 6.3 - (-11)$

$\Rightarrow \underline{x = -6,\ y = 3,\ z = -11}$

FINDING LU DECOMPOSITIONS

EX

$$\begin{pmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & \\ 1/3 & 1 & \\ 2/3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 3 & 1 \end{pmatrix}$$

Because multiply any $3 \times k$ matrix on the left by $\begin{pmatrix} 1 & & \\ 1/3 & 1 & \\ 2/3 & 0 & 1 \end{pmatrix}$

achieves
$$R_1 \longrightarrow R_1$$
$$R_2 \longrightarrow R_2 + \tfrac{1}{3} R_1$$
$$R_3 \longrightarrow R_3 + \tfrac{2}{3} R_1$$

and in the matrix on the right we have performed the opposite of these operations. Now the first column is in U form.

$$= \begin{pmatrix} 1 & & \\ 1/3 & 1 & \\ 2/3 & 1/2 & 1 \end{pmatrix} \begin{pmatrix} 6 & 18 & 3 \\ & 6 & 0 \\ & & 1 \end{pmatrix}$$

Are now fixing the 2nd column because left multiplication by $\begin{pmatrix} 1 & & \\ 1/3 & 1 & \\ 2/3 & 1/2 & 1 \end{pmatrix}$ achieves

$$R_1 \longrightarrow R_1$$
$$R_2 \longrightarrow R_2 + \tfrac{1}{3} R_1$$
$$R_3 \longrightarrow R_3 + 2/3\, R_1 + 1/2\, R_2$$

and we have made the opposite operations on the original matrix to find the matrix R

We have written $M = LU$ but the result is the not the same a before but, always for matrix multiplication

$$A \quad B \quad = \quad \underset{\text{multiply any row by } \lambda}{A'} \quad \underset{\text{multiply same row by } 1/\lambda}{B'}$$

Using this trick we have, as before

$$\begin{pmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{pmatrix} = \begin{pmatrix} 3 & & \\ 1 & 6 & \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 6 & 1 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$$

BLOCK LU DECOMPOSITIONS

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a

square matrix such that A^{-1} exists.

(So A & D are also square.)

Claim

$$M = \begin{pmatrix} 1 & 0 \\ CA^{-1} & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} 1 & A^{-1}B \\ 0 & 1 \end{pmatrix}$$

order matters!

Check

$$\text{RHS} = \begin{pmatrix} A & 0 \\ C\cancel{A^{-1}}\cancel{A} & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} 1 & A^{-1}B \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} A & AA^{-1}B \\ C & CA^{-1}B + D - CA^{-1}B \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;//$$

EX

diagonal

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}}$$

leading 1's "LDU" decomposition leading 1's

if you prefer LU $\longrightarrow = \begin{pmatrix} 1 & \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ & -2 \end{pmatrix}$

Lecture 11 Review Questions

1. Consider the linear system

$$\begin{cases} x^1 & = v^1 \\ l^2_1 x^1 + x^2 & = v^2 \\ \quad \vdots & \\ l^n_1 x^1 + l^n_2 x^2 + \dots + x^n & = v^n \end{cases}$$

(i) Find x^1

(ii) Find x^2

(iii) Find x^3

Try to write a formula for x^k

2. Let $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a square, $n \times n$ block matrix with D invertible.

(i) If D is invertible, what size are B, C & D?

(ii) Find a UDL block decomposition for this matrix. i.e. fill in the $*$'s

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & \\ * & 1 \end{pmatrix} \begin{pmatrix} * & \\ & * \end{pmatrix} \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$$

(iii) Apply your result ~~to~~ the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ for two different block partitions.

3. Write down all 2x2 bit matrices & decide which of them are singular.

(i) Suppose $\mathcal{U} = \mathbb{R}$ (real numbers).
Explain why $=$ is an equivalence relation but $\geq$ is not.

(ii) Explain why equivalence of augmented matrices is an equivalence relation.