LECTURE #12 ELEMENTARY MATRICES & DETERMINANTS

• HchIv is excellent!

Focus on square matrices and ask when are they invertible?

<u>EX</u> 1×1 matrix $M = (m)$

Invertible if $m \neq 0 \Rightarrow M^{-1} = (\frac{1}{m})$

<u>EX</u> 2×2 matrix $M = \begin{pmatrix} m^1_1 & m^1_2 \\ m^2_1 & m^2_2 \end{pmatrix}$

Invertible if $m^1_1 m^2_2 - m^1_2 m^2_1 \neq 0$

$$\Rightarrow M^{-1} = \frac{1}{m^1_1 m^2_2 - m^1_2 m^2_1} \begin{pmatrix} m^2_2 & -m^1_2 \\ -m^2_1 & m^2_2 \end{pmatrix}$$

[In earlier examples we called

$m^1_1 = a$, $m^1_2 = b$, $m^2_1 = c$, $m^2_2 = d$

and found $ad - bc \neq 0$ as the condition]

For 2×2 matrices we call the <u>determinant</u>

$$\boxed{\det M = \det \begin{pmatrix} m^1_1 & m^1_2 \\ m^2_1 & m^2_2 \end{pmatrix} = m^1_1 m^2_2 - m^1_2 m^2_1}$$

M is singular iff $\det M = 0$.

<u>EX</u> (See review question 1.) 3×3 matrix

$$M = \begin{pmatrix} m^1_1 & m^1_2 & m^1_3 \\ m^2_1 & m^2_2 & m^2_3 \\ m^3_1 & m^3_2 & m^3_1 \end{pmatrix}$$ is non-singular iff

$$\boxed{\begin{aligned} \det M = {} & m^1_1 m^2_2 m^3_3 - m^1_2 m^2_1 m^3_3 + m^1_2 m^2_3 m^3_1 \\ & - m^1_3 m^2_2 m^3_1 + m^1_3 m^2_1 m^3_2 - m^1_1 m^2_3 m^3_2 \end{aligned}} \neq 0$$

Permutations

Consider n objects and shuffle them

$$1 \quad 2 \quad 3 \quad \cdots \quad n$$
$$\downarrow$$
$$7 \quad 3 \quad 5 \qquad\quad 1$$

Each possible shuffle is called a <u>permutation</u> σ. We can encode σ as a list of numbers b/w 1 & n

$$\sigma(1) \quad \sigma(2) \quad \cdots\cdots \quad \sigma(n) = \sigma$$

saying what happened to each number.

<u>Example</u> $\sigma = 312$ is the permutation of 3 objects

$$1 \quad 2 \quad 3$$
$$\downarrow$$
$$3 \quad 1 \quad 2$$

Key facts about permutations

(i) There are $n!$ different permutations of n distinct objects.

(ii) Every permutation can be achieved by successively swapping pairs of objects.

(iii) For $n \geq 2$, $n!$ is even. In fact, there are $n!/2$ "even permutations" made from an even number of pair swappings and $n!/2$ "odd permutations" made from an odd number of pair swappings.

Call
$$\mathrm{sgn}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$

Then the determinant of an $n \times n$ matrix

$$M = \begin{pmatrix} m^1_1 & m^1_2 & \cdots & \\ m^2_1 & m^2_2 & & \\ \vdots & & & \\ & & & m^n_n \end{pmatrix}$$

is given by

$$\boxed{\det M = \sum_{\sigma} \mathrm{sgn}(\sigma)\, m^1_{\sigma(1)}\, m^2_{\sigma(2)} \cdots m^n_{\sigma(n)}}$$

* The sum is over all permutations with signs according to whether the permutation is even or odd

* The summands are products of a single entry from each row but with the column numbers shuffled by the permutation σ.

<u>EX</u>

$$\det \begin{pmatrix} m^1_1 & m^1_2 & m^1_3 & m^1_4 \\ m^2_1 & m^2_2 & m^2_3 & m^2_4 \\ m^3_1 & m^3_2 & m^3_3 & m^3_4 \\ m^4_1 & m^4_2 & m^4_3 & m^4_4 \end{pmatrix}$$

$$= \underbrace{m^1_1 m^2_2 m^3_3 m^4_4}_{\text{trivial permutation}} - \underbrace{m^1_2 m^2_1 m^3_3 m^4_4}_{\text{single swap}}$$

$$+ \underbrace{m^1_2 m^2_1 m^3_4 m^4_3}_{\text{two swaps}} - \underbrace{m^1_2 m^2_4 m^3_1 m^4_3}_{\text{three swap}}$$

$+$ (10 more even permutations) $-$ (10 more odd permutations)

<u>VERY CUMBERSOME!</u>

Easy special case, M diagonal.

Then $\boxed{\det M = m^1_1 m^2_2 \cdots m^n_n}$, the product of diagonal entries, because permutations yield vanishing off-diagonal entries. Observe, the identity has $\boxed{\det I = 1}$.

Since the determinant is meant to detect when a matrix is singular/invertible & we were able to determine invertibility using row operations, studying row operations can help us understand the determinant.

Swapping rows: The determinant formula for M with rows $i<j$ swapped reads

$$\sum_{\sigma} \operatorname{sgn}(\sigma)\, m^1_{\sigma(1)} m^2_{\sigma(2)} \cdots m^j_{\sigma(i)} \cdots m^i_{\sigma(j)} \cdots m^n_{\sigma(n)}$$

$$= \sum_{\sigma} \operatorname{sgn}(\sigma)\, \overbrace{\underbrace{m^1_{\sigma(1)} m^2_{\sigma(2)} \cdots m^i_{\sigma(j)} \cdots m^j_{\sigma(i)} \cdots m^n_{\sigma(n)}}_{\text{wrecked the order}}}^{\text{fixed the order}}$$

$$= \sum_{\sigma} \left[-\operatorname{sgn}(\tilde{\sigma})\right] m^1_{\tilde{\sigma}(1)} m^2_{\tilde{\sigma}(2)} \cdots m^i_{\tilde{\sigma}(i)} \cdots m^j_{\tilde{\sigma}(j)} \cdots m^n_{\tilde{\sigma}(n)}$$

$\tilde{\sigma}$ is permutation with i & j swapped, it has opposite sign to σ

$$= -\sum_{\tilde{\sigma}} \operatorname{sign}(\tilde{\sigma})\, m^1_{\tilde{\sigma}(1)} m^2_{\tilde{\sigma}(2)} \cdots m^n_{\tilde{\sigma}(n)} = -\det M$$

we anyway sum over all permutations

We have found an important result:
if M & M' differ by swapping a pair of rows then $\det M = -\det M'$

Lets write the matrix M as a block matrix of row vectors M^i
(NOT i^{th} power of M)

$$M = \begin{pmatrix} \vdots \\ M^i \\ \vdots \\ M^j \\ \vdots \end{pmatrix} \qquad \text{so} \quad M' = \begin{pmatrix} \vdots \\ M^j \\ \vdots \\ M^i \\ \vdots \end{pmatrix}$$

and notice

$$\begin{pmatrix} \vdots \\ M^j \\ \vdots \\ M^i \\ \vdots \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & & 1 & \\ & & & \ddots & & \\ & & 1 & & 0 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}}_{\text{identity matrix with rows } i \,\&\, j \text{ swapped}} \begin{pmatrix} \vdots \\ M^i \\ \vdots \\ M^j \\ \vdots \end{pmatrix}$$

or symbolically

$$M' = E^i_j M$$

where the elementary matrix E^i_j identity with rows i & j swapped.

Because $\det I = 1$ and swapping a pair of rows flips the sign of the determinant we have found

$$\det E^i_j = 1$$

We still need to study the other row operations, find their associated elementary matrices and their determinants.

Before doing so here are two more properties of the determinant

(i) If M has a row of zeros,

$$\det M = 0$$

(ii) If M has two identical rows

$$\det M = 0$$

The first comes directly from the determinant formula. The second is obtained by swapping the identical rows which implies $\det M = -\det M$.

Lecture 12 Review Questions

1. Consider the 3×3 matrix

$$M = \begin{pmatrix} m^1_1 & m^1_2 & m^1_3 \\ m^2_1 & m^2_2 & m^2_3 \\ m^3_1 & m^3_2 & m^3_3 \end{pmatrix}$$

Perform row operations to achieve <u>row echelon form</u>.

For simplicity assume $m^1_1 \neq 0 \neq m^1_1 m^2_2 - m^2_1 m^1_2$. Explain why the condition

$$m^1_1 m^2_2 m^3_3 - m^1_1 m^2_3 m^3_2 + m^1_3 m^2_1 m^3_2$$
$$- m^1_3 m^2_2 m^3_1 + m^1_2 m^2_3 m^3_1 - m^1_2 m^2_1 m^3_3 \neq 0$$

ensures that M is non singular.

2. What does the matrix

$$E^1_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ do to } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

by left multiplication? What about right multiplication?

2ct. Can you find elementary matrices $R^1(\lambda)$ and $R^2(\lambda)$ that multiply row 1 and row 2 by a constant λ, respectively when you compute $R^1(\lambda)M$ and $R^2(\lambda)M$?

What about the matrix $S_2^1(\lambda)$ that adds a multiple λ of row 2 to row 1 when you compute $S_2^1(\lambda)M$?

(i) Suppose $\mathcal{U} = \mathbb{R}$ (real numbers). Explain why $=$ is an equivalence relation but $\geq$ is not.

(ii) Explain why equivalence of augmented matrices is an equivalence relation.