LECTURE #12 ELEMENTARY MATRICES & DETERMINANTS ct.

Consider $M = \begin{pmatrix} M^1 \\ \vdots \\ M^n \end{pmatrix}$, M^i are row vectors,

and

$$M' = \begin{pmatrix} M^1 \\ \vdots \\ \lambda M^i \\ \vdots \\ M^n \end{pmatrix} = R^i(\lambda)\, M$$

identity matrix with i^{th} row multiplied by λ.

Then $\det M' = \sum_\sigma \operatorname{sgn}(\sigma)\, m^1_{\sigma(1)} \cdots \lambda m^i_{\sigma(i)} \cdots m^n_{\sigma(n)}$

$= \lambda \det M$

Multiplying a row by λ multiplies the determinant by λ.

The elementary matrix $R^i(\lambda)$ has determinant

$$\det R^i(\lambda) = \det \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} = \lambda$$

The final row operation is adding λR_j to R_i

$$M' = \begin{pmatrix} \vdots \\ M^i + \lambda M^j \\ \vdots \\ M^j \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & \ddots \lambda & \\ & & & 1 & \\ & & & & \ddots & 1 \end{pmatrix} \begin{pmatrix} \vdots \\ M^i \\ \vdots \\ M^j \\ \vdots \end{pmatrix}$$

or in symbols

$$M' = S^i_j(\lambda) M$$

the identity matrix with $R_i \to R_i + \lambda e_j$

This leaves the determinant unchanged because

$$\sum_\sigma \mathrm{sgn}(\sigma)\, m^1_{\sigma(1)} \cdots \left(m^i_{\sigma(i)} + \lambda m^j_{\sigma(i)}\right) \cdots m^n_{\sigma(n)}$$

$$= \sum_\sigma \mathrm{sgn}(\sigma)\, m^1_{\sigma(1)} \cdots m^n_{\sigma(n)} + \lambda \sum_\sigma \mathrm{sgn}(\sigma)\, m^1_{\sigma(1)} \cdots m^j_{\sigma(i)} \cdots m^j_{\sigma(j)} \cdots m^n_{\sigma(n)}$$

$= \det(M) + \lambda \det[\text{MATRIX WITH TWO EQUAL ROWS}]$

$= \det(M)$

Hence $\boxed{\det S^i_j(\lambda) = 1}$

Now we have elementary row matrices

E^i_j = identity $\bar{c}$ $R_i \leftrightarrow R_j$, det $= -1$

$R^i(\lambda)$ = identity $\bar{c}$ $R_i \to \lambda R_i$, det $= \lambda$

$S^i_j(\lambda)$ = identity $\bar{c}$ $R_i \to R_i + \lambda R_j$, det $= 1$

and they perform the corresponding row operation when multiplying a matrix from the left.

If E is <u>any</u> of the above elementary row matrices

$$\det(EM) = \det E \, \det M$$

because we showed above that the effect of each row operation corresponds to multiplying by the determinant of the corresponding elementary row matrix.

Now any matrix

$$M \sim \underbrace{RREF(M)}_{\text{the RREF of } M}$$

so

$$\underbrace{E_1 E_2 \cdots E_k}_{\substack{\text{some sequence of}\\ \text{elementary row}\\ \text{matrices.}}} M = RREF(M)$$

Hence by the above

$$\underbrace{\det(E_1)\cdots\det(E_k)}_{\substack{\text{but these are}\\ \text{all just } \pm 1 \text{ or } \lambda \neq 0}} \det(M) = \det(RREF(M))$$

Hence $\det M = 0$ precisely when $\det(RREF(M)) = 0$.

If M is invertible, $RREF(M) = I$

so $\det M \neq 0$ in this case since $\det I = 1$

If M is not invertible $RREF(M)$ has at least one row of zeros $\Rightarrow \det(RREF(M)) = 0$.

Therefore we have shown

$$\det M \neq 0 \text{ iff } M \text{ is invertible.}$$

i.e matrices with vanishing determinant cannot be inverted and these are the only non-invertible matrices.

Another very important result follows:

For any square matrices M & N

$$\boxed{\det MN = (\det M)(\det N)}$$

To see why, just write M & N as products of elementary matrices times their RREFs and use the above.

Lecture 13 Review Questions

1. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $N = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$

   Compute

   (i) $\det M$

   (ii) $\det N$

   (iii) MN

   (iv) $\det(MN)$

   (v) $\det M \cdot \det N$

   (vi) $\det M^{-1}$ assuming $ad-bc \neq 0$

   (vii) $\det M^T$

   (viii) $\det(M+N) - \det M - \det N$

2. Suppose $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible. Write M as a product of elementary row matrices times its RREF

(i) Suppose $\mathcal{U} = \mathbb{R}$ (real numbers).
Explain why $=$ is an equivalence relation but $\geq$ is not.

(ii) Explain why equivalence of augmented matrices is an equivalence relation.