LECTURE 14 PROPERTIES OF THE DETERMINANT

Calculating det M

$$\det M = \sum_{\sigma}{}' \operatorname{sgn}(\sigma)\, m^1_{\sigma(1)}\, m^2_{\sigma(2)} \cdots m^n_{\sigma(n)}$$

$$= m^1_1 \sum_{\hat\sigma}{}' \operatorname{sgn}(\hat\sigma)\, m^2_{\hat\sigma(2)} \cdots m^n_{\hat\sigma(n)}$$

$$- m^1_2 \sum_{\hat\sigma}{}' \operatorname{sgn}(\hat\sigma)\, m^1_{\hat\sigma(1)} \cdots m^n_{\hat\sigma(n)}$$

$$+ m^1_3 \sum_{\hat\sigma}{}' \operatorname{sgn}(\hat\sigma)\, m^1_{\hat\sigma(1)}\, m^{\cdot}_{\hat\sigma(3)} \cdots m^n_{\hat\sigma(n)}$$

$$\vdots$$

$\hat\sigma$ are permutations over $n-1$ objects

Expanding sum in 1st row. Notice signs alternate

Each sum is the determinant of the matrix left over after removing the 1st row and a column corresponding to which m^1_i is factored out front.

EXAMPLE

$$\det\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$$

$$= 1\det\begin{pmatrix}5&6\\8&9\end{pmatrix} - 2\det\begin{pmatrix}4&6\\7&9\end{pmatrix} + 3\det\begin{pmatrix}4&5\\7&8\end{pmatrix}$$

expand in 1st row

$$= (5\cdot 9 - 6\cdot 8) - 2(4\cdot 9 - 6\cdot 7) + 3\cdot(4\cdot 8 - 5\cdot 7) = 0$$

so $\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}^{-1}$ does NOT exist.

EXAMPLE

swapped rows

$$\det\begin{pmatrix}1&2&3\\4&0&0\\5&6&7\end{pmatrix} = -\det\begin{pmatrix}4&0&0\\1&2&3\\5&6&7\end{pmatrix}$$

$$= 4\det\begin{pmatrix}2&3\\6&7\end{pmatrix} = -16$$

The same method works expanding in any column because

$$\boxed{\det M^T = \det M}$$

This is true because

$$\sum_{\sigma} \operatorname{sgn}(\sigma)\, m^1_{\sigma(1)} \cdots m^n_{\sigma(n)} = \sum_{\sigma} \operatorname{sgn}(\sigma)\, m_1^{\sigma(1)} \cdots m_n^{\sigma(n)}$$

i.e. commuting row or column entries gives same sum of terms.

DETERMINANT OF THE INVERSE

Let A, B be $n \times n$ matrices

Recall we found

$$\det AB = \det A \det B$$

$$\det I = 1$$

$$\Rightarrow 1 = \det A^{-1}A = \det A^{-1} \det A$$

$$\Rightarrow \boxed{\det A^{-1} = \frac{1}{\det A}}$$

Writing matrices as products of simpler ones is a good trick in computing the determinant.

INVERSES AGAIN

The adjoint of a matrix $M = (m^i_j)$

$$\operatorname{adj} M = \left(\operatorname{cofactor}(m^i_j)\right)^T$$

where the $\operatorname{cofactor}(m^i_j)$ is the determinant of M with row i and column j removed and multiplied by $(-)^{i+j}$

EX $\operatorname{adj}\begin{pmatrix} 3 & -1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

$$= \begin{pmatrix} \det\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} & -\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \det\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ -\det\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} & \det\begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix} & -\det\begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix} \\ \det\begin{pmatrix} -1 & -1 \\ 2 & 0 \end{pmatrix} & -\det\begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} & \det\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} \end{pmatrix}^T$$

Notice That the dot product of the ith row of M & the ith row of $(\operatorname{adj} M)^T$ is just the expansion of $\det M$ in the ith row.

For the i^{th} and j^{th} rows $(i \neq j)$ the dot product amounts to expanding $\det M$ in the i^{th} row save that the j^{th} row of M is replaced by its i^{th} row. The determinants of a matrix with two equal rows vanish, so these dot products do as well.

Finally note that computing dot products of the rows of M & N^T amounts to computing the elements of MN

Hence

$$M \, \mathrm{adj}(M) = \det M \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

Thus computing $\mathrm{adj}(M)$ gives us a formula for M^{-1} when it exists

$$\boxed{M^{-1} = \frac{1}{\det M}\,\mathrm{adj}(M)}$$

when $\det M \neq 0$.

EX In the above example

$$\mathrm{adj}\begin{pmatrix} 3 & -1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ -1 & 3 & -1 \\ 1 & -3 & 7 \end{pmatrix}$$

and

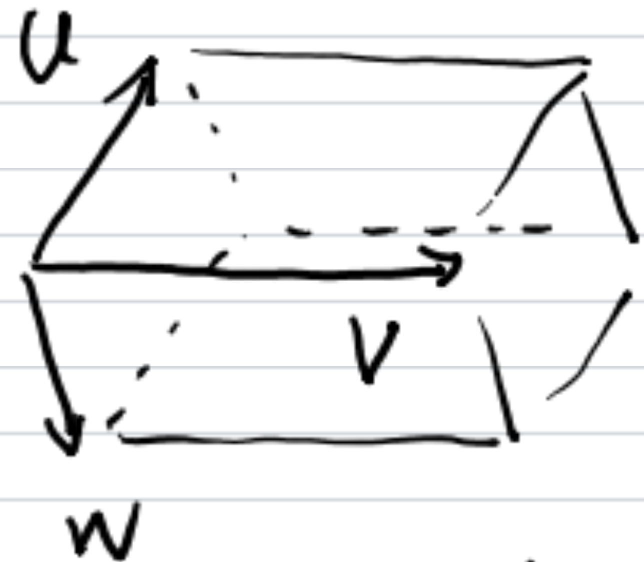
$$\begin{pmatrix} 2 & 0 & 2 \\ -1 & 3 & -1 \\ 1 & -3 & 7 \end{pmatrix}\begin{pmatrix} 3 & -1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3 & -1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \frac{1}{6}\begin{pmatrix} 2 & 0 & 2 \\ -1 & 3 & -1 \\ 1 & -3 & 7 \end{pmatrix}$$

This is sometimes called Cramer's rule.

APPLICATION

Volume of parallelepiped



$$\text{volume} = \left|\det\begin{pmatrix} u & v & w \end{pmatrix}\right|$$

This comes from the formula you know which says the volume is $|u\cdot(v\times w)|$ and using the fact that this amounts to computing the determinant by expanding in the first column.

Do you think a similar formula could work for computing higher dimensional volumes?

Lecture 14 Review Questions:

1. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

   Show

   $$\det M = \frac{1}{2} \operatorname{tr} M^2 - \frac{1}{2} (\operatorname{tr} M)^2$$

   Suppose M is a 3×3 matrix. Find and verify a similar formula for $\det M$ in terms of $\operatorname{tr} M^3$, $\operatorname{tr} M \cdot \operatorname{tr} M^2$, $(\operatorname{tr} M)^3$.

2. Suppose $M = LU$ is an LU decomposition. Explain how you would compute $\det M$ efficiently in this case.