LECTURE 15 EIGENVALUES & EIGENVECTORS

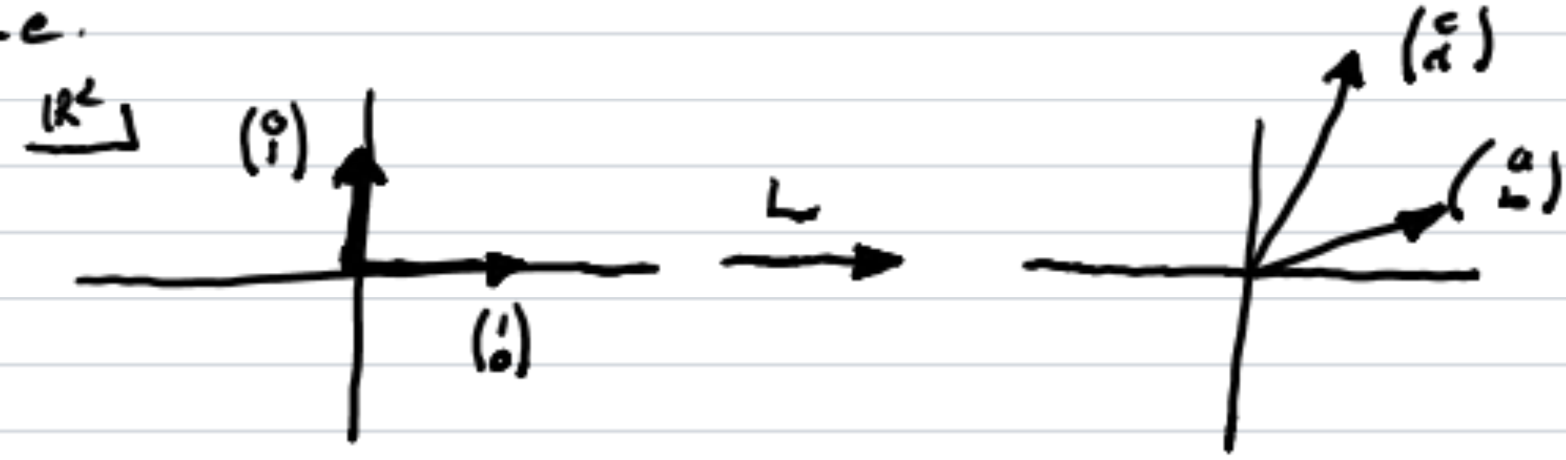
Consider a linear transformation

$$L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

First notice if we know

$$L\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\c\end{pmatrix} \;\&\; L\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}b\\d\end{pmatrix}$$

i.e.



Then we know L because

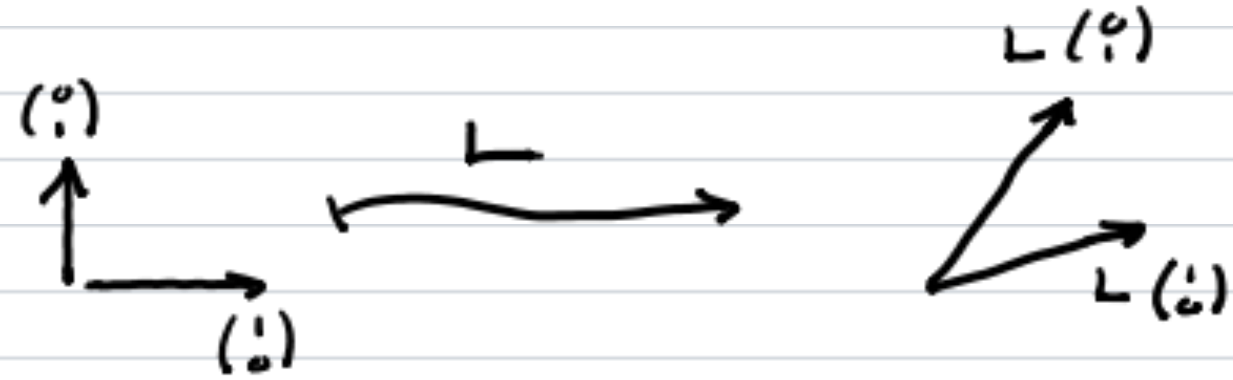
$$L\begin{pmatrix}x\\y\end{pmatrix} = L\left(x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix}\right) = L\begin{pmatrix}1\\0\end{pmatrix}x + L\begin{pmatrix}0\\1\end{pmatrix}y$$

$$= \begin{pmatrix}a\\c\end{pmatrix}x + \begin{pmatrix}b\\d\end{pmatrix}y = \begin{pmatrix}ax+by\\cx+dy\end{pmatrix} = \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

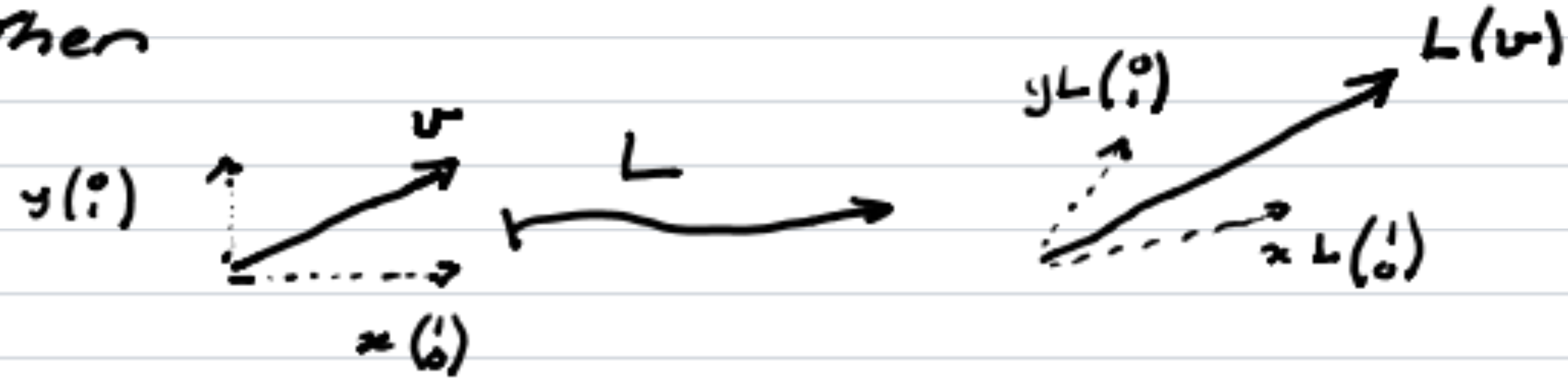
Call $\begin{pmatrix}a & b\\c & d\end{pmatrix}$ the <u>matrix of L</u> in the <u>basis</u> $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ (more on these concepts later). But maybe the is a way to represent L as a matrix that is simpler (think back to apples, oranges, fruit and sugar...)

Invariant directions

If



Then



From the above picture we see that it might be possible to choose a vector v such that $L(v)$ is in the same direction as v.

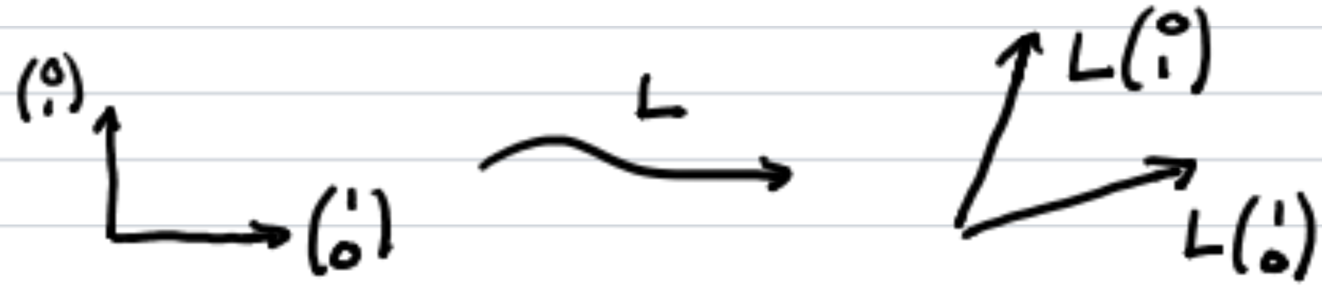
i.e. $\quad L(v) \parallel v$

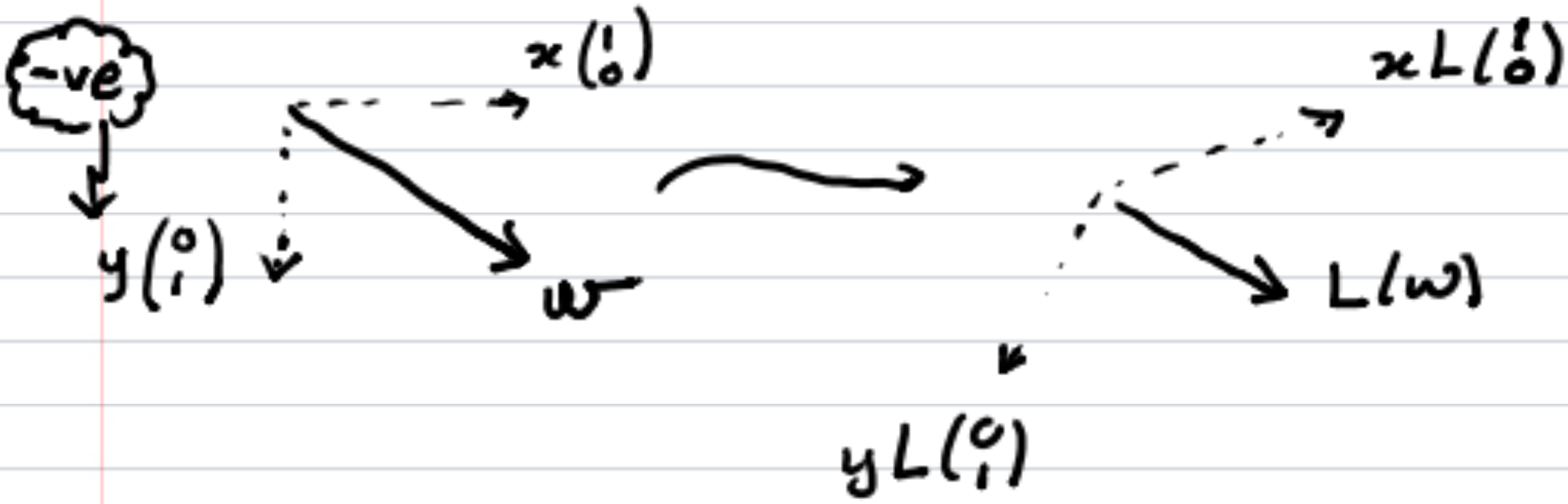
or in an equation $(\lambda \in \mathbb{R})$

$$\boxed{L(v) = \lambda v} \qquad \lambda \neq 0 \neq v$$

Some names, if v and λ exist we call the direction of v an <u>invariant direction</u> of L. v is called an <u>eigenvector</u> of L and λ its corresponding <u>eigenvalue</u>.

Look for more invariant directions:





For this linear transformation it seems that there could exist a second invariant direction

$$L(w) = \mu w$$

Previously we wrote

$$L\Big(\underbrace{x\binom{1}{0} + y\binom{0}{1}}_{\text{an arbitrary vector in the plane}}\Big) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\binom{x}{y}$$

But what if we could write
an arbitrary vector in the plane X
in terms of eigenvectors v and w?

i.e.

$X = \dot{x} v + \dot{y} w$

then

$$L(X) = L(\dot{x} v + \dot{y} w) = \dot{x} L(v) + \dot{y} L(w)$$

$$= \lambda \dot{x} + \mu \dot{y} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

The "new coordinates" give L a much simpler diagonal matrix made from eigenvalues.

This is called diagonalization and can make complicated linear systems much simpler to analyze.

__Example__ $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

where

$$L(x,y) = (2x+2y,\ 16x+6y)$$

First notice

$$\begin{pmatrix} x \\ y \end{pmatrix} \xmapsto{L} \begin{pmatrix} 2 & 2 \\ 16 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The matrix of L

We want to find an invariant direction $v = \begin{pmatrix} x \\ y \end{pmatrix}$ so solve

$$L(v) = \lambda v$$

or in matrix notation

$$\begin{pmatrix} 2 & 2 \\ 16 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 & 2 \\ 16 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2-\lambda & 2 \\ 16 & 6-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is a homogeneous system, so it only has solutions when

$$\begin{pmatrix} 2-\lambda & 2 \\ 16 & 6-\lambda \end{pmatrix} \text{ is singular.}$$

i.e.

$$\det \begin{pmatrix} 2-\lambda & 2 \\ 16 & 6-\lambda \end{pmatrix} = 0$$

$$\Rightarrow (2-\lambda)(6-\lambda) - 32 = 0$$

$$\Rightarrow \lambda^2 - 8\lambda - 20 = 0$$

$$\Rightarrow (\lambda - 10)(\lambda + 2) = 0$$

Two eigenvalues $\lambda = 10$, $\lambda = -2$.

What about eigenvectors?

Need to do both cases separately

Study

$$\begin{pmatrix} 2-\lambda & 2 \\ 16 & 6-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

when

(i) $\underline{\lambda = 10}$

$$\begin{pmatrix} -8 & 2 \\ 16 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

both equations say $y = +4x$

so $$v = \begin{pmatrix} x \\ +4x \end{pmatrix}$$

but since we only need the direction

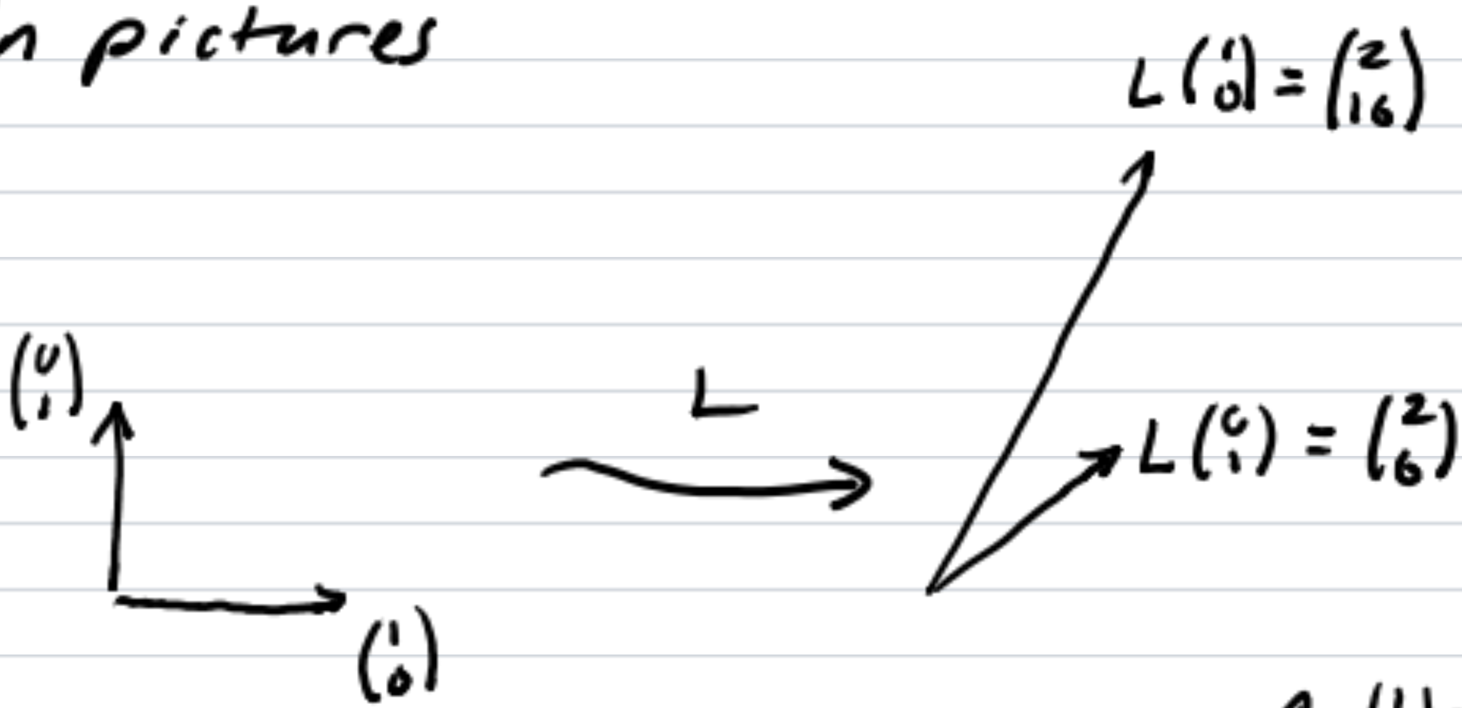
$v = \begin{pmatrix} 1 \\ +4 \end{pmatrix}$ is equally good.

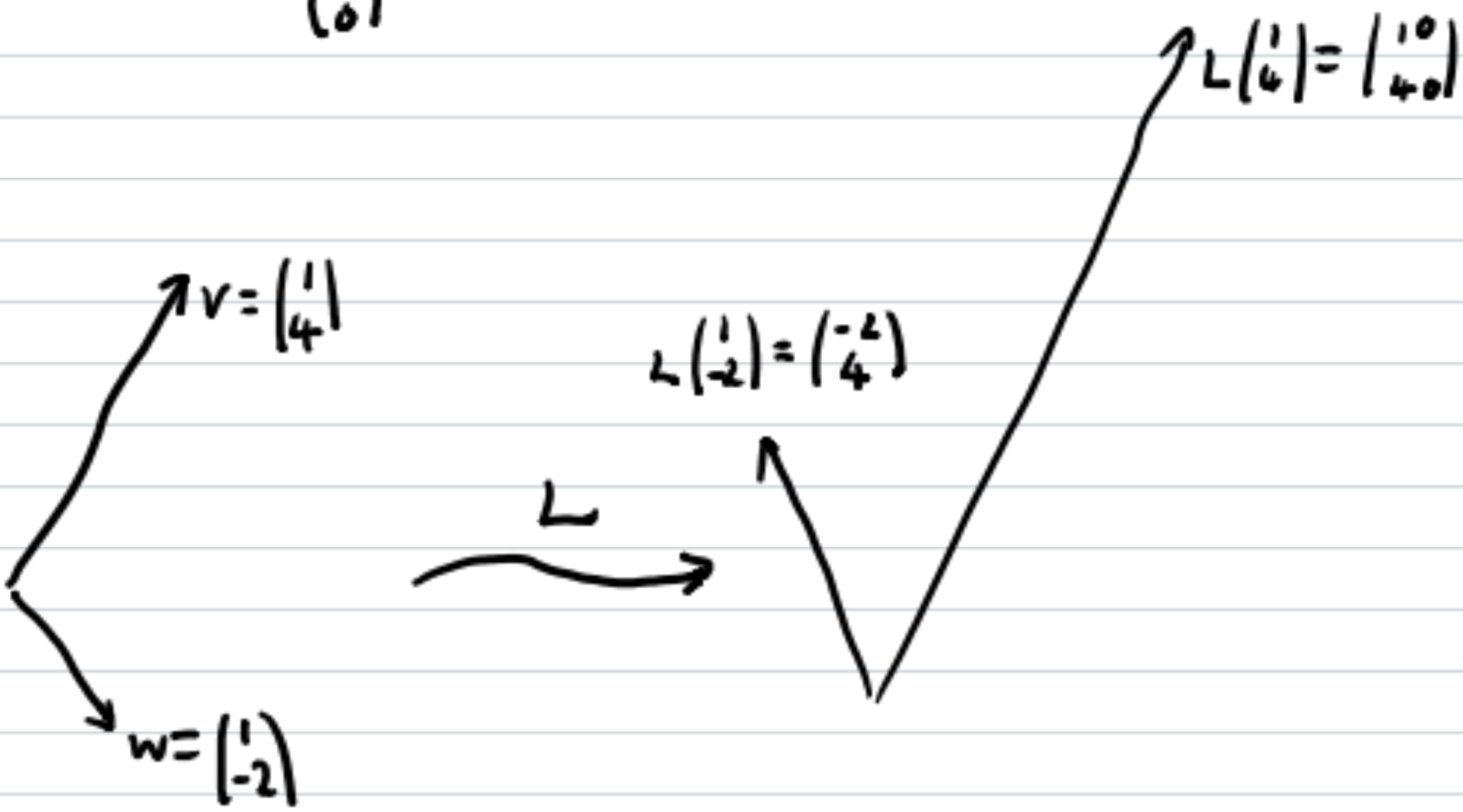
(ii) $\underline{\lambda = -2}$ $$\begin{pmatrix} 4 & 2 \\ 16 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so $y = -2x$ and $w = \begin{pmatrix} x \\ -2x \end{pmatrix}$ or

just $w = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

In pictures





Lecture 15 Review Questions:

① Consider $L: \mathbb{R}^2 \to \mathbb{R}^2$ with

$$L(x,y) = (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$$

(i) Write its matrix.

(ii) When $\theta \neq 0$, explain how it acts on the plane (draw a picture).

(iii) Do you expect L to have invariant directions?

(iv) Confirm your suspicion in (iii) by trying to solve the eigenvalue problem

$$L(v) = \lambda v$$

(v) If your answer to (iii) was no try to none the less solve $L(v) = \lambda v$ by assuming $\sqrt{-1} = i$ exists.