LECTURE 16 EIGENVALUES & EIGENVECTORS

Consider a linear transformation

$$L : V \longrightarrow V$$

If $0_V \neq v \in V$ and

$$L(v) = \lambda v$$

we say L has an eigenvector v with eigenvector λ.

This equation says (if $\lambda \neq 0$) that L leaves the direction of v invariant (unchanged).

If V is a finite dimensional vector space (again, more later) we can represent L by a square matrix M and find eigenvalues and their eigenvectors X by solving the homogeneous system

$$\boxed{(M - \lambda I)X = 0}$$

Since this system has non-zero solutions if and only if

$$M - \lambda I$$

is singular we must require

$$\det(\lambda I - M) = 0$$

The left hand side of this equation is a polynomial in the variable λ called the characteristic polynomial $P_M(\lambda)$.

For an $n \times n$ matrix the characteristic polynomial has degree n. i.e.

$$P_M(\lambda) = \lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} \cdots + c_n$$

Notice $P_M(0) = \det(-M) = (-1)^n \det M$

The fundamental theorem of algebra says every degree n polynomial can be factored over $\mathbb{C}$ (the complex numbers)

so

$$P_M(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots\cdots (\lambda - \lambda_n) \qquad \text{———} \quad (*)$$

and

$$P_M(\lambda_i) = 0.$$

i.e. the eigenvalues λ_i are roots of the characteristic polynomial. They could be real, zero or complex. They need not all be different. The number of times any λ_i repeats in the expression $(*)$ is called its multiplicity

Example $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$L(x,y,z) = (2x+y-z,\ x+2y-z,\ -x-y+2z)$$

So

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \xmapsto{\ L\ } \underbrace{\begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}}_{\text{The matrix of } L} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The characteristic polynomial is

$$\det \begin{pmatrix} \lambda-2 & -1 & 1 \\ -1 & \lambda-2 & 1 \\ 1 & 1 & \lambda-2 \end{pmatrix}$$

$$= (\lambda-2)\left[(\lambda-2)^2-1\right] + \left[-(\lambda-2)-1\right] + \left[-1-(\lambda-2)\right]$$

$$= \left[(\lambda-2)+1\right]\Big((\lambda-2)\left[(\lambda-2)-1\right] - 2\Big)$$

$$= (\lambda-1)^2(\lambda-4)$$

Eigenvalues

$\lambda = 1$

$\lambda = 4$ (multiplicity 2)

Eigenvectors (solve the homogeneous system $(M-\lambda I)X=0$):

$\underline{\lambda=4:}$
$$\begin{pmatrix} 2-4 & 1 & -1 \\ 1 & 2-4 & -1 \\ -1 & -1 & 2-4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Augmented matrix

$$\left(\begin{array}{ccc|c} -2 & 1 & -1 & 0 \\ 1 & -2 & -1 & 0 \\ -1 & -1 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right)$$

not really necessary!

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$z=t,\ y=-t,\ x=-t$

Eigenvector is $t\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$. i.e. L

leaves the line

$$\left\{ \begin{pmatrix} -t \\ -t \\ t \end{pmatrix} : t \in \mathbb{R} \right\}$$ through the origin

invariant.

$\underline{\lambda = 1}$

$$\begin{pmatrix} 2-1 & 1 & -1 \\ 1 & 2-1 & -1 \\ -1 & -1 & 2-1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e. $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

so we find the solution set

$$z = t, \quad y = s, \quad x = -s + t$$

i.e. $\left\{ s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} : s, t \in \mathbb{R} \right\}$,

a plane through the origin.

So the multiplicity 2 eigenvalue has 2 eigenvectors $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Eigenspaces

In the previous example with
$L(x,y,z) = (2x+y-z, x+2y-z, -x-y+2z)$
we found both $\begin{pmatrix}-1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\0\\1\end{pmatrix}$
had eigenvalue 1. But notice

$$L(0,1,1) = (0,1,1)$$

i.e. $\begin{pmatrix}0\\1\\1\end{pmatrix}$ is also an eigenvector
with eigenvalue 1.

This works because $\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}-1\\1\\0\end{pmatrix} + \begin{pmatrix}1\\0\\1\end{pmatrix}$

More generally if v, w are eigenvectors of a linear transformation L with the same eigenvector λ, then so is any linear combination $rv + sw$ because

$$\begin{aligned} L(rv+sw) &= rL(v) + sL(w) \\ &= r\lambda v + s\lambda w \\ &= \lambda(rv+sw) \end{aligned}$$

The space of all vectors with eigenvalue λ is called an <u>eigenspace</u>. It is itself a vector space and is an example of a more general idea called a <u>subspace</u>.

Lecture 1 Review Questions:

① Explain why the characteristic polynomial of an $n \times n$ matrix has degree n. Make your explanation easy to read by starting with some simple examples, before using properties of determinants to give a general explanation.

② Calculate the characteristic polynomial $P_M(\lambda)$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Now since M is a square matrix we can compute the matrix

$$P_M(M).$$

What do you find?

Investigate whether something similar holds for $n \times n$ matrices.