LECTURE 17 SUBSPACES & SPANNING SETS

It is time to study vector spaces more carefully & answer some fundamental questions

(i) When does a subset of a vector space form a vector space? — subspace

(ii) When are a collection of vectors independent, or are some of them just sums of the others? — linear independence

(iii) How "big" is a vector space — dimension

(iv) How do we label vectors? Can we write any vector as a sum of a basic set of vectors? — bases

__Subspace__ We say that a subset U of a vector space V is a __subspace__ of V if U is a vector space under the inherited addition & scalar multiplication operations.

__Ex__ A plane P in $\mathbb{R}^3$ through the origin:

$$ax + by + cz = 0$$

says $\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$,

a homogeneous system $MX = 0$.

If X_1 and X_2 both solve $MX = 0$ then so does $\mu X_1 + \nu X_2$

$$M(\mu X_1 + \nu X_2) = \mu M X_1 + \nu M X_2$$

linearity of matrix multiplication

$$= \mu \cdot 0 + \nu \cdot 0 = 0$$

So additive and multiplicative closure hold for P. All other requirements still hold because they do in $\mathbb{R}^3$.

Lemma Let U be a non-empty subset of a vector space. Then U is a subspace of V if and only if $\mu u_1 + \nu u_2 \in U \quad \forall u_1, u_2 \in U$, $\mu, \nu \in \mathbb{R}$.

The proof is a review exercise.

Building subspaces from linear combinations:

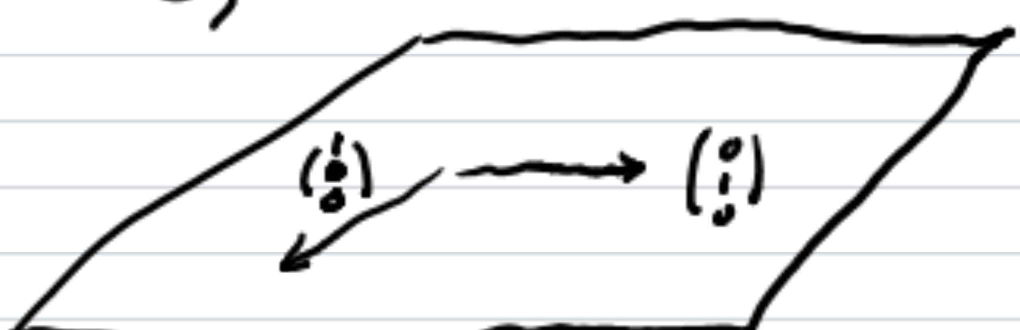
Suppose we took

$$U = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\} \subset \mathbb{R}^3$$

Since U is only two vectors it is cleary <u>not</u> a vector space.

E.g. $$\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} \notin U$$

But we know that any two vectors in $\mathbb{R}^3$ define a plane, in this case the xy plane



To obtain the xy plane from $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\\0\end{pmatrix}$ we need to take all <u>linear</u> combinations of these vectors

$$\left\{ x\begin{pmatrix}1\\0\\0\end{pmatrix} + y\begin{pmatrix}0\\1\\0\end{pmatrix} : x, y \in \mathbb{R} \right\}$$

because $x\begin{pmatrix}1\\0\\0\end{pmatrix} + y\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}x\\y\\0\end{pmatrix}$.

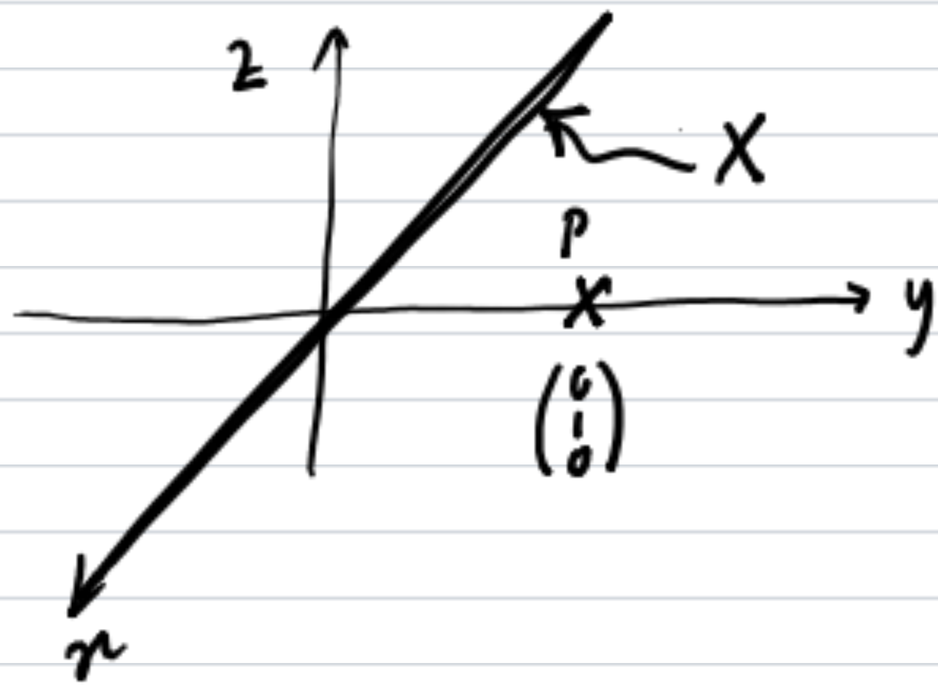
This is an example of a spanning set.

<u>Definition</u> Let V be a vector space and $S \subset V$ (any subset of V). The span of S is the set

$$\operatorname{span}(S) = \{ r_1 s_1 + \dots + r_n s_n : r_1, \dots, r_n \in \mathbb{R}, \; s_1, \dots, s_n \in S \}$$

EX $V = \mathbb{R}^3$. Let $X \subset \mathbb{R}^3$ be the x-axis and $P = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Take $S = X \cup P$, ie.



The elements of span(S) are linear combinations of vectors along the x-axis and the vector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Since the sum of any number of vectors along the x-axis is again a vector along the x-axis, the most general element is of the form

$$\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$ so span(S)

is just the xy-plane (a vectorspace!).

<u>Lemma</u> spanning sets are subspaces.

<u>Proof</u> We need to show that span(S) is a vector space. It suffices to demonstrate closure under linear combinations, i.e. if $u, v \in S$, so is $\lambda u + \mu v$.

But, by definition of S

$$\left.\begin{aligned} u &= r_1 s_1 + \dots + r_n s_n \\ \\ v &= r_{n+1} s_{n+1} + \dots + r_m s_m \end{aligned}\right\} \begin{aligned} &r_i \in \mathbb{R} \\ &s_i \in S \\ &m > n \end{aligned}$$

$$\Rightarrow \lambda u + \mu v = (\lambda r_1) s_1 + \dots + (\lambda r_n) s_n$$
$$+ (\mu r_{n+1}) s_{n+1} + \dots + (\mu r_m) s_m$$
$$\in S \text{ because } \lambda r_i, \mu r_j \in \mathbb{R}$$
$$\text{and } s_1, \dots, s_m \in S$$

Notice (like many proofs) this was <u>easy</u>, we basically just wrote out the definitions.

EX For which values of a does

$$\text{span}\left\{\begin{pmatrix}1\\0\\a\end{pmatrix}, \begin{pmatrix}1\\2\\-3\end{pmatrix}, \begin{pmatrix}a\\1\\0\end{pmatrix}\right\} = \mathbb{R}^3 \text{ ?}$$

Solution We need to check whether we can find r_1, r_2, r_3 such that

$$r_1\begin{pmatrix}1\\0\\a\end{pmatrix} + r_2\begin{pmatrix}1\\2\\-3\end{pmatrix} + r_3\begin{pmatrix}a\\1\\0\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$$

for any $\begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3$. or as a matrix

$$\begin{pmatrix}1 & 1 & a\\0 & 2 & 1\\a & -3 & 0\end{pmatrix}\begin{pmatrix}r_1\\r_2\\r_3\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$$

[Don't be confused, but r_1, r_2, r_3 are the unknowns here!!]. If $\begin{pmatrix}1 & 1 & a\\0 & 2 & 1\\a & 3 & 0\end{pmatrix}$ is invertible we can find a solution

$\begin{pmatrix}1 & 1 & a\\0 & 2 & 1\\a & 3 & 0\end{pmatrix}^{-1}\begin{pmatrix}x\\y\\z\end{pmatrix}$ <u>FOR ANY</u> $\begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3$.

but $\det\begin{pmatrix}1 & 1 & a\\0 & 2 & 1\\a & -3 & 0\end{pmatrix} = +2\det\begin{pmatrix}1 & a\\a & 0\end{pmatrix} - 1.\det\begin{pmatrix}1 & 1\\a & 3\end{pmatrix}$

$$= -2a^2 + 3 + a = -(2a-3)(a+1)$$

The span is $\mathbb{R}^3$ when $a \neq 3/2, -1$.

Lecture 17 Review Questions:

① Suppose that V is a vector space and $U \subset V$. Show that

$$\mu u_1 + \nu u_2 \in U$$

$\forall\, u_1, u_2 \in U$ and $\mu, \nu \in \mathbb{R}$ implies that U is a subspace of V. (I.e. check all the vector space requirements for U.)

② Let $P_3[x]$ be the vector space of degree 3 polynomials in the variable x. Check whether

$$x - x^3 \in \operatorname{span}\{x^2, 2x + x^2, x + x^3\}$$

③ Let U and W be subspaces of the vector space V. Are

(i) $U \cup W$

(ii) $U \cap W$

also subspaces? Explain. Draw some examples in $\mathbb{R}^3$.