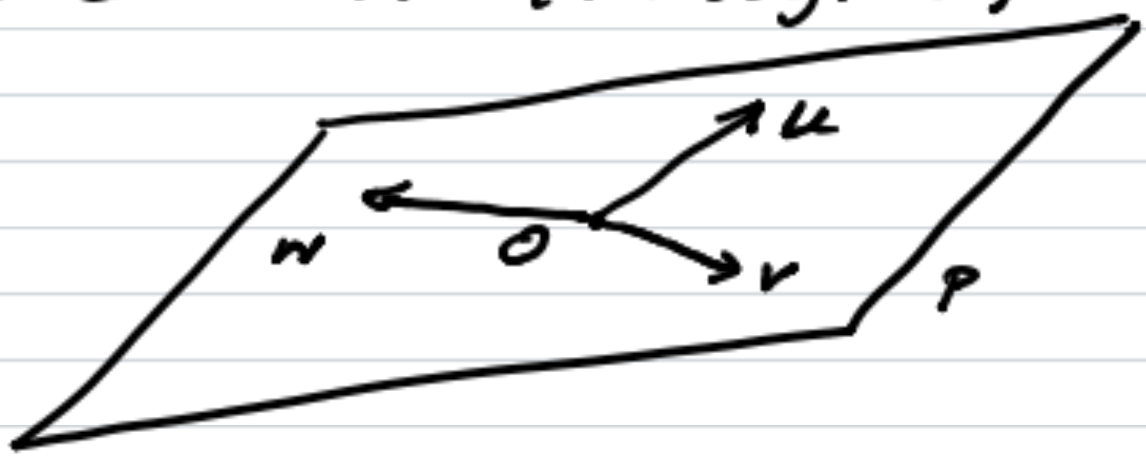
# LECTURE 18 Linear Independence

Consider three vectors in a plane $P$ in $\mathbb{R}^3$ (through $0$)




If $u, v$ & $w$ are not all parallel then $P = \text{span}\{u, v, w\}$ <u>but</u> we should be able to span a plane using only two vectors. In other words we expect a relationship

$$c^1 u + c^2 v + c^3 w = 0 \qquad c^i \in \mathbb{R}$$

expressing the fact that $u, v, w$ are not all independent.

<u>Definition</u> We say that the vectors $v_1, \dots, v_n$ are <u>linearly</u> <u>dependent</u> if there exist constants $c^1, \dots, c^n$ not all vanishing such that

$$c^1 v_1 + c^2 v_2 + \cdots + c^n v_n = 0 .$$

Otherwise $v_1, \dots, v_n$ are <u>linearly independent</u>.

**Example**: Consider vectors

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

in $\mathbb{R}^3$. Are they linearly independent?

**Sol**: We need to see if the system

$$v_1 c^1 + v_2 c^2 + v_3 c^3 = 0$$

has solutions for $c^1, c^2, c^3$

Notice that this is a homogeneous matrix system

$$\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} c^1 \\ c^2 \\ c^3 \end{pmatrix} = 0$$

with unknowns $c^1, c^2, c^3$. This has solutions if and only if the matrix $\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$ is singular so examine

$$\det \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \det \begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 3 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = 0$$

so there are solutions, lets find them

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow c_3 = \mu \;,\; c_2 = -\mu \;,\; c_3 = -2\mu$

$\Rightarrow -2\mu v_1 - \mu v_2 + \mu v_3 = 0 \Rightarrow \underline{-2v_1 - v_2 + v_3 = 0}$

<u>Linearly dependent.</u>

# Linear Dependence Theorem

The set of non-vanishing vectors $\{v_1, \ldots, v_n\}$ is linearly dependent if and only if one of the vectors $v_k$, $k \geq 2$ is a linear combination of the preceding ones.

Proof This is an if and only if statement so there are two things to show

(i) $v_k = c^1 v_1 + \cdots + c^{k-1} v_{k-1}$

$\Rightarrow$ linear dependence.

This is easy, just rewrite the assumption as

$$0 = c^1 v_1 + \cdots + c^{k-1} v_{k-1} + (-1) v_k + 0 \cdot v_{k+1} + \cdots + 0 \cdot v_n$$

so some combination of $v_1, \ldots, v_n$ with not all coefficients vanishing is zero so $v_1, \ldots, v_n$ are linearly dependent.

(ii) Linear dependence

$\Rightarrow$ some $v_k$ is a combination of the previous $v_1, \dots, v_{k-1}$.

Here the assumption says

$$c^1 v_1 + \cdots + c^k v_k + \cdots + c^n v_n = 0$$

and we may take $k$ to be the largest value for which the coefficient $c^k \neq 0$. Hence

$$c^1 v_1 + \cdots + c^k v_k = 0$$

Notice $k > 1$ because $k = 1$ would imply $v_k = 0$ contradicting our original assumption. Thus

$$v_k = -\frac{c^1}{c^k} v_1 - \cdots - \frac{c^{k-1}}{c^k} v_{k-1},$$

i.e. $v_k$ is a linear combination of the previous vectors.

<u>QED</u>

EX The vector space $P_2(t)$ with

$$v_1 = 1 + t$$
$$v_2 = 1 + t^2$$
$$v_3 = t + t^2$$
$$v_4 = 2 + t + t^2$$
$$v_5 = 1 + t + t^2$$

$\{v_1, v_2, v_3, v_4, v_5\}$ are linearly dependent because

$$v_4 = v_1 + v_2$$

Now suppose vectors $v_1, \dots, v_n$ are linearly dependent and

$$c^1 v_1 + \cdots + c^n v_n = 0$$

with $c^1 \neq 0$. Then

$$\text{span}(v_1, \dots, v_n) = \text{span}(v_2, \dots, v_n)$$

because any $x \in \text{span}(v_1, \dots, v_n)$ is given by

$$\begin{aligned} x &= d^1 v_1 + \cdots + d^n v_n \\ &= d^1\left(-\frac{c^2}{c^1} v_2 - \cdots - \frac{c^n}{c^1} v_n\right) \\ &\quad + d^2 v_2 + \cdots + d^n v_n \\ &= \left(d^2 - \frac{d^1 c^2}{c^1}\right) v_2 + \cdots + \left(d^n - \frac{d^1 c^n}{c^1}\right) v^n \end{aligned}$$

which is in $\text{span}(v_2, \dots, v_n)$.

When we write a vector space as the span of a list of vectors, we would like to use the shortest list possible.

This can be achieved by iterating the above procedure.

<u>EX</u> In the above example we found $v_4 = v_1 + v_2$ so learn

$S = \text{span}(1+t,\ 1+t^2,\ t+t^2,\ 2+t+t^2,\ 1+t+t^2)$

$= \text{span}(1+t,\ 1+t^2,\ t+t^2,\ 1+t+t^2)$

discard $v_4$

But $1+t+t^2 = \frac{1}{2}[1+t] + \frac{1}{2}[1+t^2] + \frac{1}{2}[t+t^2]$

so can write

$S = \text{span}(1+t,\ 1+t^2,\ t+t^2)$

The process terminates since there is no solution to $c^1(1+t) + c^2(1+t^2) + c^3(t+t^2) = 0$

This means the vectors $\{1+t, 1+t^2, t+t^2\}$ are special, they are <u>linearly independent</u> and <u>span</u> the vector space $S$. This notion is called a <u>basis</u>.

EX Let $B^3$ be the space bit-valued $3 \times 1$ matrices (column vectors). Are

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

linearly independent?

Sol Need to solve

$$c^1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c^2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c^3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} c^1 + c^2 \\ c^1 + c^3 \\ c^2 + c^3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c^1 \\ c^2 \\ c^3 \end{pmatrix} = 0 \quad \text{for bits } c^1, c^2, c^3.$$

But

$$\det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 1 . \det \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} - 1 . \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= -1 - 1 = 1 + 1 = 0$$

So there are non-trivial solutions and the above vectors are linearly dependent!

<u>Remark:</u> Contrast the 3-vectors above to those in $P_2(t)$ in the previous example!

# Lecture 18 Review Questions:

① How many different vectors are there in $\mathbb{B}^3$?
Find three of them that span $\mathbb{B}^3$ and are linearly independent.
Write all other $\mathbb{B}^3$ vectors in terms of these.
Would it be possible to span $\mathbb{B}^3$ with only two vectors?

② Let

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdots, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

be vectors in $\mathbb{R}^n$ and $v$ be an arbitrary vector in $\mathbb{R}^n$.
Show $e_1, \ldots, e_n$ are linearly independent.

PTO

Show

$$v = \sum_{i=1}^{n} (v \cdot e_i) e_i$$

What does this say about $\text{span}(e_1, e_2, \ldots e_n)$ ?