Lecture #19 Bases & Dimension

A set of vectors that is linearly independent and spans a vector space V is called a basis for V.

If the number of vector in a basis is finite and equals n we say that V is a finite-dimensional vector space of dimension $n \equiv \dim V$

Ex $P_n(t)$ has a basis $\{1, t, \dots, t^n\}$ because every degree n polynomial is a sum

$$a^0 \cdot 1 + a^1 \cdot t + \dots + a^n \cdot t^n$$

so $P_n(t) = \text{span}\{1, t, \dots, t^n\}$ and this set of vectors is linearly independant. (This is true because $c^0 \cdot 1 + c^1 \cdot t + \dots + c^n \cdot t = 0$ means that the LHS is the zero polynomial $\Rightarrow c^0 = c^1 \dots = c^n = 0$.)

Hence $\dim P_n(t) = n+1$.

Theorem If $\{v_1, \dots, v_n\}$ is a basis for a vector space V, then every vector v can be written uniquely as a linear combination

$$v = c^1 v_1 + c^2 v_2 + \dots + c^n v_n .$$

Proof $\{v^1, \dots, v^n\}$ is a basis so must span V which means any $v \in V$ can be written

$$v = c^1 v_1 + \dots + c^2 v_2$$

Now suppose we found a different linear combination equaling v

$$v = d^1 v_1 + \dots + d^n v_n$$

$$\Rightarrow 0 = v - v = (c^1 - d^1) v_1 + \dots + (c^n - d^n) v_n$$

with not all $c^i - d^i$ vanishing. Then $v_1, \dots, v_n$ would be linearly independant and could not be a basis. This is a contradiction. $\blacksquare$

<u>WARNING</u> Bases give you a <u>unique</u> way to express vectors, but bases are themselves unique.

<u>EX</u> $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ and $\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}$ are both bases for $\mathbb{R}^2$.

Bases in $\mathbb{R}^n$

A review question asked you to show

$$\text{span}\left\{\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}, \dots, \begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}\right\} = \mathbb{R}^n$$

and that this set of vectors were independent. That means

$$\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}, \dots \begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix} \text{ is a basis for } \mathbb{R}^n$$

and that $\boxed{\dim \mathbb{R}^n = n}$ This is often called the standard or canonical basis for $\mathbb{R}^n$. It corresponds to a unit (length 1) vector pointing in the direction of each Cartesian coordinate axis. In $\mathbb{R}^3$ you called this basis $\{i, j, k\}$.

Testing if vectors $v_1, \dots, v_m$ form a basis for $\mathbb{R}^n$:

(i) Must have $m = n = \dim(\mathbb{R}^n)$

(ii) We need to check if $v_1, \dots, v_n$ are linearly independent and whether the span $\mathbb{R}^n$.

Spanning $\mathbb{R}^n$ requires we solve

$$V = x^1 v_1 + x^2 v_2 + \dots + x^n v_n$$

for an arbitrary V in $\mathbb{R}^n$ and linear independence requires

$$0 = x^1 v_1 + x^2 v_2 + \dots + x^n v_n$$

has No non-zero solution for $x^1, \dots, x^n$.

Notice if we call

$M = (v_1 \dots v_n)$ $n \times n$ matrix of column vectors and

$X = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}$ column vector of coefficients

the we have two linear systems

$MX = V$ solution for spanning

$MX = 0$ no non-zero solutions for linear independence

This is really a linear system & its associated homogeneous system. We are requiring that there is only a single particular system.

Hence, we simply need to ensure that M^{-1} exists so that

$$X = M^{-1} V \quad \text{uniquely.}$$

We could bring M to RREF or simpler just require $\det M \neq 0$.

Thus, $v_1, \dots, v_n$ is a basis for $\mathbb{R}^n$ if and only if

$$\det (v_1 \ v_2 \cdots v_n) \neq 0.$$

Ex $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ and $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ are both bases for $\mathbb{R}^2$ because

$$\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \neq 0$$

$$\det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2 \neq 0 .$$

We defined the dimension of a finite dimensional vector space as the number of vectors in a basis. Could different bases have differing number of vectors?

<u>Lemma</u> If $v_1, \ldots, v_n$ is a basis for V and $w_1, \ldots, w_m$ are linearly independent then $m \leq n$.

<u>Proof</u> Clearly

$$V = \text{span}\{w_1, v_1, \ldots, v_n\}$$

and this set is linearly dep$\underline{t}$

so we can use the linear independence theorem to discard one of the v's

$$V = \text{span}\{w_1, v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n\}$$

where this set is also a basis.

Now by linear independence of the w's we can continue successively replacing v's by w's. If we ever run out of w's then $m \leq n$ which agrees with the lemma. But, suppose some w's are left over, then we would have

$$V = \text{span}\{w_1, \dots, w_n\}$$

where $w_{n+1}, \dots, w_m$ could be written in terms of these w's. This contradicts linear independence of the w's.

<u>Corollary</u> If $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_m\}$ are both bases for V then $n = m$.

<u>Proof</u> Apply the lemma twice to learn

$$m \leq n \ \& \ n \leq m \Rightarrow m = n. \quad \text{QED}$$

<u>Remark</u>: Contrast the 3-vectors above to those in $P_2(t)$ in the previous example!

Lecture 18 Review Questions:

1. Prove the converse to the theorem in the Lecture. Namely suppose every $v \in V$ can be expressed uniquely as a linear combination of $v^1, \dots, v^n$ then $\{v^1, \dots, v^n\}$ is a basis for V.

   Hint: First explain why $v^1, \dots, v^n$ span V. Then show that $v^1, \dots, v^n$ are linearly independent.

2. Show that the set of all linear transformations mapping $\mathbb{R}^3 \longrightarrow \mathbb{R}$ is itself a vector space. Find a basis for this vector space.

   Hint represent $\mathbb{R}^3$ as column vectors and argue that linear transformations $\mathbb{R}^3 \longrightarrow \mathbb{R}$ are really just row vectors.

   Hint If you are really stuck, look up "dual space" (but this might be more confusing than helpful!)