LECTURE #2

GAUSSIAN ELIMINATION

Last time we studied the linear system

$$\left.\begin{aligned} x + y &= 27 \\ 2x - y &= 0 \end{aligned}\right\} \text{——} ①$$

and found

$$\left.\begin{aligned} x \quad\quad &= 9 \\ y &= 18 \end{aligned}\right\} \text{——} ②$$

We wrote ① as

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}}_{\text{Matrix}} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\text{Vector}} = \underbrace{\begin{pmatrix} 27 \\ 0 \end{pmatrix}}_{\text{Vector}}$$

so ② can be written as

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{"IDENTITY MATRIX"}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 18 \end{pmatrix}$$

NOTATIONS: AUGMENTED MATRIX

Write ① as $\left(\begin{array}{cc|c} 1 & 1 & 27 \\ 2 & -1 & \end{array}\right)$

and ② as $\left(\begin{array}{cc|c} 1 & & 9 \\ & 1 & 18 \end{array}\right)$

(zeroes are blank, don't need x & y).

Another example

$$\left(\begin{array}{cccc|c} 1 & 3 & 2 & & 1 \\ 6 & 2 & & -2 & \\ -1 & & 1 & 1 & 3 \end{array}\right) \text{ denotes } \begin{array}{rcrcrcrcl} x & + & 3y & + & 2z & & & = & 1 \\ 6x & + & 2y & & & - & 2w & = & 0 \\ -x & & & + & z & + & w & = & 3 \end{array}$$

GENERAL CASE

$$r \text{ equations} \updownarrow \left(\begin{array}{cccc|c} a^1{}_1 & a^1{}_2 & \cdots & a^1{}_k & b^1 \\ a^2{}_1 & a^2{}_2 & \cdots & a^2{}_k & b^2 \\ \vdots & \vdots & & \vdots & \vdots \\ a^r{}_1 & a^r{}_2 & \cdots & a^r{}_k & b^r \end{array}\right)$$

$$\longleftrightarrow$$

k unkowns

AN IDEA

$$\left(\begin{array}{cc|c} 3 & & 27 \\ 2 & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 1 & 27 \\ 2 & -1 & \end{array}\right)$$

$$\left(\begin{array}{cc|c} 2 & & 6 \\ & 1 & 18 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & & 9 \\ & 1 & 18 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 1 & 27 \\ & -3 & -54 \end{array}\right)$$

FRED

Equivalence Relation for Linear Systems

Sometimes mathematical objects are not "equal" but seem to encode the same information.

<u>Example</u> augmented matrices

$$\left(\begin{array}{cc|c} 1 & 1 & 27 \\ 2 & -1 & \end{array}\right) \neq \left(\begin{array}{cc|c} 1 & & 9 \\ & 1 & 18 \end{array}\right)$$

but both say $x = 9$, $y = 18$.

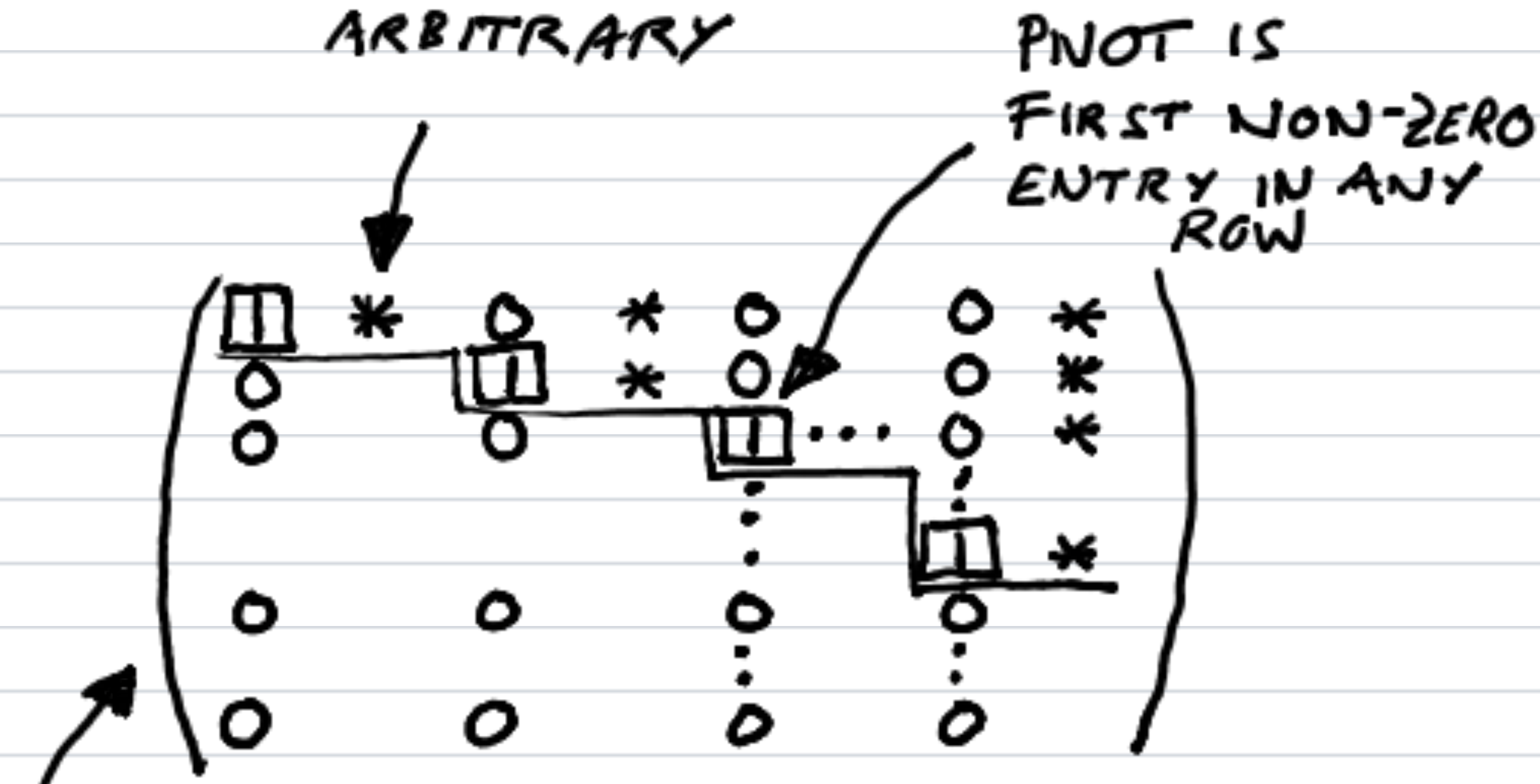
<u>Instead</u>, we will write

$$\left(\begin{array}{cc|c} 1 & 1 & 27 \\ 2 & -1 & \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & & 9 \\ & 1 & 18 \end{array}\right)$$

and say that two augmented matrices are (row) <u>"equivalent"</u> if they have the same solutions.

l'Echelon (Fr. noun) step.

REDUCED ROW ECHELON FORM (RREF)



ZERO ROWS COME LAST

- PIVOT IS ALWAYS 1
- PIVOT IS ALWAYS TO RIGHT OF PREVIOUS PIVOT
- THE PIVOT IS THE ONLY NON-ZERO ENTRY IN ITS COLUMN

Example

$$\begin{pmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Non-Example

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

WHY DO WE CARE?

EXAMPLE

$$\left(\begin{array}{cccc|c} 1 & 0 & 7 & 0 & 4 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \text{says}$$

$$\begin{aligned} x \qquad + 7z \qquad &= 4 \\ y + 3z \qquad &= 1 \\ w &= 2 \end{aligned}$$

solve for pivot variables bottom up

$w = 2$

$z = \lambda$ (NOT pivot, <u>undetermined</u>)

$y = 1 - 3\lambda$

$x = 4 - 7\lambda$

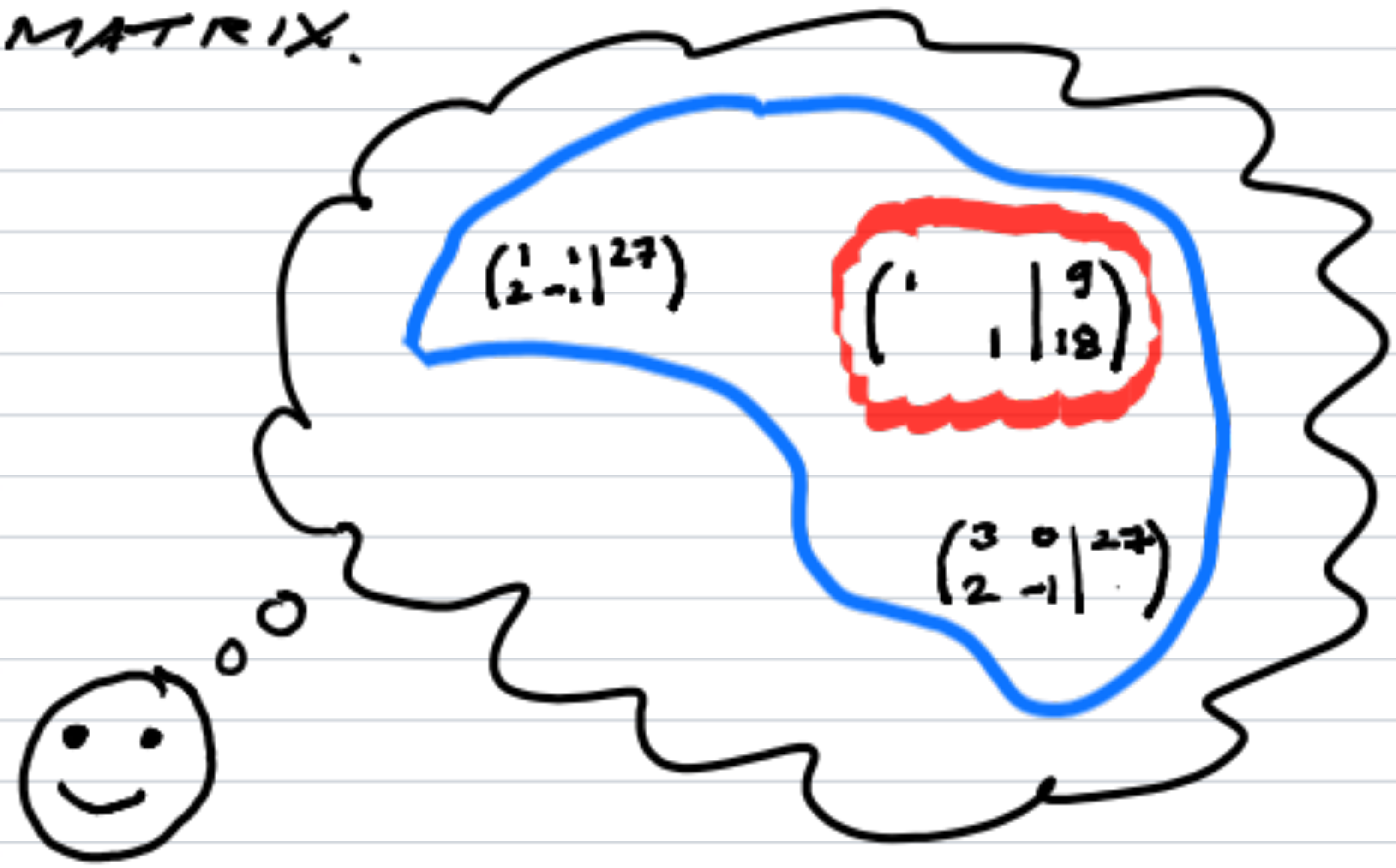
<u>OR</u>

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -7 \\ -3 \\ 1 \\ 0 \end{pmatrix}$$

CAN READ OFF SOLUTION SET DIRECTLY FROM RREF

EVEN BETTER

THEOREM EVERY AUGMENTED MATRIX IS (ROW) EQUIVALENT TO A **UNIQUE** RREF AUGMENTED MATRIX.



NEXT TIME:

We would like to see why this theorem is true. We also want a method to find RREFs. Solve both problems with "elementary row operations."

Lecture 2 Review Questions

1. Show that this pair of augmented matrices are row equivalent

$$\left(\begin{array}{cc|c} a & b & x \\ c & d & y \end{array}\right) \quad \& \quad \left(\begin{array}{cc|c} 1 & 0 & \frac{dx-by}{ad-bc} \\ 0 & 1 & \frac{ax-cy}{ad-bc} \end{array}\right)$$

(assuming $ad - bc \neq 0$).

2. List as many manipulations of augmented matrices as you can think of that <u>preserve</u> row equivalence. Explain your answers. Also give some example manipulations that <u>break</u> row equivalence.

3. Equivalence relations:

We defined $\sim$ for augmented matrices but the idea is more general. The complete definition of an equivalence relation $\sim$ on a set of objects $\mathcal{U}$ is given by three requirements

(i) REFLEXIVE If $u \in \mathcal{U}$ then

$$u \sim u$$

(ii) SYMMETRIC If $u, v \in \mathcal{U}$ then

$$u \sim v \Rightarrow v \sim u$$

(iii) TRANSITIVE If $u, v, w \in \mathcal{U}$ then

$$u \sim v \ \& \ v \sim w \Rightarrow u \sim w$$

(i) Suppose $\mathcal{U} = \mathbb{R}$ (real numbers).
Explain why $=$ is an equivalence relation but $\geqslant$ is not.

(ii) Explain why equivalence of augmented matrices is an equivalence relation.