Lecture #20 Diagonalization

Let $e_1, \cdots, e_n$ be a basis for the vector space V and $f_1, \dots, f_m$ be a basis for the vector space W. Consider a linear transformation

$$L : V \longrightarrow W$$

Arbitrary vectors $v \in V$, $w \in W$ can be written uniquely as

$$v = v^1 e_1 + v^2 e_2 + \cdots + v^n e_n$$

$$w = w^1 f_1 + w^2 f_2 + \cdots + w^m f_m$$

We call the coefficients $v^1, \dots, v^n$ the <u>components</u> of the vector v in the basis $\{e_1, \dots, e_n\}$.

<u>EX</u> $P_1(t)$. In the basis $\{1-t, 1+t\}$ the vector $v = 2t$ has components $v^1 = -1$, $v^2 = +1$ because $v = 2t = -1 \cdot (1-t) + 1 \cdot (1+t)$

We may view these components as vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$

$$\begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{pmatrix} \in \mathbb{R}^n, \quad \begin{pmatrix} w^1 \\ w^2 \\ \vdots \\ w^m \end{pmatrix}$$

Therefore, we expect to represent L as a $m \times n$ matrix. Write

$$L(v) = L(v^1 e_1 + \cdots v^n e_n)$$

$$= L(e_1)v^1 + \cdots + L(e_n)v^n$$

This is a vector in W. Lets compute its components in the basis $f_1, \ldots, f_m$.

Firstly

$$L(e_j) = f_1 M^1_j + \cdots + f_m M^m_j = \sum_{i=1}^m f_i M^i_j$$

where M^i_j are the components of the vector $L(e_j)$ in the basis $f_1, \ldots, f_m$

$\boxed{\text{$M^i_j$ is the i^{th} component of L in the W-basis acting on j^{th} V-basis vector!}}$

Hence

$$L(v) = \sum_{j=1}^{n} \left(f_1 M_j^1 + \cdots + f_m M_j^m \right) v^j$$

$$= \sum_{i=1}^{m} f_i \left[\sum_{j=1}^{n} M_j^i v^j \right]$$

matrix mult$^{\underline{n}}$

So

$$\begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} \xmapsto{L} \begin{pmatrix} M_1^1 & \cdots & M_m^1 \\ \vdots & & \vdots \\ M_1^n & \cdots & M_m^n \end{pmatrix} \begin{pmatrix} v^1 \\ \vdots \\ v^m \end{pmatrix}$$

and $M = (M_j^i)$ is called the matrix of L. It is only defined by specifying bases for V and W.

<u>EX</u> $L : P^1(t) \longrightarrow P^1(t)$ $\quad L(a+bt) = (a+b)t$

with basis $1-t$, $1+t$ for both $V = P^1(t) = W$

$$\begin{array}{ccccc} & f_1 & M_1^1 & f_2 & M_1^2 \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ L(1-t) = 0 = & (1-t) \cdot & 0 \; + & (1+t) \cdot & 0 \end{array} = \begin{pmatrix} 1-t & 1+t \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{ccccc} L(1+t) = 2t = & (1-t) \cdot & 1 \; + & (1+t) \cdot & 1 \\ & \uparrow & \uparrow & \uparrow & \uparrow \\ & f_1 & M_2^1 & f_2 & M_2^2 \end{array} = \begin{pmatrix} 1-t & 1+t \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad M = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}}_{\text{columns say which basis vector } L \text{ acts on}} \Bigg\} \text{rows say which component of } L(e_j).$$

Now suppose we are "lucky":

$$L : V \longrightarrow V$$

and the basis $v_1, \dots, v_n$ is a set of linearly independent eigenvectors $\bar{c}$ eigenvalues $\lambda_1, \dots, \lambda_n$. Then

$$L(v_1) = \lambda_1 v_1$$
$$L(v_2) = \qquad \lambda_2 v_2$$
$$L(v_n) = \qquad \qquad \lambda_n v_n$$

$\Rightarrow$ L has matrix

$$M = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{pmatrix}$$

diagonal.

We call the $n \times n$ matrix of a linear transformation <u>diagonalizable</u> if it has n linearly independent eigenvectors because in that case its eigenvectors form a basis and its matrix is diagonal with eigenvalues along the diagonal.

Notice if the matrix of a linear transformation is diagonal, then the basis with components $\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \ldots \begin{pmatrix} 0 \\ 0 \\ \vdots \\ n \end{pmatrix}$ are already n-linearly independent eigenvectors.

Suppose $\{u_1, \dots, u_n\}$ and $\{v_1, \dots, v_n\}$ are both bases for a vector space V. Then we may write each v_i uniquely as a linear combination of u_j:

$$v_i = \sum_j u_j \underbrace{P^j_i}_{\text{coefficients}}$$

We may view the coefficients P^j_i as a matrix $P = (P^k_i)$. It must have an inverse since we can also uniquely write

$$u_j = \sum_k v_k Q^k_j \quad \text{for some coeffs } Q^k_j$$

$$\Rightarrow v_i = \sum_k \sum_j v_k Q^k_j P^j_i$$

but $\sum_j Q^k_j P^j_i$ is matrix multiplication of $(Q^k_j)(P^j_i)$ and must equal the identity because v_i can't be written as a sum of other basis elements so the RHS $= v_i$

Now suppose $L : V \to V$ has matrix $M = (M^j_i)$ in the basis $u_1, \dots, u_n$

so $$L(u_i) = \sum_k M^k_i u_k .$$

Now suppose $v_1, \dots, v_n$ are n linearly independent eigenvectors with eigenvalues $\lambda_1, \dots, \lambda_n$. Then

$$L(v_i) = \lambda_i v_i = \sum_k v_k D^k_i \quad \text{— } (*)$$

where $(D^j_i) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ is diagonal. But $v_i = \sum_j u_j P^j_i$. so LHS $(*)$ is

$$L\Big(\sum_j u_j P^j_i\Big) = \sum_j L(u_j) P^j_i = \sum_j \sum_k u_k \underbrace{M^k_j P^j_i}_{\text{MATRIX MULT}}$$

and RHS of $(*)$ is

$$\sum_k \sum_j u_j P^j_k D^k_i$$

i.e. $$\boxed{MP = PD} \quad \text{or} \quad \boxed{D = P^{-1} M D}$$

If two matrices M, N are related by $N = P^{-1} M P$ for some invertible P, we say they are similar. Diagonalizable matrices are similar to diagonal matrices!

Review Exercise

① Show that similarity of matrices is an equivalence relation (give link here!)

② When is the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ diagonalizable? Include examples in your answer.

③ Let $P_n(t)$ be degree n polynomials and

$$\frac{d}{dt} : P_n(t) \longrightarrow P_{n-1}(t)$$

be the derivative operator.

Find the matrix of $\frac{d}{dt}$ in the bases

$\{1, t, \ldots, t^n\}$ for $P_n(t)$

and

$\{1, t, \ldots, t^{n-1}\}$ for $P_{n-1}(t)$.