Lecture #21 ORTHONORMAL BASES

The canonical/natural basis

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

has some useful properties

* unit lengths

$$\|e_i\| = \sqrt{e_i \cdot e_i} = \sqrt{e_i^T e_i} = 1$$

* orthogonal (right angled)

$$e_i \cdot e_j = e_i^T e_j = 0 \quad \text{when } i \neq j$$

These are summarized by

$$e_i^T e_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

The symbol δ_{ij} is called the Kronecker delta and equals the matrix entries of the identity matrix.

Given column vectors v and w, the product $v^T w$ is called the inner product & equals the dot product.

We can also form their outer product vw^T which is a square matrix.

These are particularly interesting for the basis vectors $e_1, \dots, e_n$

Call

$$\Pi_1 = e_1 e_1^T = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} (1\ 0\ 0 \cdots) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & & \\ 0 & \cdots & & 0 \end{pmatrix}$$

$$\vdots$$

$$\Pi_n = e_n e_n^T = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} (0\ 0 \cdots 1) = \begin{pmatrix} 0 & \cdots & & 0 \\ \vdots & & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

<u>Notice</u> $\Pi_i \Pi_j = e_i e_i^T e_j e_j^T$

$$= e_i (\delta_{ij}) e_j^T$$

$$\Rightarrow \Pi_i \Pi_j = \begin{cases} \Pi_i & i = j \\ 0 & i \neq j \end{cases}$$

MOREOVER IF D is a diagonal matrix with entries $\lambda_1, \dots, \lambda_n$ we have

$$D = \lambda_1 \Pi_1 + \lambda_2 \Pi_2 + \cdots + \lambda_n \Pi_n$$

This is a very useful way to express a diagonal matrix.

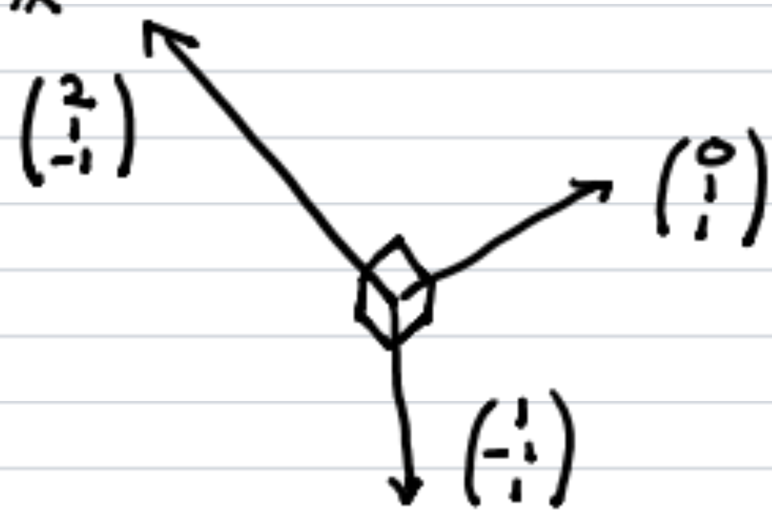
We can mimic properties of the canonical basis by searching for

(i) <u>Orthogonal bases</u> $\{v_1, \dots, v_n\}$

$$v_i \cdot v_j = 0 \quad i \neq j$$

i.e. all basis vectors perpendicular

EX in $\mathbb{R}^3$



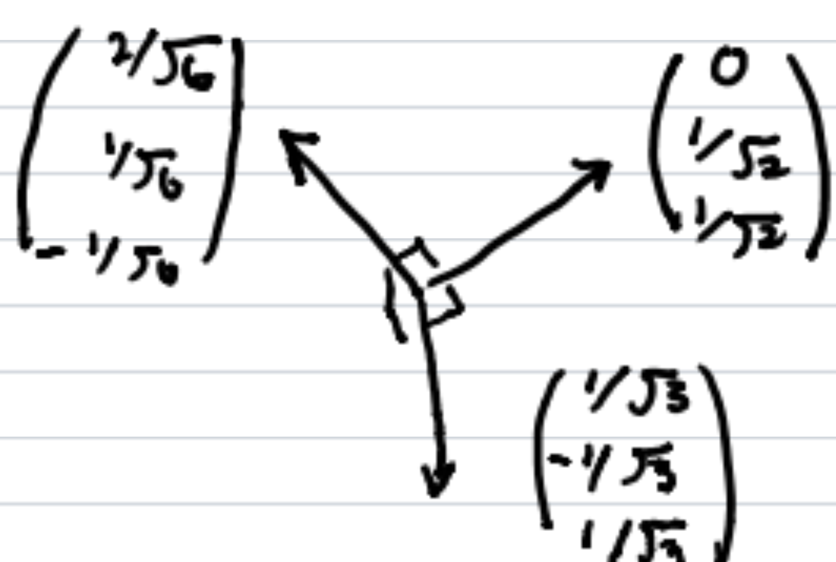
Q. How did I find these?

(iii) <u>Orthonormal bases</u> $\{u_1, \dots, u_n\}$

orthogonal bases whose elements have unit length $|u_i| = 1$ i.e.

$$u_i \cdot u_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

EX in $\mathbb{R}^3$



Suppose $\{u_1, \dots, u_n\}$ is an orthonormal basis for $\mathbb{R}^n$.

Since this is a basis, we can <u>uniquely</u> write a vector v as

$$v = c^1 u_1 + c^2 u_2 + \dots + c^n u_n$$

But notice

$$u_1 \cdot v = c^1 \underbrace{u_1 \cdot u_1}_{\substack{1 \\ \text{orthonormal}}} + c^2 \underbrace{u_1 \cdot u_2}_{0} + \dots + c^n \underbrace{u_1 \cdot u_n}_{0}$$

perpendicular

$\Rightarrow c^1 = u_1 \cdot v.$

similarly $c^2 = u_2 \cdot v$, $c^3 = u_3 \cdot v, \dots$

$$\Rightarrow v = (u_1 \cdot v)\, u_1 + (u_2 \cdot v)\, u_2 + \dots + (u_n \cdot v)\, u_n$$

or in short hand

$$\boxed{v = \sum_{i=1}^{n} (u_i \cdot v)\, u_i \,,}$$

when $\{u_1, \dots, u_n\}$ orthonormal basis

Relating orthonormal bases:

Suppose $\{u_1, \dots, u_n\}$ and $\{w_1, \dots, w_n\}$ are both orthonormal bases for $\mathbb{R}^n$.

Then

$$w_1 = (w_1 \cdot u_1) u_1 + \cdots + (w_1 \cdot u_n) u_n$$
$$\vdots$$
$$w_n = (w_n \cdot u_1) u_1 + \cdots + (w_n \cdot u_n) u_n$$

OR

$$w_i = \sum_j u_j (u_j \cdot w_i)$$

So the matrix for the change of basis $\{u\}$ to $\{w\}$

$$P = (p_i^j) = (u_j \cdot w_i)$$

Claim that $P^{-1} = P^T$ because

$$\sum_i (u_j \cdot w_i)(w_i \cdot u_k) = \sum_i u_j^T w_i w_i^T u_k$$

If we could show $\sum_i w_i w_i^T = I$, we would be done because

$u_j^T u_k = u_j \cdot u_k = \delta_{jk}$ the entries of the identity.

To see that $\sum_i w_i w_i^T = I$ we need to check whether $\sum_i w_i w_i^T v = v$ for any $v \in \mathbb{R}^n$.

This is easy because

$$v = \sum_j c^j w_j \quad \text{for some } j$$

so

$$\sum_i w_i w_i^T \sum_j c^j w_j = \sum_j c^j \sum_i w_i w_i^T w_j$$
$$= \sum_j c^j \sum_i w_i \, w_i \cdot w_j$$
$$= \sum_j c^j w_i \delta_{ij}$$
$$= \sum_j c^j w_j$$
$$= v$$

EX $\mathbb{R}^3$ with $u_1 = \begin{pmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ -1/\sqrt{6} \end{pmatrix}$ $u_2 = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ $u_3 = \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

$$w_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = e_1, \quad w_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e_2, \quad w_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = e_3$$

The matrix for the change of basis $\{w\}$ to $\{u\}$

$$P = (p^j_i) = (w_j \cdot u_i) = \begin{pmatrix} e_1 \cdot u_1 & e_1 \cdot u_2 & e_1 \cdot u_3 \\ e_2 \cdot u_1 & e_2 \cdot u_2 & e_2 \cdot u_3 \\ e_3 \cdot u_1 & e_3 \cdot u_2 & e_3 \cdot u_3 \end{pmatrix}$$

$$= \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}$$

$$P^T = \begin{pmatrix} u_1^T \\ u_2^T \\ u_3^T \end{pmatrix} = \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}$$

Explicitly can check $P^T P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

or from first principles

$$P^T P = \begin{pmatrix} u_1^T \\ u_2^T \\ u_3^T \end{pmatrix} (u_1 \ u_2 \ u_3)$$

$$= \begin{pmatrix} u_1^T u_1 & u_1^T u_2 & u_1^T u_3 \\ u_2^T u_1 & u_2^T u_2 & u_2^T u_3 \\ u_3^T u_1 & u_3^T u_2 & u_3^T u_3 \end{pmatrix}$$

$$= \begin{pmatrix} u_1 \cdot u_1 & u_1 \cdot u_2 & u_1 \cdot u_3 \\ u_2 \cdot u_1 & u_2 \cdot u_2 & u_2 \cdot u_2 \\ u_3 \cdot u_1 & u_3 \cdot u_2 & u_3 \cdot u_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\{u_1, u_2, u_3\}$ orthonormal

<u>Finally</u> Suppose D is a diagonal matrix and $P = (P^{-1})^T$ is an orthogonal matrix. Then changing basis

$D \longrightarrow PDP^{-1} = PDP^T$ which is symmetric because $(PDP^T)^T = (P^T)^T D^T P^T = PDP^T$

Review Exercises

① Let $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

(i) Write D in terms of the vectors $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and their transposes.

(ii) Suppose $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible.

Show D is similar to

$$M = \frac{1}{ad-bc} \begin{pmatrix} \lambda_1 ad - \lambda_2 bc & (\lambda_1 - \lambda_2) bd \\ (\lambda_2 - \lambda_1) ac & -\lambda_1 bc + \lambda_2 ad \end{pmatrix}$$

(iii) Suppose the vectors $(a \ b)$ and $(c \ d)$ are orthogonal. What can you say about M in that case?

② Suppose $\{v_1, \dots, v_n\}$ is an <u>orthogonal</u> basis for $\mathbb{R}^n$

There fore we can write uniquely

$$v = c^1 v_1 + c^2 v_2 + \dots + c^n v_n$$

Find a formula for $c^1, c^2, \dots, c^n$ interms of v and $v^1, v^2, \dots, v^n$.

③ Let u, v be vectors in $\mathbb{R}^3$ and $P = \text{span}\{u, v\}$ be the <u>plane</u> spanned by u & v (assume $v \neq \lambda u$)

(i) Is $v^\perp = v - \frac{u \cdot v}{u \cdot u} u$ in the plane P? (Explain)

(ii) What is the angle between $v^\perp$ and u

(iii) How could you find a third vector perpendicular to both u and $v^\perp$.

(iv) Construct an orthonormal basis for $\mathbb{R}^3$ from u and v

(v) Test your abstract formulae starting with $u = (1,1,0)$ and $v = (0,1,1)$.