Lecture #22 DIAGONALIZING SYMMETRIC MATRICES

Symmetric matrices have many applications

EXAMPLE

	Sacramento	Seattle	San Francisco
Sacramento	0	2000	80
Seattle	2000	0	2010
San Francisco	80	2010	0

is encoded in a matrix

$$M = \begin{pmatrix} 0 & 2000 & 80 \\ 2000 & 0 & 2010 \\ 80 & 2010 & 0 \end{pmatrix} = M^T$$

If M is an $n\times n$ matrix that obeys $M = M^T$, we say M is symmetric.

Symmetric matrices have a nice property. They always have real eigenvalues.

EXAMPLE

$$P_\lambda\begin{pmatrix} a & b \\ b & d\end{pmatrix} = \det\begin{pmatrix} \lambda - a & b \\ b & \lambda - d\end{pmatrix}$$

$$= (\lambda - a)(\lambda - d) - b^2$$

$$= \lambda^2 - (a+d)\lambda - b^2 + ad$$

$$\Rightarrow \lambda = a + d \pm \sqrt{\underbrace{4b^2 + (a-d)^2}_{\text{Always positive}}}$$

The general proof is review exercise ①.

Now suppose a symmetric matrix M has two distinct eigenvalues λ and μ $\bar{c}$ eigenvectors X and Y resp.

$$MX = \lambda X \quad , \quad MY = \lambda Y.$$

Then the dot product

$$X \cdot Y = X^T Y = Y^T X.$$

But

$$X^T M Y = X^T \mu Y = \mu X \cdot Y$$

Yet

$$X^T M Y = (X^T M Y)^T \quad \sim \text{transpose of } 1\times 1 \text{ matrix}$$

$$= Y^T M^T X$$

$$= Y^T M X \quad \sim M \text{ is symmetric}$$

$$= Y^T \lambda X$$

$$= \lambda X \cdot Y$$

subtracting these two results

$$0 = X^T M Y - X^T M Y = \mu X \cdot Y - \lambda X \cdot Y$$

$$\Leftrightarrow \quad 0 = \underbrace{(\mu - \lambda)}_{\neq 0} X \cdot Y$$

Hence $X \cdot Y = 0$. Eigenvectors of distinct eigenvalues of a symmetric matrix are orthogonal!

EX $M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$$\det(M - \lambda) = (2-\lambda)^2 - 1$$

$\Rightarrow$ Eigenvalues $\lambda = 3, 1$ distinct

Eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

and $(1,1) \cdot (1,-1) = 0$ orthogonal.

Last lecture we saw that the matrix P built from orthonormal basis vectors $\{v_1, \dots, v_n\}$

$$P = (v_1 \;\; v_2 \;\; \cdots \;\; v_n)$$

was an orthogonal matrix, ie.

$$P^{-1} = P^T \quad (\text{or } PP^T = I)$$

Moreover, given any vector X_1 you can always find vectors $X_2, \dots, X_n$ such that $\{X_1, \dots, X_n\}$ is an orthonormal basis. (Later we will learn a procedure "Gram-Schmidt" for this.)

Now suppose M is a symmetric $n \times n$ matrix and λ_1 is an eigenvalue with eigenvector X_1.

Let $P = (X_1\ X_2 \dots X_n)$

(note $X_2, \dots, X_n$ might not be eigenvectors).

Then $MP = (\lambda_1 X_1\ MX_2 \dots MX_n)$

But $P^{-1} = P^T = \begin{pmatrix} X_1^T \\ X_2^T \\ \vdots \\ X_n^T \end{pmatrix}$ so

$$P^TMP = \left(\begin{array}{c|c} X_1^T \lambda X_1 & \\ X_2^T \lambda X_1 & \overset{\text{stuff}}{*} \\ \vdots & \\ X_n^T \lambda X_1 & \end{array}\right)$$

$$\underset{\text{orthonormal}}{=} \left(\begin{array}{c|c} \lambda & \\ 0 & * \\ \vdots & \\ 0 & \end{array}\right) \underset{\substack{P^TMP \\ \text{is symmetric}}}{=} \left(\begin{array}{c|c} \lambda & 0 \cdots 0 \\ \hline 0 & \\ \vdots & \hat{M} \\ 0 & \end{array}\right)$$

Here we know nothing about $\hat{M}$ except that it is symmetric and $(n-1) \times (n-1)$. But we could now iterate this procedure and successively put eigenvalues along the diagonal. In this way we see

<u>Theorem</u> Every symmetric matrix is similar to a diagonal matrix of its eigenvalues.

i.e. $M = M^T$

$\Rightarrow M = PDP^T$

where P is an orthogonal matrix.

To diagonalize a real symmetric matrix, begin by building an orthogonal matrix from an orthonormal basis of eigenvectors

EX $M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has eigenvalues and eigenvectors $\lambda = 3$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = 1$ $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Build $P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

Notice $P^T P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Then $MP = \begin{pmatrix} \frac{3}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

i.e. $MP = DP$

$\Rightarrow \quad D = P^{-1} M P = P^T M P$

$= \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ "diagonalized M"

by an orthogonal change of basis.

Review Exercises

① Reality of eigenvalues.

(i) Suppose $z = x + iy$ and

$\bar{z} = x - iy$. $\quad x, y \in \mathbb{R}$, $i^2 = -1$

What can you say about $\bar{z} z$?

(ii) What can you say about complex numbers λ that obey $\lambda = \bar{\lambda}$?

(iii) Let

$$X = \begin{pmatrix} z^1 \\ z^2 \\ \vdots \\ z^n \end{pmatrix} \in \mathbb{C}^n$$

and call

$$X^\dagger = (\bar{z}^1, \bar{z}^2, \ldots, \bar{z}^n)$$

Compute $X^\dagger X$. What can you say about the result.

(iv) Now suppose $M = M^T$ is an $n \times n$ symmetric, real matrix, with eigenvalue λ, and a corresponding eigenvector X. i.e.

$$MX = \lambda X$$

Compute

$$\frac{\overline{X}^T M X}{\overline{X}^T X}$$

(v) Suppose Λ is a 1×1 matrix. What is Λ^T?

(vi) Viewing $\overline{X}^T, M, X$ as matrices, what size matrix is $\overline{X}^T M X$?

(vii) Compute $\overline{(\overline{X}^T)^T}$. Then compute

$$\overline{(\overline{X}^T M X)^T}$$

(viii) Explain why $\lambda = \overline{\lambda}$. What does this mean?

(2) Let $X_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

where $a^2+b^2+c^2=1$.

Find vectors X_2 and X_3 such that X_1, X_2, X_3 form an orthonormal basis for $\mathbb{R}^3$.

(3) What can you say about the dimensions of the eigenspaces of a real symmetric matrix?