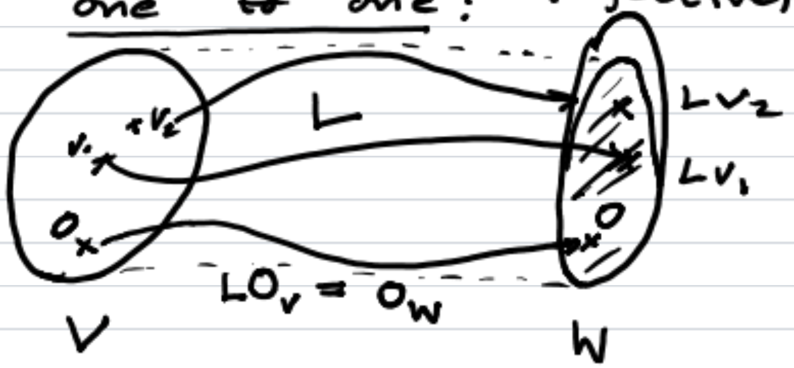
Lecture #23 KERNEL, RANK & RANGE

Study linear transformations in more detail, in particular when is a linear transformation

one to one? (injective)

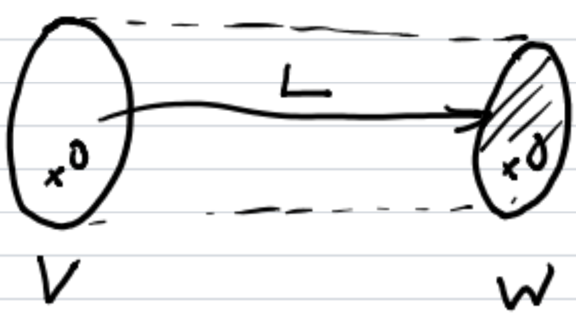


i.e. $v_1 \neq v_2 \Rightarrow Lv_1 \neq Lv_2$

or equivalently

$Lv_1 = Lv_2 \Rightarrow v_1 = v_2$

onto?



i.e. for every $w \in W$, there is at least one $v \in V$ s.t. $L(v) = w$. Write $L(V) = W$

Suppose L is <u>not</u> injective.
Then we can find $v_1 \neq v_2$
such that $Lv_1 = Lv_2$. i.e.

$$L(v_1 - v_2) = 0$$

The set of all vectors $v \in V$ obeying $Lv = 0_W$ is called the kernel of L:

$$\ker L = \{v \in V : Lv = 0_W\}$$

<u>Theorem</u> $\ker L = \{0_V\}$ if and only if L is one to one.

Notice if L has matrix M in some basis, finding the kernel is just solving the homogeneous system

$$MX = 0.$$

<u>EX</u> Is $L(x,y) = (x+y, x+2y, y)$ one to one?

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

so only solution is $x = y = 0$

$\therefore L$ is injective.

Notice if $L(v) = 0$ and
$L(u) = 0$ then

$$L(\lambda u + \nu v) = 0$$

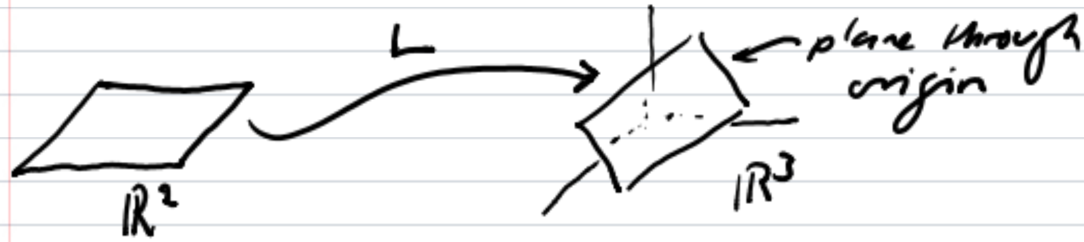
i.e. if $u, v \in \ker V$

then $\lambda u + \nu v \in \ker V$.

So by the subspace theorem, $\ker V$ is a subspace of V (i.e. $\ker V$ is a vector space).

EX Let $L : V \longrightarrow V$ with a zero eigenvalue. Then the eigenspace of the zero eigenvalue equals $\ker L$.

In the example before last,
$L : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ was certainly not onto because $\mathbb{R}^2$ is a plane, and L was one to one, pictorially



We call the <u>set</u> of all possible outputs for a linear transformation is <u>range</u> or <u>image</u> written

$$L(V) = \{L(v) : v \in V\}$$

This is the whole vector space!

Not $L(V) \subset W$ is a subset of W when $L: V \to W$. Other notations include $L(V) = \operatorname{Im} L = \operatorname{ran} L$.

Observe, if $w = L(v)$ and
$z = L(u)$ then

$L(v), L(u) \in L(V)$ but

$L(V) \ni L(\lambda u + \nu v) = \lambda L(u) + \nu L(v)$

so the subspace theorem strikes again and we learn that $L(V)$ is a subspace of W.

To find a basis for $L(V)$ we can start with a basis $\{v_1, \dots, v_n\}$ for V and notice

$$L(V) = \mathrm{span}\{L(v_1), L(v_2), \dots, L(v_n)\}$$

But these may not be linearly independent, so solve

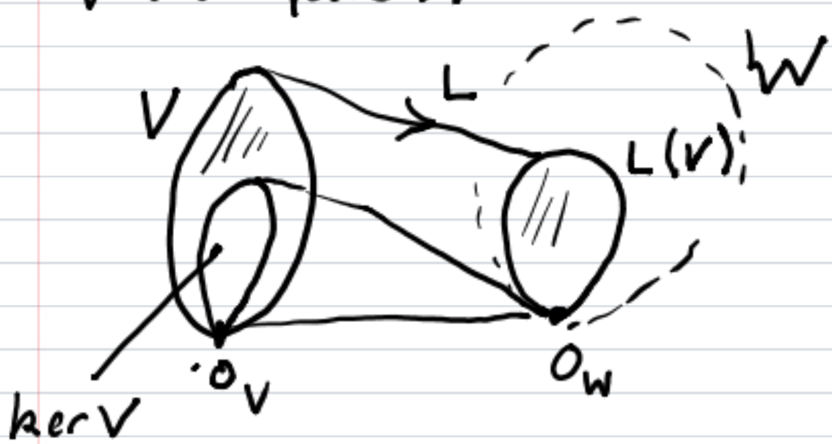
$$c^1 L(v_1) + c^2 L(v_2) + \cdots + c^n L(v_n) = 0$$

for the c^i to find relations among the $L(v_i)$ and reduce them to a linearly independent spanning set.

The Dimension Formula

$$\dim(V) = \dim \ker V + \dim L(V)$$

relates the dimension of three vector spaces:



Proof: Pick a basis for V

$$\{v_1, \ldots, v_p, u_1, \ldots, u_q\}$$

where $v_1, \ldots, v_p$ is also a basis for $\ker V$. This can always be done. Notice that

$$p = \dim \ker L \quad \text{and} \quad p+q = \dim V$$

so we need to show $q = \dim L(V)$.

The idea is to show that $\{L(u_1), \ldots, L(u_q)\}$ is a basis for $L(V)$.

To see this is a spanning set, notice that if $w = L(v)$ (so that $w \in L(V)$) then

$$w = L(c^1 v_1 + c^2 v_2 + \cdots c^p v_p + d^1 u_1 + \cdots + d^q u_q)$$

using basis for V

$$= c^1 \underbrace{L(v_1)}_{0} + \cdots + c^p \underbrace{L(v_p)}_{0} + d^1 L(u_1) + \cdots + d^q L(u_q)$$

so $L(V) = \text{span}\{L(u_1), \ldots, L(u_q)\}$ & it remains to check linear independence.

Suppose $d^1 L(u_1) + \cdots + d^q L(u_q) = 0$

But this says $L(d^1 u_1 + \cdots + d^q u_q) = 0$

i.e. $d^1 u_1 + \cdots + d^q u_q \in \ker L$

But then we could write this vector as a linear combination of the $\ker L$ basis $\{v_1, \ldots, v_q\}$ which is impossible unless

$d^1 = d^2 = \cdots = d^q = 0$ QED

Review Exercises

① Let $L: V \to W$ be a linear transformation. Show $\ker L = \{0_V\}$ if and only if L is one to one.

② Let $\{v_1, \ldots, v_n\}$ be a basis for V. Explain why

$$L(V) = \text{span}\{L(v_1), \ldots, L(v_n)\}.$$

③ Suppose $L: \mathbb{R}^4 \to \mathbb{R}^3$ whose matrix M (in the natural bases) is row equivalent to

$$M \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Explain why the first three columns of the original matrix M form a basis for $L(V)$.

Describe a general procedure for finding a basis for $L(V)$ when $L: \mathbb{R}^n \to \mathbb{R}^m$ (linear).

④ Suppose $\{v_1, \ldots, v_p\}$ is a basis for $\ker L$ where $L: V \to W$, then it is always possible to extend this set to a basis for V. Pick a simple yet non-trivial linear transformation with a non-trivial kernel and verify the above claim.

(viii) Explain why $\lambda = \bar{\lambda}$. What does this mean?

(2) Let $X_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

where $a^2+b^2+c^2=1$.

Find vectors X_2 and X_3 such that X_1, X_2, X_3 form an orthonormal basis for $\mathbb{R}^3$.

(3) What can you say about the dimensions of the eigenspaces of a real symmetric matrix?