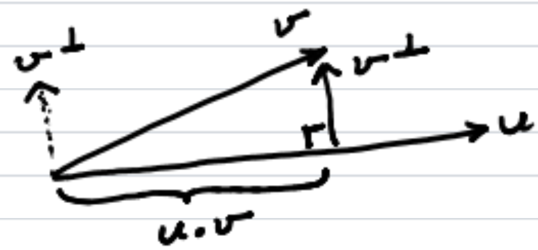
Lecture #24 ORTHOGONAL COMPLEMENTS & GRAM SCHMIDT

From two vectors u and v



the vector

$$\boxed{v^\perp = v - \frac{u \cdot v}{u \cdot u} u} \text{ --- } (*)$$

is orthogonal to u because

$$u \cdot v^\perp = u \cdot v - \frac{u \cdot v}{\cancel{u \cdot u}} \cancel{u \cdot u} = 0$$

Hence $\left\{ \frac{u}{|u|}, \frac{v^\perp}{|v^\perp|} \right\}$ is an

orthonormal basis for span$\{u, v\}$.

Sometimes people rewrite $(*)$ as

$$\boxed{v = v^\perp + v^{\parallel}} \text{ --- } (**)$$

where
$$v^\perp = v - \frac{u \cdot v}{u \cdot v} u$$
$$v^{\parallel} = \frac{u \cdot v}{u \cdot v} u$$

Eq$\underline{n}$ $(**)$ is called an <u>orthogonal decomposition</u> because we have written v as a sum of a vector perpendicular and a vector parallel to u.

If u, v are vectors in $\mathbb{R}^3$,
then $\{u, v^\perp, u \times v^\perp\}$ would be
an orthogonal basis (unless $v \parallel u$
in which case $v^\perp = 0$).

<u>Remark</u>, once you have an orthogonal
basis, an orthonormal one is obtained
easily by dividing by lengths,
for example, in $\mathbb{R}^3$ $\left\{\frac{u}{|u|}, \frac{v^\perp}{|v^\perp|}, \frac{u \times v^\perp}{|u \times v^\perp|}\right\}$.

But suppose u, v are vectors in
some vector space with dimension
<u>greater than 3</u> so that the cross
product is unavailable to us.
Given a third vector w, could
we find an orthogonal basis
for span$\{u, v, w\}$? Step 1. is
obvious, start with

$$u, \quad v^\perp = v - \frac{u \cdot v}{u \cdot u} u.$$

Now from $u, v^\perp, w$ how can we build three orthogonal vectors?

Consider (assuming $v^\perp \neq 0$).

$$w^\perp = w - \frac{u\cdot w}{u\cdot u}\,u - \frac{v^\perp\cdot w}{v^\perp\cdot v^\perp}\,v^\perp$$

Then

$$u\cdot w^\perp = u\cdot w - \frac{u\cdot w}{\cancel{u\cdot u}}\,\cancel{u\cdot u} - \frac{v^\perp\cdot w}{v^\perp\cdot v^\perp}\,\underbrace{\cancel{u\cdot v^\perp}}_{0} = 0$$

$$v^\perp\cdot w^\perp = v^\perp\cdot w - \frac{u\cdot w}{u\cdot u}\,\underbrace{\cancel{v^\perp\cdot u}}_{0} - \frac{v^\perp\cdot w}{\cancel{v^\perp\cdot v^\perp}}\,\cancel{v^\perp\cdot v^\perp} = 0$$

Hence $\{u, v^\perp, w^\perp\}$. If u, v, w are linearly independent so are $\{u, v^\perp, w^\perp\}$, hence this is an orthogonal basis for span $\{u, v, w\}$.

A simple question to ask yourself is when does $v^\perp = 0$? What about $w^\perp$?

In fact, given linearly indep$\underline{t}$ vectors $v_1, \dots, v_n$ an <u>orthogonal basis</u> for $\text{span}\{v_1, \dots, v_n\}$ is

$$\left\{\begin{array}{l}
v_1, \\
v_2^\perp = v_2 - \dfrac{v_1 \cdot v_2}{v_1 \cdot v_1} v_1, \\
v_3^\perp = v_3 - \dfrac{v_1 \cdot v_3}{v_1 \cdot v_1} v_1 - \dfrac{v_2^\perp \cdot v_3}{v_2^\perp \cdot v_2^\perp} v_2^\perp, \\
\quad\vdots \\
v_n^\perp = v_n - \dfrac{v_1 \cdot v_n}{v_1 \cdot v_1} v_1 - \dfrac{v_2^\perp \cdot v_n}{v_2^\perp \cdot v_2^\perp} v_2^\perp - \cdots - \dfrac{v_{n-1}^\perp \cdot v_n}{v_{n-1}^\perp \cdot v_{n-1}^\perp} v_{n-1}^\perp
\end{array}\right\}$$

This algorithm is called the <u>Gram-Schmidt</u> procedure.

<u>EXAMPLE</u> $u = (1,1,0)$, $v = (1,1,1)$, $w = (3,1,1)$

$$v^\perp = (1,1,1) - \frac{2}{2}(1,1,0) = (0,0,1)$$

$$w^\perp = (3,1,1) - \frac{4}{2}(1,1,0) - \frac{1}{1}(0,0,1) = (1,-1,0)$$

Orthogonal basis for $\mathbb{R}^3$

$$\{(1,1,0), (0,0,1), (1,-1,0)\}$$

Orthonormal basis for $\mathbb{R}^3$

$$\left\{\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right), (0,0,1), \left(\tfrac{1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}, 0\right)\right\}$$

Let U and V be subspaces of a vector space W. We saw in a review exercise that $U \cap V$ was also a subspace but $U \cup V$ was **not** a subspace. However

$$\text{span}\, U \cup V$$

is **certainly** a subspace of V.

Notice, elements of span $U \cup V$ all take the form

$$u + v \quad \text{with } u \in U, v \in V$$

hence we call the subspace

$$U + V = \text{span}\, U \cup V,$$

the **sum** of U and V.

When $U \cap V = \{0\}$ we write

$$U + V = U \oplus V$$

which is called the **direct sum** of U and V.

Notice that if $0 = u + v \in U \oplus V$ and $u \in U$, $v \in V$ then $u = -v$ but $-v$ is in V so $u \in V$ which implies $u = v = 0$.

Hence elements of $U \oplus V$ can be written **uniquely** as $u + v$ with $u \in U$, $v \in V$.

Suppose U is a subspace of a vector space V. Call
$U^\perp = \{v \in V : v \cdot u = 0 \text{ for all } u \in U\}$
The space "U-perp" is the set of all vectors in V that are orthogonal to every vector in U.
It is often called the orthogonal complement of U.

<u>Theorem</u> $U^\perp$ is a subspace of V and $V = U \oplus U^\perp$.

<u>Proof</u> To see $U^\perp$ is a subspace just requires closure which is easy.
$U^\perp \cap U = \{0\}$ holds because if $u \in U$ and $u \in U^\perp$, $u \cdot u = 0 \Leftrightarrow u = 0$
Finally, if $e_1, \dots, e_m$ is an orthonormal basis for U we can write any v as

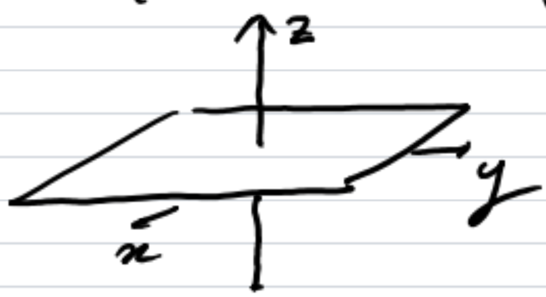
$$v = u + v^\perp \quad \text{where}$$

$$u = (v \cdot e_1)e_1 + \cdots + (v \cdot e_m)e_m \in U$$

$$v^\perp = v - (v \cdot e_1)e_1 - \cdots - (v \cdot e_n)e_n$$

and it is easy to check $v^\perp \in U^\perp$ //

EX $\mathbb{R}^3$ is the direct sum of any plane and a line orthogonal to that plane. For example
$$\mathbb{R}^3 = \{(x,y,0) : x,y \in \mathbb{R}\} \oplus \{(0,0,z) : z \in \mathbb{R}\}$$



Notice since every vector in U is perpendicular to every vector in $U^\perp$ we have $U = (U^\perp)^\perp$, another involution!

Review Exercises

① Suppose u and v are linearly independent, Show that u and $v^{\perp}$ are also linearly independent. Explain why $\{u, v^{\perp}\}$ are a basis for span $\{u, v\}$.

② Repeat the same problem for three vectors u, v, w that are linearly independent with $v^{\perp}, w^{\perp}$ as defined in the Lecture.

③ Consider $v^{\perp}, w^{\perp}$ as defined in the Lecture. When do these vectors vanish?

PTO

④ Explain why elements of span $U \cup V$ all have the form $u + v$ with $u \in U$, $v \in V$.

⑤ Show that elements of $U \oplus V$ can be expressed uniquely as $u + v$ with $u \in U$, $v \in V$.
[Hint: suppose the opposite were true and derive a contradiction]

⑥ Use the subspace theorem to show that $U^\perp$ is a subspace.

(viii) Explain why $\lambda = \bar{\lambda}$. What does this mean?