LECTURE #3

ELEMENTARY ROW OPERATIONS

Want to achieve RREF

$$\begin{pmatrix}
\boxed{1} & * & 0 & * & 0 & & 0 & * \\
0 & & \boxed{1} & * & 0 & & 0 & * \\
0 & & 0 & & \boxed{1} & \cdots & 0 & * \\
 & & & & \vdots & & \vdots & \\
 & & & & & & \boxed{1} & * \\
0 & & 0 & & 0 & & 0 & \\
0 & & 0 & & 0 & & 0 &
\end{pmatrix}$$

Must respect row equivalence (don't want to change solution set).

Three elementary row operations:

① $R_i \leftrightarrow R_j$ swap i^{th} & j^{th} rules

② λR_i multiply i^{th} row by $\lambda \neq 0$

③ $R_i + \lambda R_j$ add multiple of j^{th} row to i^{th} row.

Why are these allowed?

Example

$$R_1 \qquad ax + by = c$$
$$R_2 \qquad dx + ey = f$$

① $R_1 \leftrightarrow R_2$

$dx + ey = f$ just swaps
$ax + by = c$ two equations

② λR_1

$\lambda ax + \lambda by = \lambda c$ rescales an
$dx + ey = f$ equation

③ $R_1 + \lambda R_2$

$(a+\lambda d)x + (b+\lambda e)y = c+\lambda f$ add one
$dx + ey = f$ equation to another

Note In ③ $\lambda = -b/e$ is a clever choice

WHY?

A PLAN & AN EXAMPLE

Ex

$$
\begin{aligned}
3x_3 &= 9 \\
x_1 + 5x_2 - 2x_3 &= 2 \\
\tfrac{1}{3}x_1 + 2x_2 \quad &= 3
\end{aligned}
$$

Augmented matrix

Keep track of operations

$$\left(\begin{array}{ccc|c} & & 3 & 9 \\ 1 & 5 & -2 & 2 \\ 1/3 & 2 & & 3 \end{array}\right)$$

WORK COLUMN BY COLUMN

$R1 \leftrightarrow R3$

$$\sim \left(\begin{array}{ccc|c} 1/3 & 2 & & 3 \\ 1 & 5 & -2 & 2 \\ & & 3 & 9 \end{array}\right)$$

Swap rows to make sure first entry in C1 is not zero.

pivot

$3R1$

$$\sim \left(\begin{array}{ccc|c} \boxed{1} & 6 & & 9 \\ 1 & 5 & -2 & 2 \\ & & 3 & 9 \end{array}\right)$$

Multiply R1 with λ to make sure pivot is one.

$R2-R1$

$$\sim \left(\begin{array}{ccc|c} 1 & 6 & & 9 \\ & -1 & -2 & -7 \\ & & 3 & 9 \end{array}\right)$$

Add multiples of pivot row to other rows to make all other elements in pivot column vanish.

IGNORE FIRST ROW & COLUMN

AND REPEAT ABOVE PROCEDURE

$$\overset{-R_2}{\sim} \left(\begin{array}{ccc|c} 1 & 6 & & 9 \\ & \boxed{1} & 2 & 7 \\ & & 3 & 9 \end{array}\right)$$

$$\overset{R_1-6R_2}{\sim} \left(\begin{array}{ccc|c} 1 & & -12 & -33 \\ & \boxed{1} & 2 & 7 \\ & & 3 & 9 \end{array}\right)$$

As last step of iteration remove entries <u>above</u> pivot.

ITERATE UNTIL RREF

$$\overset{\frac{1}{3}R_3}{\sim} \left(\begin{array}{ccc|c} 1 & & -12 & -33 \\ & 1 & 2 & 7 \\ & & \boxed{1} & 3 \end{array}\right)$$

Making pivot one

$$\overset{\substack{R_1+12R_3 \\ R_2-2R_3}}{\sim} \left(\begin{array}{ccc|c} 1 & & & 3 \\ & 1 & & 1 \\ & & 1 & 3 \end{array}\right)$$

RREF

The system says

$$\begin{aligned} x_1 \qquad\qquad &= 1 \\ x_2 \qquad &= 3 \\ x_3 &= 1 \end{aligned}$$

It happens to have a unique solution.

ANOTHER EXAMPLE

$$\left(\begin{array}{cccc|c} \boxed{1} & & -1 & 2 & -1 \\ 1 & 1 & 1 & -1 & 2 \\ & -1 & -2 & 3 & -3 \\ 5 & 2 & -1 & 4 & 1 \end{array}\right)$$

$$\begin{array}{c} R_2 - R_1 \\ R_4 - 5R_1 \\ \sim \end{array} \left(\begin{array}{cccc|c} 1 & & -1 & 2 & -1 \\ & \boxed{1} & 2 & -3 & 3 \\ & -1 & -2 & 3 & -3 \\ & 2 & 4 & -6 & 6 \end{array}\right)$$

$$\begin{array}{c} R_3 + R_2 \\ R_4 - 2R_2 \\ \sim \end{array} \left(\begin{array}{cccc|c} 1 & & -1 & 2 & -1 \\ & 1 & 2 & -3 & 3 \\ & & & & \\ & & & & \end{array}\right)$$

RREF

$$\begin{array}{rcl} x_1 \quad\quad - x_3 + 2x_4 &=& -1 \\ x_2 + 2x_3 - 3x_4 &=& 3 \end{array}$$

$$\underbrace{x_1 \quad x_2}_{\text{pivot variables}}$$

<u>No</u> unique solution, set $x_3 = \lambda$, $x_4 = \mu$

$$\begin{cases} x_1 = \lambda - 2\mu - 1 \\ x_2 = -2\lambda + 3\mu + 3 \end{cases} \quad \text{OR} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ \\ \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \\ \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 3 \\ \\ 1 \end{pmatrix}$$

UNIQUENESS OF GAUSS-JORDAN ELIMINATION (Simple proof of W. Holzman)

Suppose Alice & Bob compute the RREF for a linear system but get different results, A & B (say).

Working from left keep only columns with a leading one (pivots) plus the first column in which A & B differ. Call the new matrices $\hat{A}$ & $\hat{B}$

Example

$$A = \begin{pmatrix} 1 & 3 & & 6 & 2 \\ & & 1 & 1 & \\ & & & & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 & & 7 & 4 \\ & & 1 & 1 & 3 \\ & & & & 2 \end{pmatrix}$$

$$\hat{A} = \begin{pmatrix} 1 & & 6 \\ & 1 & 1 \\ & & \end{pmatrix} \qquad \hat{B} = \begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & \end{pmatrix}$$

Note that $\hat{R} \sim \hat{S}$ because removing a column does not affect row equivalence.

View $\hat{R}$ & $\hat{S}$ as augmented matrices.

Observe

$$\hat{R} = \left(\begin{array}{c|c} I & r \\ \hline & \end{array}\right) \text{ or } \left(\begin{array}{c|c} I & \\ \hline & 1 \end{array}\right)$$

$$\hat{S} = \left(\begin{array}{c|c} I & s \\ \hline & \end{array}\right) \text{ or } \left(\begin{array}{c|c} I & \\ \hline & 1 \end{array}\right)$$

where I is an identity matrix & r, s are vectors. It is now easy to see that if $\hat{R}$ & $\hat{S}$ have the same solution (view $\emptyset$ as a "solution") then $\hat{R} = \hat{S}$.

This is a contradiction!

We are forced to conclude $A = B$ $\blacksquare$

Lecture 3 Review Questions

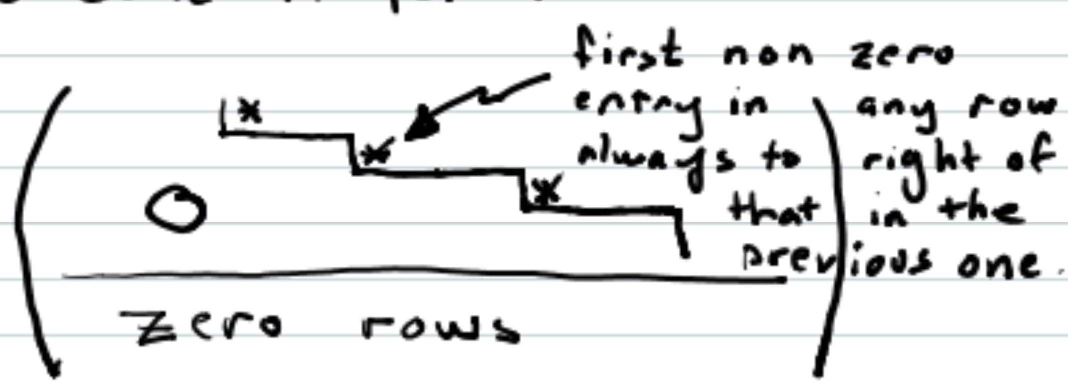
1. Explain why row equivalence is not affected by removing columns.

   Is the same true for rows?

   (If your answer is no, give a simple counterexample.)

2. **Gaussian Elimination:**

   Another method for solving linear systems is to use row operations to bring the augmented matrix to row echelon form:



Do you think row echelon form is unique (give a counter example)?

Once a system is in row echelon form it can be solved by "back - substitution".

Write the following row echelon matrix as a system of equations

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ & 1 & 1 & 2 \\ & & 3 & 3 \end{array}\right).$$

Solve these using back substitution.

3. Explain why the linear system $\left(\begin{array}{ccc|c} 1 & & 3 & 1 \\ & 1 & 2 & 4 \\ & & & 6 \end{array}\right)$

has <u>NO</u> solution.

For which values of k does the system below have a solution

$$\begin{aligned} x - 3y \qquad &= 6 \\ x \qquad + 3z &= -3 \\ 2x + kx \quad + (3-k)z &= 1 \end{aligned}$$

(i) Suppose $U = \mathbb{R}$ (real numbers).
Explain why $=$ is an equivalence relation but $\geqslant$ is not.

(ii) Explain why equivalence of augmented matrices is an equivalence relation.