LECTURE #4

SOLUTION SETS   H I.2

The space of all solutions to a linear system is called the <u>solution set</u>

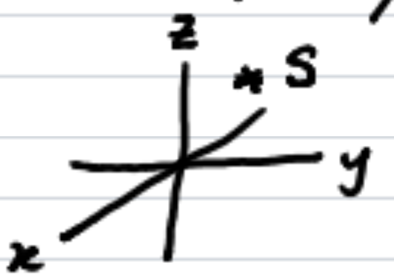
<u>Ex</u> 3 unknowns, r equations

$$\left(\begin{array}{ccc|c} a^1{}_1 & a^1{}_2 & a^1{}_3 & b^1 \\ a^2{}_1 & a^2{}_2 & a^2{}_3 & b^2 \\ & \vdots & & \vdots \\ a^r{}_1 & a^r{}_2 & a^r{}_3 & b^3 \end{array}\right)$$

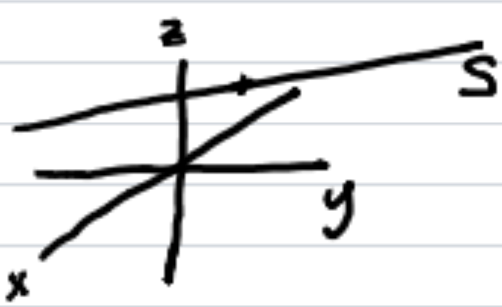
has 5 possible types of solution sets S.
They can be visualized in $\mathbb{R}^3$ as

(i) $S = \phi$ empty set (No solutions)

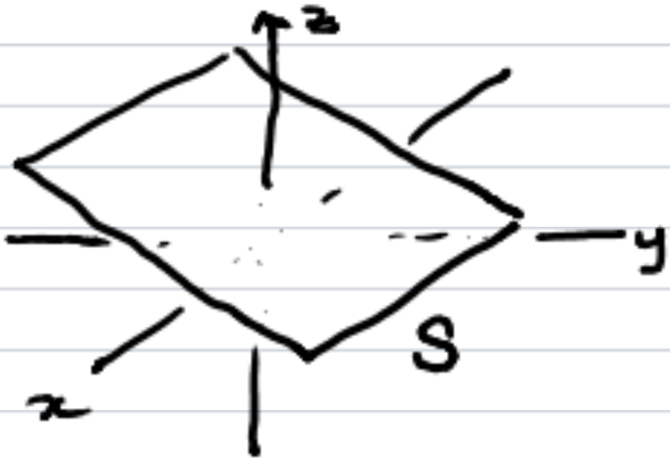
(ii) S = point (unique solution)



(iii) S = line (one free parameter)



(iv) S = plane (two free parameters)



(v) $S = \mathbb{R}^3$ (three free parameters)

For systems with k unknowns there are $k+2$ types of solution sets in $\mathbb{R}^k$ — these are hard to visualize. ("hyperplanes")

Non Leading Variables

Variables that are not pivot variables in RREF are free, we set them equal to arbitrary parameters $\mu_1, \mu_2, \dots$

Ex.
$$\left(\begin{array}{cccc|c} 1 & & 1 & -1 & 1 \\ & 1 & -1 & 1 & -1 \\ & & & & \end{array}\right)$$

says
$$\begin{array}{l} x \quad\quad + z - w = 1 \\ \quad\;\; y - z + w = 1 \end{array}$$

pivot variables

non-leading variables (not pivot)

Solution $w = \mu$, $z = \nu$, $y = 1 + \nu - \mu$, $x = 1 - \nu + \mu$

i.e. $S = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \\ \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ \\ 1 \end{pmatrix} + \nu \begin{pmatrix} -1 \\ 1 \\ 1 \\ \end{pmatrix} : \mu, \nu \in \mathbb{R} \right\}$

is the solution set.

General, particular & homogeneous solutions

We could write the previous system as

$$MX = V \quad \text{matrix multiplication}$$

where

$$M = \begin{pmatrix} 1 & & 1 & -1 \\ & 1 & -1 & 1 \\ & & & \\ \end{pmatrix} \quad \text{Matrix}$$

$$X = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \qquad V = \begin{pmatrix} 1 \\ -1 \\ \\ \end{pmatrix}$$

unknowns constant vector

if we knew how to define matrix matrix multiplication for systems of arbitrary size.

Defining addition of vectors componentwise

$$\begin{pmatrix} a^1 \\ a^2 \\ \vdots \\ a^r \end{pmatrix} + \begin{pmatrix} b^1 \\ b^2 \\ \vdots \\ b^r \end{pmatrix} \equiv \begin{pmatrix} a^1 + b^1 \\ a^2 + b^2 \\ \vdots \\ a^r + b^r \end{pmatrix}$$

and scalar multiplication

$$\lambda \begin{pmatrix} a^1 \\ a^2 \\ \vdots \\ a^r \end{pmatrix} \equiv \begin{pmatrix} \lambda a^1 \\ \lambda a^2 \\ \vdots \\ \lambda a^r \end{pmatrix}$$

we can write our solution above as

$$\boxed{X = X_0 + \mu Y_1 + \nu Y_2}$$

where

$$X_0 = \begin{pmatrix} 1 \\ 1 \\ \\ \\ \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 1 \\ -1 \\ \\ 1 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ \\ \end{pmatrix}$$

Now <u>suppose</u> our rule for

Matrix × Vector = Vector

obeyed

$$\boxed{M(\alpha X + \beta Y) = \alpha M X + \beta M Y}$$

"Linearity"

(this is the key property of a <u>linear</u> <u>transformation</u>)

then $\begin{cases} MX = V \\ X = X_0 + \mu Y_0 + \nu Y_1 \end{cases}$

together say

$$MX_0 + \mu M Y_0 + \nu M Y_1 = V$$

FOR ANY $\mu, \nu \in \mathbb{R}$.

Some interesting choices

$\mu = \nu = 0 \Rightarrow \boxed{MX_0 = V}$

$\Rightarrow \mu M Y_0 + \nu M Y_1 = V$

$\mu = 1, \nu = 0 \Rightarrow \boxed{MY_0 = 0}$

$\mu = 0, \nu = 1 \Rightarrow \boxed{MY_1 = 0}$

X_0 is an example of a particular solution

Y_0, Y_1 are called homogeneous solutions

We didn't prove linearity of matrix multiplication, so lets <u>at least</u> check our claims for the example

$\underline{MX_0 = V}$ says $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \\ \end{pmatrix}$

solves

$$\begin{aligned} x \quad\quad + z - w &= 1 \\ y - z + w &= 1 \end{aligned}$$

$\underline{MY_1 = 0}$ says $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ \\ 1 \end{pmatrix}$

solves

$$\begin{aligned} x \quad\quad + z - w &= 0 \\ y - z + w &= 0 \end{aligned}$$

$\underline{MY = 0}$ says $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ \end{pmatrix}$

solves

$$\begin{aligned} x \quad\quad + z - w &= 0 \\ y - z + w &= 0 \end{aligned}$$

Given a linear system

$$\underbrace{M}_{\text{Matrix of coefficients}}\ \underbrace{X}_{\text{unknowns}} = \underbrace{V}_{\text{constant vector}}$$

We call X_0 a <u>particular solution</u> if

$$MX_0 = V$$

We call Y a <u>homogeneous solution</u> if

$$MY = 0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and $MX = 0$ is called the (associated) <u>homogeneous system</u>

If X_0 is a particular solution, the general solution set is

$$\boxed{S = \{X_0 + Y : MY = 0\}}$$

or in words

general solution = particular + homogeneous

Lecture 4 Review Questions

1. Write down examples of augmented matrices corresponding to each of the 5 types of solution sets for systems $\bar{c}$ 3 unknowns.

2. Let

$$M = \begin{pmatrix} a^1_1 & a^1_2 & \cdots & a^1_k \\ \vdots & & & \\ a^r_1 & & \cdots & a^r_k \end{pmatrix}$$

$$X = \begin{pmatrix} x^1 \\ \vdots \\ x^k \end{pmatrix}$$

Propose a rule for MX so that

$$MX = 0$$

is equivalent to the linear system

$$\begin{aligned} a^1_1 x^1 + \cdots + a^1_k x^k &= 0 \\ \vdots \qquad\qquad & \;\vdots \\ a^r_1 x^1 + \cdots + a^r_k x^k &= 0 \end{aligned}$$

Does your rule for a matrix $\times$ vector obey the linearity property? Explain

(i) Suppose $\mathcal{U} = \mathbb{R}$ (real numbers).
Explain why $=$ is an equivalence relation but $\geqslant$ is not.

(ii) Explain why equivalence of augmented matrices is an equivalence relation.