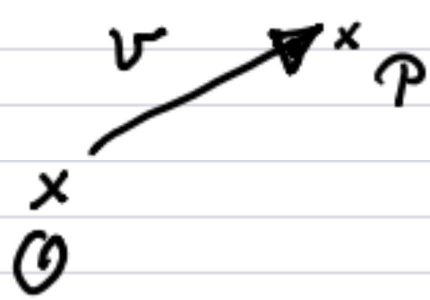
LECTURE #5

VECTORS in SPACE, n-VECTORS H II.1

Just as in $\mathbb{R}^3$, vectors in $\mathbb{R}^n$ describe a direction and magnitude.

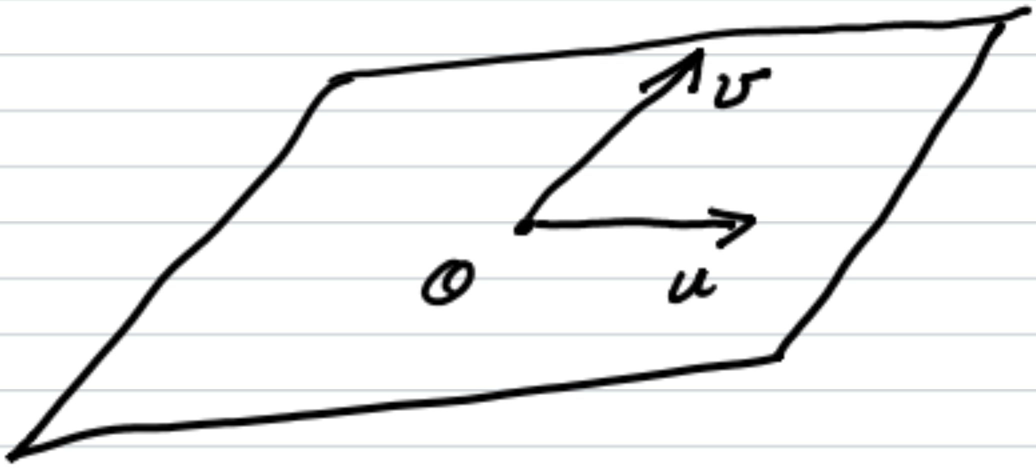
If you specify an origin O you can use them to specify positions P



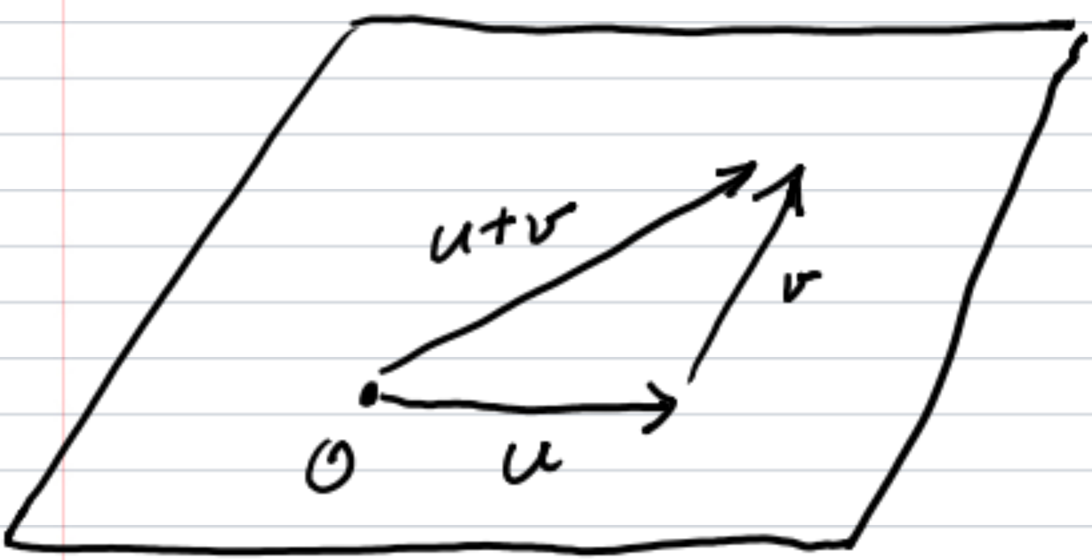
If P has coordinates $(y^1, \ldots, y^n)$ & O has coordinates $(x^1, \ldots, x^n)$ the components of v are

$$v = \begin{pmatrix} y^1 - x^1 \\ y^2 - x^2 \\ \vdots \\ y^n - x^n \end{pmatrix}$$

Any two vectors u & v determine a plane through O



Their sum $u+v$ corresponds to adding them head to tail and lies in the same plane:



i.e. $S = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \\ \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ \\ 1 \end{pmatrix} + \nu \begin{pmatrix} -1 \\ 1 \\ 1 \\ \end{pmatrix} : \mu, \nu \in \mathbb{R} \right\}$

is the solution set.

Vectors can be used to describe geometrical objects in $\mathbb{R}^n$:

<u>Ex</u>: $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}$

describes a line in 4-dimensional space $\mathbb{R}^4$.

<u>Ex</u>: $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} : s, t \in \mathbb{R} \right\}$

describes a plane (// to xy plane) in $\mathbb{R}^3$.

<u>Ex</u>: Let $X, Y_1, \ldots, Y_k$ be vectors in $\mathbb{R}^d$.

Then $\left\{ X + \sum_{i=1}^{k \leq d} \lambda_i Y_i \mid \lambda_i \in \mathbb{R} \right\}$

describes a k-dimensional hyperplane in $\mathbb{R}^d$ (usually – what could go wrong ??)

PROPERTIES OF VECTORS H II.2

Lengths of vectors

There are many useful definitions for the "length" of a vector.

EX Einstein $\mathbb{R}^4 =$ space $\mathbb{R}^3 \times$ time $\mathbb{R}$

suggested that $X = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$ should

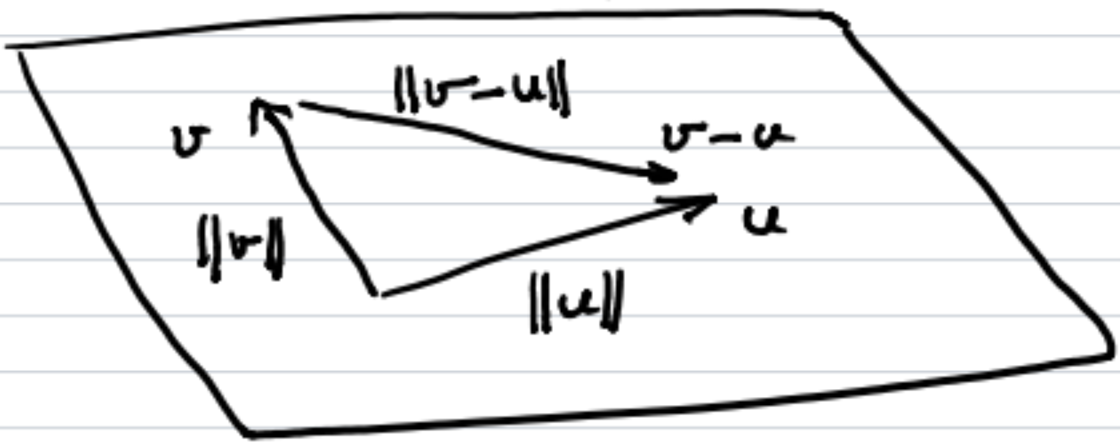
have $(\text{length})^2 = -t^2 + x^2 + y^2 + z^2$.

Most common definition is Pythagorean length for $V = \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{pmatrix} \in \mathbb{R}^n$

length $\|v\| = \sqrt{(v^1)^2 + (v^2)^2 + \cdots + (v^n)^2}$

Angles b/w vectors

Since any two vectors $u, v \in \mathbb{R}^n$ span a plane we have a triangle:



But for any triangle the cosine law says *

$$\|v-u\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\,\|v\| \cos\theta$$

We would like to calculate $\cos\theta$

$$\begin{aligned}
&\|v-u\|^2 - \|u\|^2 - \|v\|^2 \\
&\quad = (v^1-u^1)^2 + (v^2-v^2)^2 + \cdots + (v^n-u^n)^2 \\
&\qquad - \left((u^1)^2 + (u^2)^2 + \cdots + (u^n)^2\right) \\
&\qquad - \left((v^1)^2 + (v^2)^2 + \cdots + (v^n)^2\right) \\
&\quad = -2v^1u^1 - 2v^2u^2 \cdots - 2u^nv^n
\end{aligned}$$

$$\Rightarrow \|u\|\,\|v\| \cos\theta = v^1u^1 + v^2u^2 + \cdots v^nu^n$$

The RHS is called the <u>dot product</u> (or inner/scalar product)

$$\boxed{u \cdot v = u^1 v^1 + u^2 v^2 + \cdots + u^n v^n}$$

It is used to compute the angle θ b/w vectors in $\mathbb{R}^n$

$$\boxed{\cos\theta = \frac{u \cdot v}{\|u\| \, \|v\|}}$$

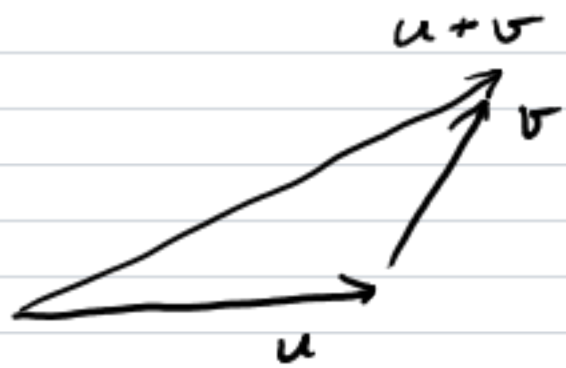
Notice that $|\cos\theta| \leq 1$

so we learn

$$\boxed{\frac{|u \cdot v|}{\|u\| \, \|v\|} \leq 1}$$ "Cauchy Schwartz" inequality.

* Fine print, in the above we <u>assumed</u> that $\|u\|, \|v\|$ gave the usual lengths in the plane spanned by u & v. This is true, see HII.2

TRIANGLE INEQUALITY



$$\|u+v\| \leq \|u\| + \|v\|$$

PROOF $(LHS)^2 = (u+v)\cdot(u+v)$ USE $\|v\|=v.v$

$= u\cdot u + 2u\cdot v + v\cdot v$ USE dot product distributes and is symmetric

$= \|u\|^2 + \|v\|^2 + 2\|u\|\|v\|\cos\theta$

$= (\|u\|+\|v\|)^2 + \underbrace{2\|u\|\|v\|}_{\geq 0}\underbrace{(\cos\theta - 1)}_{\leq 0}$

$\leq (\|u\|+\|v\|)^2$

$= (RHS)^2$

So $(LHS)^2 \leq (RHS)^2$ and both LHS & RHS are positive so LHS $\leq$ RHS QED

Lecture 5 Review Questions

1. (H II.1, 2.15)

(i) Find the angle b/w the diagonal of the unit square in $\mathbb{R}^2$ and one of the coordinate axes

(ii) Find the angle b/w the diagonal of the unit cube in $\mathbb{R}^3$ and one of the coordinate axes

(n) Find the angle b/w the diagonal of the unit cube in $\mathbb{R}^n$ and one of the coordinate axes

(∞) What is the limit as $n \to \infty$ of the angle b/w the diagonal of the unit cube in $\mathbb{R}^n$ and one of the axes.

2. Consider the matrix

$$M = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and the vector

$$X = \begin{pmatrix} x \\ y \end{pmatrix}$$

(i) Sketch X & MX in $\mathbb{R}^2$

(ii) Compute $\|MX\| / \|X\|$.

Explain your result.

(iii) compute $\dfrac{(MX) \cdot X}{\|X\|^2}$ and

explain this result too.

(i) Suppose $U = \mathbb{R}$ (real numbers).
Explain why $=$ is an equivalence relation but $\geq$ is not.

(ii) Explain why equivalence of augmented matrices is an equivalence relation.