LECTURE #6

VECTOR SPACES H Ch2 I.1

So far we have studied vectors in $\mathbb{R}^n$ written as

$$v = \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{pmatrix} \text{ ——— } (*)$$

This notation is cumbersome & conceptually limiting.

Instead define vectors by their key properties.

i.e. want $(*)$ to be just an example of a more general & interesting structure.

The key property of vectors is that you can add them

So we DEFINE:

A <u>vector space</u> (over $\mathbb{R}$) as a set V with two operations $+$ and $\cdot$ that obey

(+i) $u+v \in V$ for all $u, v \in V$

[ADDING VECTORS GIVES A VECTOR]

(+ii) $u+v = u+v$ for all $u, v \in V$

[THE ORDER OF ADDITION DOES NOT MATTER]

(+iii) $(u+v)+w = u+(v+w)$

for all $u, v, w \in V$

[ASSOCIATIVE]

(+iv) There is a zero vector $0 \in V$

such that $u+0=u$

for all u in V

(+v) Every $v \in V$ has an additive inverse w

such that $w+v=0$

(·i) for all $r \in \mathbb{R}$, $v \in V$

$$r \cdot v \in V$$

[MULTIPLYING A VECTOR BY A SCALAR IS A VECTOR]

(·ii) for all $r, s \in \mathbb{R}$, and $v \in V$

$$(r+s) \cdot v = r \cdot v + s \cdot v$$

[SCALAR ADDITION DISTRIBUTES]

(·iii) for all $r \in \mathbb{R}$ and $u, v \in V$

$$r \cdot (u+v) = r \cdot u + r \cdot v$$

[VECTORS ADDITION DISTRIBUTES]

(·iv) for all $r, s \in \mathbb{R}$, $v \in V$

$$(rs) \cdot v = r \cdot (s \cdot v)$$

(·v) for all $v \in V$

$$1 \cdot v = v$$

REMARKS

* It is not hard to devise strange rules for addition or multiplication that break any of the above rules.

* Don't confuse $\cdot$ with the dot product. It is really just multiplying a vector by a number. Often we will write just

$$r v \quad \text{for} \quad r \cdot v$$

once we have checked the rules above.

Example

$$V = \{ f \mid f : \mathbb{N} \to \mathbb{R} \}$$

i.e. The set of real-valued functions of one natural number variable

with addition

$f_1 + f_2$ being the function

$$f_1(n) + f_2(n)$$

and scalar multiplication $r \cdot f$ being the function

$$r f(n)$$

Notice we can think of these as infinite column vectors

Ex $f(n) = n^3$

n	$f(n)=n^3$
0	0
1	1
2	8
3	27
$\vdots$	$\vdots$

$$\longleftrightarrow \begin{pmatrix} 0 \\ 1 \\ 8 \\ 27 \\ \vdots \end{pmatrix}$$

Alternatively, V is the space of sequences.

Lets check an axiom

(+i) ADDITIVE CLOSURE

$f_1(n) + f_2(n)$ is a function $\mathbb{N} \to \mathbb{R}$

since $f_1(n) + f_2(n)$ is real.

(+iv) We need to propose the zero vector 0. The function

$f(n) = 0$ works

because $f(n) + g(n) = 0 + g(n) = 0$

As for many examples, checking the other examples amounts to simple properties of real numbers.

REMARK

Here we defined vector spaces over real numbers.

The same definitions work for multiplication by complex numbers $z \in \mathbb{C}$.

In quantum physics vector spaces over $\mathbb{C}$ describe all possible states a system can have.

For example

$$V = \left\{ \begin{pmatrix} \lambda \\ \mu \end{pmatrix} : \lambda, \mu \in \mathbb{C} \right\}$$

describes states of an electron where $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a spin "up" ~~state~~ & $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is spin "down".

Ref. Wiki spin $1/2$.

Lecture 6 Review Questions

1. Check that $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\} = \mathbb{R}^2$

   with addition

   $$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x+u \\ y+v \end{pmatrix}$$

   and scalar multiplication

   $$r \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx \\ ry \end{pmatrix}$$

   is a vector space.

2. If we changed the set of sequences $V = \{ f \mid f : \mathbb{N} \to \mathbb{R} \}$ to the set of convergent sequences, would the same addition and multiplication rules give a vector space. <u>Explain</u>.

3. Let $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$.

Propose as many rules for addition and multiplication that you can think of that break some of the vector space conditions while satisfying at the same time others.

Hefferon p. 88

$$\left.\begin{array}{l} 1.18\ a,b \\ 1.19\ a \\ 1.20\ a \\ 1.21 \end{array}\right\} \text{perhaps?}$$

(i) Suppose $\mathcal{U} = \mathbb{R}$ (real numbers).
Explain why $=$ is an equivalence relation but $\geqslant$ is not.

(ii) Explain why equivalence of augmented matrices is an equivalence relation.