LECTURE

LINEAR TRANSFORMATIONS

The key properties of vector spaces are

<u>vector addition</u> $u, v \in V$

$$u + v \in V$$

<u>scalar multiplication</u> $r \in \mathbb{R}, u \in V$

$$r \cdot u \in V$$

Now suppose we have two vector spaces V, W and a map

$$L : V \longrightarrow W$$

So if $u, v \in V$, $L(u), L(v) \in W$

We want to preserve the addition & multiplication properties

$$L(u + v) = L(u) + L(v)$$

$$\underset{\substack{\uparrow \\ \text{operations} \\ \text{in } V}}{L(r \cdot u)} = \underset{\substack{\uparrow \\ \text{operations} \\ \text{in } W}}{r \cdot L(u)}$$

Putting these together—
we define a linear trans-
formation $L: V \to W$ between
vector spaces V & W by the
requirements

"Linearity"

$$\boxed{L(r.u+s.v) = r.L(u)+s.L(v)}$$

for all $u, v \in U$, $r, s \in \mathbb{R}$

Example

$$L: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

with

$$L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ y+z \\ 0 \end{pmatrix}$$

Call $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Check linearity, LHS

$$L(ru+sv) = L\left(r\begin{pmatrix}x\\y\\z\end{pmatrix} + s\begin{pmatrix}a\\b\\c\end{pmatrix}\right)$$

$$= L\left(\begin{pmatrix}rx\\ry\\rz\end{pmatrix} + \begin{pmatrix}sa\\sb\\sc\end{pmatrix}\right) = L\begin{pmatrix}rx+sa\\ry+sb\\rz+sc\end{pmatrix}$$

$$= \begin{pmatrix}rx+sa+ry+sb\\ry+sb+rz+sc\\0\end{pmatrix}$$

compare with RHS

$$rL(u)+sL(v) = rL\begin{pmatrix}x\\y\\z\end{pmatrix} + sL\begin{pmatrix}a\\b\\c\end{pmatrix}$$

$$= r\begin{pmatrix}x+y\\y+z\\0\end{pmatrix} + s\begin{pmatrix}a+b\\b+c\\0\end{pmatrix}$$

$$= \begin{pmatrix}rx+ry\\ry+rz\\0\end{pmatrix} + \begin{pmatrix}sa+sb\\sb+sc\\0\end{pmatrix} = \begin{pmatrix}rx+ry+sa+sb\\ry+rz+sb+sc\\0\end{pmatrix}$$

Notice LHS = RHS. Therefore L is a linear transformation

<u>Remark</u> We can write $L(u)$ using a matrix

$$L\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x+y\\y+z\\0\end{pmatrix} = \begin{pmatrix}1&1&0\\0&1&1\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

and you already checked that matrices multiply vectors in $\mathbb{R}^n$ obeyed

$$M(ru+sv) = rMu + sMv$$

so the above check was guaranteed to work!

<u>I.e.</u> Matrix multiplication is a linear transformation!

Example

Let V be the vector space of polynomials of finite degree with standard addition and multiplication rules

i.e. $V = \{a_0 + a_1 x + \dots + a_n x^n : n \in \mathbb{N},$
$a_0, \dots, a_n \in \mathbb{R}\}$

Let $L : V \longrightarrow V$ be the derivative operator $\frac{d}{dx}$.

L is a linear operator because if p_1, p_2 are polynomials, $r, s \in \mathbb{R}$

$$\frac{d}{dx}(r p_1 + r p_2) = r \frac{dp_1}{dx} + s \frac{dp_2}{dx}$$

by the rules of differentiation

Notice we could represent polynomials as "semi-infinite" column vectors

$$a_0 + a_1 x + \cdots + a_n x^n \longleftrightarrow \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

Then since

$$\frac{d}{dx}\left(a_0 + a_1 x + \cdots + a_n x^n\right) \cdot \longleftrightarrow \begin{pmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ \vdots \\ na_n \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

$$= a_1 + 2a_2 x + \cdots + na_n x^{n-1}$$

we could represent $\frac{d}{dx}$ as an infinite matrix

$$\frac{d}{dx} \longleftrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ \vdots & & & & \end{pmatrix}$$

Remark

Some of our vector space examples have been "finite dimensional" (Ex: $\mathbb{R}^n$) while others such as the previous polynomial examples are "infinite dimensional".

To understand dimensionality we need to think about the smallest number n of independent vectors $e_1, \dots, e_n$ required to build an arbitrary vector

$$v = r_1 e_1 + \cdots + r_n e_n$$

In the finite dimensional case, (dimension of $V = n < \infty$) it is always possible to represent linear transformations as <u>matrices</u> — the next subject of our studies.

Lecture 7 Review Questions

1. Show why the pair of conditions

① $\begin{cases} L(u+v) = L(u) + L(v) \\ L(r.u) = r.L(u) \end{cases}$

is equivalent to the single condition

② $L(r.u) + L(s.v) = r.L(u) + s.L(v)$

(your answer should have two parts (i) show ① $\Rightarrow$ ② and (ii) show ② $\Rightarrow$ ①).

2. Let P_n be the vector space of degree n polynomials in the variable t. Suppose

$L: P_2 \longrightarrow P_3$

is a linear transformation and $L(1) = 4$, $L(t) = t^3$, $L(t^2) = t-1$

(i) Find $L(1+t+2t^2)$

(ii) Find $L(a+bt+ct^2)$

(iii) Find all values a, b & c in (ii) such that $L(a+bt+ct^2) = 1+3t+2t^3$

3. Let $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$.

Propose as many rules for addition and multiplication that you can think of that break some of the vector space conditions while satisfying at the same time others.

Hefferon p. 88 $\left.\begin{array}{l} \text{1.18 a, b} \\ \text{1.19 a} \\ \text{1.20 a} \\ \text{1.21} \end{array}\right\}$ perhaps?

(i) Suppose $U = \mathbb{R}$ (real numbers).
Explain why $=$ is an equivalence relation but $\geq$ is not.

(ii) Explain why equivalence of augmented matrices is an equivalence relation.