LECTURE #8

MATRICES

An $r \times k$ matrix $M = (m^i_j)$ $i = 1, \ldots, r$; $j = 1, \ldots, k$ is a rectangular array of real (or complex*) numbers

$$M = \begin{pmatrix} m^1_1 & m^1_2 & \cdots & m^1_k \\ m^2_1 & m^2_2 & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ m^r_1 & m^r_2 & \cdots & m^r_k \end{pmatrix}$$

The numbers

labels row

$$m^i_j$$

labels column

are called entries.

---

* It is often useful to consider matrices with more general entries

A $r\times 1$ matrix $V=(v^r_1)\equiv(v^r)$

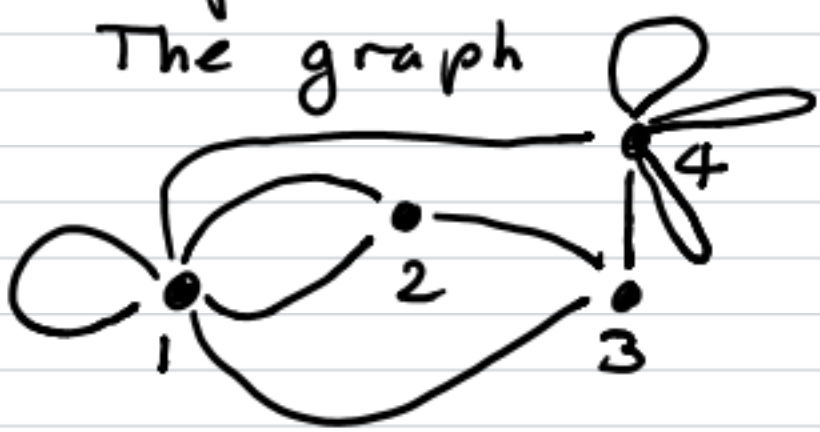
is called a <u>column vector</u>

$$V=\begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^r \end{pmatrix}$$

and $1\times k$ matrix is a <u>row vector</u>.

Matrices are a useful way to store information:

<u>Ex</u> The graph



can be represented as

$$M=\begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 3 \end{pmatrix}$$ where $m^i_j=$ the number of links b/w node i & node j.

Because $m^i_j=m^j_i$, we call M a <u>symmetric matrix.</u>

The space of $r \times k$ matrices M^r_k is a <u>vector space</u> with addition and scalar multiplication

$$M + N = (m^i_j) + (n^i_j) = (m^i_j + n^i_j)$$

$$r \cdot M = r \cdot (m^i_j) = (r m^i_j)$$

Notice that $M^n_1 = \mathbb{R}^n$ (column vectors)

Just as $r \times k$ matrices could be used for linear transformations

$$\mathbb{R}^k \longrightarrow \mathbb{R}^r$$

via

$$MV = \left(\sum_{j=1}^{k} m^i_j v^j\right)$$

$r \times k$ matrix (m^i_j)

$k \times 1$ matrix = column vector (v^j)

$$(\longrightarrow)(\downarrow)$$

row size = column size

We can use matrices for linear transformations

$$M_k^s \xrightarrow{L} M_k^r$$

via

$$\underset{\substack{\uparrow \\ r\times s \\ (l_j^i)}}{L}\ \underset{\substack{\uparrow \\ s\times k \\ (m_l^j)}}{M} = \underbrace{\left(\sum_{j=1}^{s} l_j^i m_l^j\right)}_{r\times k}$$

This rule obeys linearity

$$L(rM_1 + sM_2) = rLM_1 + sLM_2$$

Notice the rule

$$(r\times l)(l\times k) = (r\times k)$$

columns & rows match

<u>Example</u> $(3\times 1)(1\times 2) = (3\times 2)$

$$\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \end{pmatrix} = \begin{pmatrix} 1.2 & 1.3 \\ 3.2 & 3.3 \\ 2.2 & 2.3 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 6 & 9 \\ 4 & 6 \end{pmatrix}$$

1 row

1 column

The entries m^i_i are called <u>diagonal</u> and the set $\{m^1_1, m^2_2, \dots\}$ is called the <u>diagonal of a matrix</u>.
(The diagonal of the RHS above is $\{1, 9\}$)
An $r\times r$ matrix is called a <u>square matrix</u>. The square matrix whose only non-vanishing entries are diagonal is called a diagonal matrix

<u>Ex</u>

$$M = \begin{pmatrix} 4 & & & \\ & 1 & & \\ & & 6 & \\ & & & 2 \end{pmatrix}$$

The diagonal matrix with all diagonal entries $a^i_i = 1$ is called the identity matrix

$$I = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \quad (\text{or } \mathbb{1})$$

It is special because

$$\boxed{I_r M = M I_k} \quad \forall M \in M^r_k$$

(We give I a subscript r to specify its number of rows/columns)

The <u>transpose</u> of an $r \times k$ matrix $M = (m^i_j)$ is the $k \times r$ matrix with entries

$$M^T = (\overline{m}^i_j) \text{ with } \overline{m}^i_j = m^j_i$$

<u>Example</u>

$$\begin{pmatrix} 3 & 1 & 2 \\ 6 & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 3 & 6 \\ 1 & 0 \\ 2 & 1 \end{pmatrix}$$

Symmetric matrices obey

$$M = M^T$$

The transpose is called a <u>involution</u> because for <u>any</u> matrix

$$(M^T)^T = M$$

<u>Notice</u> that the transpose of a column vector is a row vector & vice versa.

<u>Ex</u> $\begin{pmatrix} 3 & 1 & 2 \end{pmatrix}^T = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

<u>Theorem</u> Let M, N be matrices such that MN makes sense. Then

$$(MN)^T = N^T M^T$$

Lecture 8 Review Questions

1. In the lecture we showed that <u>left</u> multiplication by $r \times s$ matrices L was linear transformation

$$\begin{array}{ccc} L: M^s_k & \longrightarrow & M^r_k \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ M & \longmapsto & LM \end{array}$$

Show that right multiplication by $k \times l$ matrices R is a linear transformation

$$\begin{array}{ccc} R: M^s_k & \longrightarrow & M^s_l \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ M & \longmapsto & MR \end{array}$$

(i.e. Check that right matrix multiplication obeys linearity)

2. Prove the theorem

$$(MN)^T = N^T M^T$$

3. Explain what happens to a matrix when

   (i) You multiply it by a diagonal matrix from the left.

   (ii) When you multiply it by a diagonal matrix from the right.

Give a few simple examples before you start explaining.

(i) Suppose $U = \mathbb{R}$ (real numbers).
Explain why $=$ is an equivalence relation but $\geqslant$ is not.

(ii) Explain why equivalence of augmented matrices is an equivalence relation.