LECTURE #9 MATRIX PROPERTIES

BLOCK MATRICES

It is often convenient to partition a matrix M into smaller matrices called blocks using a system of horizontal matrices.

Ex

$$M = \left(\begin{array}{cc:c} 1 & 2 & 0 \\ 3 & 0 & 1 \\ \hdashline 2 & 1 & 0 \end{array}\right) = \left(\begin{array}{c:c} A & B \\ \hline C & D \end{array}\right)$$

where $A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $C = (2\ 1)$, $D = (0)$

* Blocks must fit together correctly (e.g. $\left(\begin{array}{c|c} B & A \\ \hline D & C \end{array}\right)$ makes sense but $\begin{pmatrix} C & A \\ D & B \end{pmatrix}$ doesn't)

* There are many choices for writing M as a "block matrix"

* Matix operations on block matrices can be carried out by treating the blocks as matrix entries.

Ex

$$M^2 = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

$$= \left(\begin{array}{c|c} A^2 + BC & AB + BD \\ \hline CA + DC & CB + D^2 \end{array}\right)$$

$$= \left(\begin{array}{c|c} \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(2\ \ 1) & \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0) \\ \hline (2\ \ 1)\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} + (0)(2\ \ 1) & (2\ \ 1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + (0)(0) \end{array}\right)$$

$$= \left(\begin{array}{c|c} \begin{pmatrix} 7 & 2 \\ 3 & 6 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} & \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \hline (5\ \ 4) + (0\ \ 0) & (1) + (0) \end{array}\right)$$

$$= \left(\begin{array}{cc|c} 7 & 2 & 2 \\ 5 & 7 & 0 \\ \hline 5 & 4 & 1 \end{array}\right)$$

Versus

$$\begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 7 & 2 & 2 \\ 5 & 7 & 0 \\ 5 & 4 & 1 \end{pmatrix}$$

The Algebra of Square Matrices

* Not every pair of matrices can be added or multipied - require $(r\times k)+(r\times k)$ or $(r\times k)\times(k\times \ell)$.

* For $n\times n$ square matrices there is no such problem; if $M \in M^n_n$ we can form
$$M^2 = MM, \quad M^3 = MMM, \text{ etc...}$$
and define $M^0 = I$ (identity).

Then if $f(x)$ is any function with convergent Taylor series
$$f(x) = f(0) + f'(0)x + \tfrac{1}{2!} f''(0)\, x^2 + \cdots$$
we can define the matrix-function
$$f(M) = f(0)I + f'(0)M + \tfrac{1}{2!} f''(0) M^2 + \cdots$$

Example $f(x) = \log(1+x)$
$$f(x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \cdots$$
Let $M = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \Rightarrow M^2 = \begin{pmatrix} 1 & 2t \\ 0 & 1 \end{pmatrix} \Rightarrow M^3 = \begin{pmatrix} 1 & 3t \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow f(M) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - \tfrac{1}{2}\begin{pmatrix} 1 & 2t \\ 0 & 1 \end{pmatrix} + \tfrac{1}{3}\begin{pmatrix} 1 & 3t \\ 0 & 1 \end{pmatrix} - \cdots$$
$$= \begin{pmatrix} 1 & 1-t+t^2-t^3\cdots \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{1+t} \\ 0 & 1 \end{pmatrix}$$

Notice, require $|t|<1$ for series ~~to~~ converge, this situation is similar to that for regular functions.

Matrix multiplication does <u>not</u> commute

I.e. if M & N are square matrice, generically

$$MN \neq NM$$

<u>Example</u>

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \text{ but } \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Notice that since $n \times n$ matrices are linear transformations $\mathbb{R}^n \longrightarrow \mathbb{R}^n$, the order of successive linear transformations matters.

Ie. if $L: \mathbb{R}^n \to \mathbb{R}^n$ & $K: \mathbb{R}^n \to \mathbb{R}^n$ are linear transformations, for $v \in \mathbb{R}^n$

$$L(K(v)) \neq K(L(v))$$

in general!

Finding matrices such that $MN = NM$ is a important problem in mathematics!

The trace

Matrices are rather complicated, so any way of condensing their essential features is useful.

The <u>trace</u> of a square matrix $M = (m^i_j)$ is defined to be the sum of its diagonal entries

$$\boxed{\operatorname{tr} M = \sum_{i=1}^{n} m^i_i}$$

<u>Ex</u> $\operatorname{tr}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 1+5+9 = 15$

Notice

$$\begin{aligned}
\operatorname{tr}(MN) &= \operatorname{tr}\Big(\sum_l M^i_l N^l_j\Big) \\
&= \sum_i \sum_l M^i_l N^l_i \\
&= \sum_l \Big(\sum_i N^l_i M^i_l\Big) \\
&= \operatorname{tr}\Big(\sum_i N^l_i M^i_r\Big) \\
&= \operatorname{tr}(NM)
\end{aligned}$$

entries are just numbers

The trace lets you swap the order

$$\operatorname{tr}(MN) = \operatorname{tr}(NM)$$

In the previous example

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad , \quad N = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and

$$MN = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \neq NM = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

but

$$\operatorname{tr} MN = 2+1 = 3 = 1+2 = \operatorname{tr} NM$$

Another useful property of the trace is $\operatorname{tr} M = \operatorname{tr} M^T$ because the trace only sees the diagonal elements.

<u>Ex</u> $\operatorname{tr}\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = 4 = \operatorname{tr}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = \operatorname{tr}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}^T$

<u>Finally</u> trace is a <u>linear</u> mapping

$$\operatorname{tr} : M^n_n \longrightarrow \mathbb{R}$$

This is easy to check.

Linear systems again

We can view a Linear system as a matrix equation

$$\underset{\substack{r \times k \\ \text{matrix} \\ \text{of coefficients}}}{M} \; \underset{\substack{k \times 1 \\ \text{matrix} \\ \text{of unknowns}}}{X} = \underset{\substack{r \times 1 \\ \text{matrix} \\ \text{of constants}}}{V} \quad \text{—} \; (*)$$

If M is a square matrix there are the same number of equations (r) as unknowns (k) so we could hope to have a unique solution.

For square matrices we discussed functions of matrices. Perhaps the nicest function would be

$$f(M) = \frac{1}{M} \quad \text{where } \frac{1}{M} M = I$$

because we could solve $(*)$ immediately by multiplying both sides by $\frac{1}{M}$ and find

$$X = \frac{1}{M} V$$

<u>Sometimes</u> $\frac{1}{M}$ can be found, it is called the <u>inverse</u> of M, most often denoted M^{-1}.

Lecture 9 Review Questions

1. Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{pmatrix}$$

find

(i) AA^T

(ii) A^TA

What can you say about the matrices AA^T and A^TA in general. Explain.

2. Compute $\exp(A)$ for

(i) $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

(ii) $A = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$

(iii) $A = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$

3. Suppose $ad - bc \neq 0$ and Let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

<u>Find</u> a matrix M^{-1} such that

$$M^{-1}M = I$$

Compute MM^{-1} and comment. Explain why your result explains what you found in a previous homework exercise.

(i) Suppose $U = \mathbb{R}$ (real numbers).
Explain why $=$ is an equivalence relation but $\geq$ is not.

(ii) Explain why equivalence of augmented matrices is an equivalence relation.