11. LU Decomposition

Certain matrices are easier to work with than others. In this section, we will see how to write any square matrix $M$ as the product of two matrices that are easier to work with. We’ll write $M = LU$, where:

- $L$ is lower triangular. This means that all entries above the main diagonal are zero. In notation, $L = (l_{ij})$ with $l_{ij} = 0$ for all $j > i$.

$$L = \begin{pmatrix}
  l_{11} & 0 & 0 & \ldots \\
  l_{12} & l_{22} & 0 & \ldots \\
  l_{13} & l_{23} & l_{33} & \ldots \\
  \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

- $U$ is upper triangular. This means that all entries below the main diagonal are zero. In notation, $U = (u_{ij})$ with $u_{ij} = 0$ for all $j < i$.

$$U = \begin{pmatrix}
  u_{11} & u_{12} & u_{13} & \ldots \\
  0 & u_{22} & u_{23} & \ldots \\
  0 & 0 & u_{33} & \ldots \\
  \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

$M = LU$ is called an LU decomposition of $M$.

This is a useful trick for many computational reasons. It is much easier to compute the inverse of an upper or lower triangular matrix. Since inverses are useful for solving linear systems, this makes solving any linear system associated to the matrix much faster as well. We haven’t talked about determinants yet, but suffice it to say that they are important and very easy to compute for triangular matrices.

**Example** Linear systems associated to triangular matrices are very easy to solve by back substitution.

$$\begin{pmatrix}
  a & b & 1 \\
  0 & c & e
\end{pmatrix} \Rightarrow y = \frac{e}{c}, x = \frac{1}{a}(1 - \frac{be}{c})$$

$$\begin{pmatrix}
  1 & 0 & 0 & d \\
  a & 1 & 0 & e \\
  b & c & 1 & f
\end{pmatrix} \Rightarrow x = d, \quad y = e - ad, \quad z = f - bd - c(e - ad)$$

For lower triangular matrices, back substitution gives a quick solution; for upper triangular matrices, forward substitution gives the solution.
Using \textit{LU} Decomposition to Solve Linear Systems

Suppose we have \( M = LU \) and want to solve the system

\[
MX = LUX = V.
\]

- Step 1: Set \( W = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = UX \).

- Step 2: Solve the system \( LW = V \). This should be simple by forward substitution since \( L \) is lower triangular. Suppose the solution to \( LW = V \) is \( W_0 \).

- Step 3: Now solve the system \( UX = W_0 \). This should be easy by backward substitution, since \( U \) is upper triangular. The solution to this system is the solution to the original system.

We can think of this as using the matrix \( L \) to perform row operations on the matrix \( U \) in order to solve the system; this idea will come up again when we study determinants.

Example Consider the linear system:

\[
\begin{align*}
6x + 18y + 3z &= 3 \\
2x + 12y + &= 19 \\
4x + 15y + 3z &= 0
\end{align*}
\]

An \textit{LU} decomposition for the associated matrix \( M \) is:

\[
\begin{pmatrix}
6 & 18 & 3 \\
2 & 12 & 1 \\
4 & 15 & 3 \\
\end{pmatrix} =
\begin{pmatrix}
3 & 0 & 0 \\
1 & 6 & 0 \\
2 & 3 & 1 \\
\end{pmatrix}
\begin{pmatrix}
2 & 6 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}.
\]

- Step 1: Set \( W = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = UX \).

- Step 2: Solve the system \( LW = V \):

\[
\begin{pmatrix}
2 & 0 & 0 \\
1 & 6 & 0 \\
2 & 3 & 1 \\
\end{pmatrix}
\begin{pmatrix}
u \\ v \\ w \end{pmatrix} =
\begin{pmatrix}
3 \\
19 \\
0 \\
\end{pmatrix}
\]
By substitution, we get $u = 1$, $v = 3$, and $w = -11$. Then
\[
W_0 = \begin{pmatrix} 1 \\ 3 \\ -11 \end{pmatrix}
\]

- Step 3: Solve the system $UX = W_0$.
\[
\begin{pmatrix} 2 & 6 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -11 \end{pmatrix}
\]

Back substitution gives $z = -11$, $y = 3$, and $x = -6$.

Then $X = \begin{pmatrix} -6 \\ 3 \\ -11 \end{pmatrix}$, and we’re done.

Finding an LU Decomposition.

For any given matrix, there are actually many different $LU$ decompositions. However, there is a unique $LU$ decomposition in which the $L$ matrix has ones on the diagonal; then $L$ is called a lower unit triangular matrix.

To find the $LU$ decomposition, we’ll create two sequences of matrices $L_0, L_1, \ldots$ and $U_0, U_1, \ldots$ such that at each step, $L_i U_i = M$. Each of the $L_i$ will be lower triangular, but only the last $U_i$ will be upper triangular.

Start by setting $L_0 = I$ and $U_0 = M$, because $L_0 U_0 = M$.

Next, use the first row of $U_0$ to zero out the first entry of every row below it. For our running example, $U_0 = M = \begin{pmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{pmatrix}$, so the second row minus $\frac{1}{3}$ of the first row will zero out the first entry in the second row. Likewise, the third row minus $\frac{2}{3}$ of the first row will zero out the first entry in the third row.

Set $L_1$ to be the lower triangular matrix whose first column is filled with the constants used to zero out the first column of $M$. Then $L_1 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & 0 & 1 \end{pmatrix}$. Set $U_1$ to be the matrix obtained by zeroing out the first column of $M$. Then $U_1 = \begin{pmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 3 & 1 \end{pmatrix}$. 

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Now repeat the process by zeroing the second column of $U_1$ below the diagonal using the second row of $U_1$, and putting the corresponding entries into $L_1$. The resulting matrices are $L_2$ and $U_2$. For our example, $L_2 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & \frac{1}{2} & 1 \end{pmatrix}$, and $U_2 = \begin{pmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Since $U_2$ is upper-triangular, we’re done. Inserting the new number into $L_1$ to get $L_2$ really is safe: the numbers in the first column don’t affect the second column of $U_1$, since the first column of $U_1$ is already zeroed out.

If the matrix you’re working with has more than three rows, just continue this process by zeroing out the next column below the diagonal, and repeat until there’s nothing left to do.

The fractions in the $L$ matrix are admittedly ugly. For two matrices $LU$, we can multiply one entire column of $L$ by a constant $\lambda$ and divide the corresponding row of $U$ by the same constant without changing the product of the two matrices. Then:

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & \frac{1}{2} & 1 \end{pmatrix} I \begin{pmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 6 & 0 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 6 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The resulting matrix looks nicer, but isn’t in standard form.

For matrices that are not square, $LU$ decomposition still makes sense. Given an $m \times n$ matrix $M$, for example we could write $M = LU$ with $L$ a square lower unit triangular matrix, and $U$ a rectangular matrix. Then $L$ will be an $m \times m$ matrix, and $U$ will be an $m \times n$ matrix (of the same shape as $M$). From here, the process is exactly the same as for a square matrix. We create a sequence of matrices $L_i$ and $U_i$ that is eventually the $LU$ decomposition. Again, we start with $L_0 = I$ and $U_0 = M$.

**Example** Let’s find the $LU$ decomposition of $M = U_0 = \begin{pmatrix} -2 & 1 & 3 \\ -4 & 4 & 1 \end{pmatrix}$.
Since $M$ is a $2 \times 3$ matrix, our decomposition will consist of a $2 \times 2$ matrix and a $2 \times 3$ matrix. Then we start with $L_0 = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The next step is to zero-out the first column of $M$ below the diagonal. There is only one row to cancel, then, and it can be removed by subtracting $2$ times the first row of $M$ to the second row of $M$. Then:

$L_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad U_1 = \begin{pmatrix} -2 & 1 & 3 \\ 0 & 6 & -5 \end{pmatrix}$

Since $U_1$ is upper triangular, we’re done. With a larger matrix, we would just continue the process.

**Block LU Decomposition**

Let $M$ be a square block matrix with square blocks $X,Y,Z,W$ such that $X^{-1}$ exists. Then $M$ can be decomposed with an $LDU$ decomposition, where $D$ is block diagonal, as follows:

$$M = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$$

Then:

$$M = \begin{pmatrix} I & 0 \\ ZX^{-1} & I \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & W - ZX^{-1}Y \end{pmatrix} \begin{pmatrix} I & X^{-1}Y \end{pmatrix}.$$

This can be checked simply by multiplying the product on the right.

**Example** For a $2 \times 2$ matrix, we can regard each entry as a block.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

By multiplying the diagonal matrix by the upper triangular matrix, we get the standard $LU$ decomposition of the matrix.

**References**

Wikipedia:

- [LU Decomposition](#)
- [Block LU Decomposition](#)
Review Questions

1. Consider the linear system:

\[
\begin{align*}
    x^1 &= v^1 \\
    l_1^2 x^1 + l_2^2 x^2 &= v^2 \\
    & \vdots \\
    l_n^1 x^1 + l_2^n x^2 + \ldots + x^n &= v^n
\end{align*}
\]

i. Find \( x^1 \).
ii. Find \( x^2 \).
iii. Find \( x^3 \).
iv. Try to find a formula for \( x^k \).

2. Let \( M = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \) be a square \( n \times n \) block matrix with \( W \) invertible.

i. If \( W \) is invertible, what size are \( X, Y, \) and \( Z? \)
ii. Find a \( UDL \) decomposition for \( M \). In other words, fill in the stars in the following equation:

\[
\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \begin{pmatrix} I & * \\ 0 & I \end{pmatrix} \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \begin{pmatrix} I & 0 \\ * & I \end{pmatrix}
\]