11. LU Decomposition

Certain matrices are easier to work with than others. In this section, we will see how to write any square matrix M as the product of two matrices that are easier to work with. We'll write $M = LU$, where:

- L is *lower triangular*. This means that all entries above the main diagonal are zero. In notation, $L = (l^i_j)$ with $l^i_j = 0$ for all $j > i$.

$$L = \begin{pmatrix} l^1_1 & 0 & 0 & \cdots \\ l^2_1 & l^2_2 & 0 & \cdots \\ l^3_1 & l^3_2 & l^3_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- U is *upper triangular*. This means that all entries below the main diagonal are zero. In notation, $U = (u^i_j)$ with $u^i_j = 0$ for all $j < i$.

$$U = \begin{pmatrix} u^1_1 & u^1_2 & u^1_3 & \cdots \\ 0 & u^2_2 & u^2_3 & \cdots \\ 0 & 0 & u^3_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$M = LU$ is called an *LU decomposition* of M.

This is a useful trick for many computational reasons. It is much easier to compute the inverse of an upper or lower triangular matrix. Since inverses are useful for solving linear systems, this makes solving any linear system associated to the matrix much faster as well. We haven't talked about determinants yet, but suffice it to say that they are important and very easy to compute for triangular matrices.

Example Linear systems associated to triangular matrices are very easy to solve by back substitution.

$$\left(\begin{array}{cc|c} a & b & 1 \\ 0 & c & e \end{array}\right) \Rightarrow y = \frac{e}{c}, x = \frac{1}{a}(1 - \frac{be}{c})$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & d \\ a & 1 & 0 & e \\ b & c & 1 & f \end{array}\right) \Rightarrow x = d. \qquad y = e - ad, \qquad z = f - bd - c(e - ad)$$

For lower triangular matrices, *back* substitution gives a quick solution; for upper triangular matrices, *forward* substitution gives the solution.

Using LU Decomposition to Solve Linear Systems

Suppose we have $M = LU$ and want to solve the system

$$MX = LUX = V.$$

- Step 1: Set $W = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = UX$.
- Step 2: Solve the system $LW = V$. This should be simple by forward substitution since L is lower triangular. Suppose the solution to $LW = V$ is W_0.
- Step 3: Now solve the system $UX = W_0$. This should be easy by backward substitution, since U is upper triangular. The solution to this system is the solution to the original system.

We can think of this as using the matrix L to perform row operations on the matrix U in order to solve the system; this idea will come up again when we study determinants.

Example Consider the linear system:

$$\begin{array}{rrrl} 6x & +18y & +3z = & 3 \\ 2x & +12y & + & = 19 \\ 4x & +15y & +3z = & 0 \end{array}$$

An LU decomposition for the associated matrix M is:

$$\begin{pmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 6 & 0 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 6 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- Step 1: Set $W = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = UX$.
- Step 2: Solve the system $LW = V$:

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 6 & 0 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 3 \\ 19 \\ 0 \end{pmatrix}$$

By substitution, we get $u = 1$, $v = 3$, and $w = -11$. Then

$$W_0 = \begin{pmatrix} 1 \\ 3 \\ -11 \end{pmatrix}$$

- Step 3: Solve the system $UX = W_0$.

$$\begin{pmatrix} 2 & 6 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -11 \end{pmatrix}$$

Back substitution gives $z = -11, y = 3$, and $x = -6$.

Then $X = \begin{pmatrix} -6 \\ 3 \\ -11 \end{pmatrix}$, and we're done.

Finding an LU Decomposition.

For any given matrix, there are actually many different LU decompositions. However, there is a unique LU decomposition in which the L matrix has ones on the diagonal; then L is called a *lower unit triangular matrix*.

To find the LU decomposition, we'll create two sequences of matrices $L_0, L_1, \dots$ and $U_0, U_1, \dots$ such that at each step, $L_i U_i = M$. Each of the L_i will be lower triangular, but only the last U_i will be upper triangular.

Start by setting $L_0 = I$ and $U_0 = M$, because $L_0 U_0 = M$.

Next, use the first row of U_0 to zero out the first entry of every row below it. For our running example, $U_0 = M = \begin{pmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{pmatrix}$, so the second row minus $\frac{1}{3}$ of the first row will zero out the first entry in the second row. Likewise, the third row minus $\frac{2}{3}$ of the first row will zero out the first entry in the third row.

Set L_1 to be the lower triangular matrix whose first column is filled with the constants used to zero out the first column of M. Then $L_1 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & 0 & 1 \end{pmatrix}$. Set U_1 to be the matrix obtained by zeroing out the first column of M. Then $U_1 = \begin{pmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 3 & 1 \end{pmatrix}$.

Now repeat the process by zeroing the second column of U_1 below the diagonal using the second row of U_1, and putting the corresponding entries into L_1. The resulting matrices are L_2 and U_2. For our example, $L_2 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & \frac{1}{2} & 1 \end{pmatrix}$, and $U_2 = \begin{pmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Since U_2 is upper-triangular, we're done. Inserting the new number into L_1 to get L_2 really is safe: the numbers in the first column don't affect the second column of U_1, since the first column of U_1 is already zeroed out.

If the matrix you're working with has more than three rows, just continue this process by zeroing out the next column below the diagonal, and repeat until there's nothing left to do.

The fractions in the L matrix are admittedly ugly. For two matrices LU, we can multiply one entire column of L by a constant λ and divide the corresponding row of U by the same constant without changing the product of the two matrices. Then:

$$\begin{aligned} LU &= \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & \frac{1}{2} & 1 \end{pmatrix} I \begin{pmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 & 0 \\ 1 & 6 & 0 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 6 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

The resulting matrix looks nicer, but isn't in standard form.

For matrices that are not square, LU decomposition still makes sense. Given an $m \times n$ matrix M, for example we could write $M = LU$ with L a square lower unit triangular matrix, and U a rectangular matrix. Then L will be an $m \times m$ matrix, and U will be an $m \times n$ matrix (of the same shape as M). From here, the process is exactly the same as for a square matrix. We create a sequence of matrices L_i and U_i that is eventually the LU decomposition. Again, we start with $L_0 = I$ and $U_0 = M$.

Example Let's find the LU decomposition of $M = U_0 = \begin{pmatrix} -2 & 1 & 3 \\ -4 & 4 & 1 \end{pmatrix}$.

Since M is a 2×3 matrix, our decomposition will consist of a 2×2 matrix and a 2×3 matrix. Then we start with $L_0 = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The next step is to zero-out the first column of M below the diagonal. There is only one row to cancel, then, and it can be removed by subtracting 2 times the first row of M to the second row of M. Then:

$$L_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \qquad U_1 = \begin{pmatrix} -2 & 1 & 3 \\ 0 & 6 & -5 \end{pmatrix}$$

Since U_1 is upper triangular, we're done. With a larger matrix, we would just continue the process.

Block LU Decomposition

Let M be a square block matrix with square blocks X, Y, Z, W such that X^{-1} exists. Then M can be decomposed with an LDU decomposition, where D is block diagonal, as follows:

$$M = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$$

Then:

$$M = \begin{pmatrix} I & 0 \\ ZX^{-1} & I \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & W - ZX^{-1}Y \end{pmatrix} \begin{pmatrix} I & X^{-1}Y \\ 0 & I \end{pmatrix}.$$

This can be checked simply by multiplying the product on the right.

Example For a 2×2 matrix, we can regard each entry as a block.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

By multiplying the diagonal matrix by the upper triangular matrix, we get the standard LU decomposition of the matrix.

References

Wikipedia:

- LU Decomposition
- Block LU Decomposition

Review Questions

1. Consider the linear system:

$$\begin{array}{llll} x^1 & & & = v^1 \\ l^2_1 x^1 & +l^2_2 x^2 & & = v^2 \\ & \vdots & & \vdots \\ l^n_1 x^1 & +l^n_2 x^2 & \ldots + x^n & = v^n \end{array}$$

   i. Find x^1.

   ii. Find x^2.

   iii. Find x^3.

   k. Try to find a formula for x^k.

2. Let $M = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ be a square $n \times n$ block matrix with W invertible.

   i. If W is invertible, what size are X, Y, and Z?

   ii. Find a UDL decomposition for M. In other words, fill in the stars in the following equation:

$$\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \begin{pmatrix} I & * \\ 0 & I \end{pmatrix} \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \begin{pmatrix} I & 0 \\ * & I \end{pmatrix}$$