13. Elementary Matrices and Determinants II

In the last Section, we saw the definition of the determinant and derived an elementary matrix that exchanges two rows of a matrix. Next, we need to find elementary matrices corresponding to the other two row operations, multiplying a row by a scalar, and adding a multiple of one row to another. As a consequence, we will derive some important properties of the determinant.

Consider $M = \begin{pmatrix} R^1 \\ \vdots \\ R^n \end{pmatrix}$, where R^i are row vectors. Let $R^i(\lambda)$ be the identity matrix, with the ith diagonal entry replaced by λ, not to be confused with the row vectors. Then:

$$M' = \begin{pmatrix} R^1 \\ \vdots \\ \lambda R^i \\ \vdots \\ R^n \end{pmatrix} = R^i(\lambda)M$$

What effect does multiplication by $R^i(\lambda)$ have on the determinant?

$$\begin{aligned} \det M' &= \sum_\sigma \operatorname{sgn}(\sigma) m^1_{\sigma(1)} \dots \lambda m^i_{\sigma(i)} \dots m^n_{\sigma(n)} \\ &= \lambda \sum_\sigma \operatorname{sgn}(\sigma) m^1_{\sigma(1)} \dots m^i_{\sigma(i)} \dots m^n_{\sigma(n)} \\ &= \lambda \det M \end{aligned}$$

Thus, multiplying a row by λ multiplies the determinant by λ.

Since $R^i(\lambda)$ is just the identity matrix with a single row multiplied by λ, then by the above rule, the determinant of $R^i(\lambda)$ is λ. Thus:

$$\det R^i(\lambda) = \det \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} = \lambda$$

The final row operation is adding λR^j to R^i. This is done with the matrix $S^i_j(\lambda)$, which is an identity matrix but with a λ in the i, j position.

$$S^i_j(\lambda) = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & \lambda & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

Then multiplying $S^i_j(\lambda)$ by M gives the following:

$$\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & \lambda & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \begin{pmatrix} \vdots \\ R^i \\ \vdots \\ R^j \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ R^i + \lambda R^j \\ \vdots \\ R^j \\ \vdots \end{pmatrix}$$

What is the effect of multiplying by $S^i_j(\lambda)$ on the determinant? Let $M' = S^i_j(\lambda)M$, and let M'' be the matrix M but with R^i replaced by R^j.

$$\begin{aligned} \det M' &= \sum_\sigma \operatorname{sgn}(\sigma) m^1_{\sigma(1)} \dots (m^i_{\sigma(i)} + \lambda m^j_{\sigma(j)}) \dots m^n_{\sigma(n)} \\ &= \sum_\sigma \operatorname{sgn}(\sigma) m^1_{\sigma(1)} \dots m^i_{\sigma(i)} \dots m^n_{\sigma(n)} \\ &\qquad + \sum_\sigma \operatorname{sgn}(\sigma) m^1_{\sigma(1)} \dots \lambda m^j_{\sigma(j)} \dots m^j_{\sigma(j)} \dots m^n_{\sigma(n)} \\ &= \det M + \lambda \det M'' \end{aligned}$$

Since M'' has two identical rows, its determinant is 0. Then $\det S^i_j(\lambda)M = \det M$.

Notice that if M is the identity matrix, then we have $\det S^i_j(\lambda) = \det(S^i_j(\lambda)I) = \det I = 1$.

We now have an elementary matrices associated to each of the row operations.

$$
\begin{aligned}
E^i_j &= I \text{ with rows } i,j \text{ swapped}; \quad \det E^i_j = -1 \\
R^i(\lambda) &= I \text{ with } \lambda \text{ in position } i,i; \quad \det R^i_j(\lambda) = \lambda \\
S^i_j(\lambda) &= I \text{ with } \lambda \text{ in position } i,j; \quad \det S^i_j(\lambda) = 1
\end{aligned}
$$

We have also proved the following theorem along the way:

Theorem. *If E is* any *of the elementary matrices $E^i_j, R^i(\lambda), S^i_j(\lambda)$, then* $\det(EM) = \det E \det M$*.*

We have seen that any matrix M can be put into reduced row echelon form via a sequence of row operations, and we have seen that any row operation can be emulated with left matrix multiplication by an elementary matrix. Suppose that $\text{RREF}(M)$ is the reduced row echelon form of M. Then $\text{RREF}(M) = E_1 E_2 \ldots E_k M$ where each E_i is an elementary matrix.

What is the determinant of a square matrix in reduced row echelon form?

- If M is not invertible, then some row of $\text{RREF}(M)$ contains only zeros. Then we can multiply the zero row by any constant λ without changing M; by our previous observation, this scales the determinant of M by λ. Thus, if M is not invertible, $\det \text{RREF}(M) = \lambda \det \text{RREF}(M)$, and so $\det \text{RREF}(M) = 0$.

- Otherwise, every row of $\text{RREF}(M)$ has a pivot on the diagonal; since M is square, this means that $\text{RREF}(M)$ is the identity matrix. Then if M is invertible, $\det \text{RREF}(M) = 1$.

- Additionally, notice that $\det \text{RREF}(M) = \det(E_1 E_2 \ldots E_k M)$. Then by the theorem above, $\det \text{RREF}(M) = \det(E_1) \ldots \det(E_k) \det M$. Since each E_i has non-zero determinant, then $\det \text{RREF}(M) = 0$ if and only if $\det M = 0$.

Then we have shown:

Theorem. *For any square matrix M,* $\det M \neq 0$ *if and only if M is invertible.*

Since we know the determinants of the elementary matrices, we can immediately obtain the following:

Corollary. *Any elementary matrix $E^i_j, R^i(\lambda), S^i_j(\lambda)$ is invertible, except for $R^i(0)$. In fact, the inverse of an elementary matrix is another elementary matrix.*

To obtain one last important result, suppose that M and N are square $n \times n$ matrices, with reduced row echelon forms such that, for E_i and F_i elementary matrices, $M = E_1 E_2 \dots E_k \,\mathrm{RREF}(M)$, $N = F_1 F_2 \dots F_k \,\mathrm{RREF}(N) = N$. If $\mathrm{RREF}(M)$ is the identity matrix (ie, M is invertible), then:

$$\begin{aligned}
\det(MN) &= \det(E_1 E_2 \dots E_k \,\mathrm{RREF}(M) F_1 F_2 \dots F_k \,\mathrm{RREF}(N)) \\
&= \det(E_1 E_2 \dots E_k I F_1 F_2 \dots F_k \,\mathrm{RREF}(N)) \\
&= \det(E_1) \dots \det(E_k) \det(I) \det(F_1) \dots \det(F_k) \det(\mathrm{RREF}(N) \\
&= \det(M) \det(N)
\end{aligned}$$

Otherwise, M is not invertible, and $\det M = 0 = \det ref(M)$. Then there exists a row of zeros in $ref(M)$, so $R^n(\lambda) ref(M) = ref(M)$. Then:

$$\begin{aligned}
\det(MN) &= \det(E_1 E_2 \dots E_k ref(M) N) \\
&= \det(E_1 E_2 \dots E_k ref(M) N) \\
&= \det(E_1) \dots \det(E_k) \det(ref(M) N) \\
&= \det(E_1) \dots \det(E_k) \det(R^n(\lambda) ref(M) N) \\
&= \det(E_1) \dots \det(E_k) \lambda \det(ref(M) N) \\
&= \lambda \det(MN)
\end{aligned}$$

Which implies that $\det(MN) = 0 = \det M \det N$.

Thus we have shown that for any matrices M and N,

$$\det(MN) = \det M \det N$$

References

Hefferon, Chapter Four, Section I.1 and I.3

Wikipedia:

- Determinant
- Elementary Matrix

Review Questions

1. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $N = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Compute the following:

- $\det M$.
- $\det N$.
- $\det(MN)$.
- $\det M \det N$.
- $\det(M^{-1})$ assuming $ab - cd \neq 0$.
- $\det(M^T)$
- $\det(M + N) - \det M - \det N$

2. Suppose $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible. Write M as a product of elementary row matrices times $\mathrm{RREF}(M)$.

3. Find the inverses of each of the elementary matrices, $E^i_j, R^i(\lambda), S^i_j(\lambda)$. Make sure to show that the elementary matrix times its inverse is actually the identity.