## 14. Properties of the Determinant

Last time we showed that the determinant of a matrix is non-zero if and only if that matrix is invertible. We also showed that the determinant is a *multiplicative* function, in the sense that $\det(MN) = \det M \det N$. Now we will devise some methods for calculating the determinant.

Recall that:

$$\det M = \sum_{\sigma} \operatorname{sgn}(\sigma) m^1_{\sigma(1)} m^2_{\sigma(2)} \dots m^n_{\sigma(n)}.$$

A *minor* of an $n \times n$ matrix $M$ is any square matrix obtained from $M$ by deleting rows and columns. In particular, any entry $m^i_j$ of $M$ is associated to a minor obtained by deleting the $i$th row and $j$th column of $M$.

It is possible to write the determinant of a matrix in terms of the determinants of its minors as follows:

$$\begin{aligned}
\det M &= \sum_{\sigma} \operatorname{sgn}(\sigma) m^1_{\sigma(1)} m^2_{\sigma(2)} \dots m^n_{\sigma(n)} \\
&= m^1_1 \sum_{\hat\sigma} \operatorname{sgn}(\hat\sigma) m^2_{\hat\sigma(2)} \dots m^n_{\hat\sigma(n)} \\
&\quad - m^1_2 \sum_{\hat\sigma} \operatorname{sgn}(\hat\sigma) m^1_{\hat\sigma(1)} m^3_{\hat\sigma(3)} \dots m^n_{\hat\sigma(n)} \\
&\quad + m^1_3 \sum_{\hat\sigma} \operatorname{sgn}(\hat\sigma) m^1_{\hat\sigma(1)} \dots m^n_{\hat\sigma(n)} \pm \dots
\end{aligned}$$

Here the symbols $\hat\sigma$ refer to permutations of $n-1$ objects. What we're doing here is collecting up all of the terms of the original sum that contain the term $m^1_j$ for some fixed $j$. Each term in that collection is associated to a permutation sending $1 \to j$. The rest of any such permutation maps the set $\{2, \dots, n\} \to \{1, \dots, \hat{j}, \dots, n\}$. We call this partial permutation $\hat\sigma = \begin{bmatrix} \sigma(2) & \dots & \sigma(n) \end{bmatrix}$.

The last issue is that the permutation $\hat\sigma$ may not have the same sign as $\sigma$. From previous homework, we know that a permutation has the same parity as its inversion number. Removing $1 \to j$ from a permutation the reduces the inversion number by the number of elements right of $j$ that are less than $j$. Since $j$ comes first in the permutation $\begin{bmatrix} j & \sigma(2) & \dots & \sigma(n) \end{bmatrix}$, the inversion number of $\hat\sigma$ is reduced by $j-1$. Then the sign of $\sigma$ differs from the sign of $\hat\sigma$ if $\sigma$ sends 1 to an even number.

Graphically, to expand by minors we pick an entry $m^1_j$ of the first row, then add $(-1)^{j-1}$ times the determinant of the matrix with row $i$ and column $j$ deleted.

**Example** Let's compute the determinant of $M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ using expansion by minors.

$$\begin{aligned}
\det M &= 1 \det \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} - 2 \det \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} + 3 \det \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix} \\
&= 1(5 \cdot 9 - 8 \cdot 6) - 2(4 \cdot 9 - 7 \cdot 6) + 3(4 \cdot 8 - 7 \cdot 5) \\
&= 0
\end{aligned}$$

Then $M^{-1}$ does not exist.

**Example** Sometimes the entries of a matrix allows us to simplify the calculation of the determinant. Take $N = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix}$. Then we can switch the first and second rows of $N$ to get:

$$\begin{aligned}
\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix} &= -\det \begin{pmatrix} 4 & 0 & 0 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix} \\
&= 4 \det \begin{pmatrix} 2 & 3 \\ 8 & 9 \end{pmatrix} \\
&= -16
\end{aligned}$$

**Theorem.** *For any square matrix $M$, we have:*

$$\det M^T = \det M$$

*Proof.* By definition,

$$\det M = \sum_\sigma \operatorname{sgn}(\sigma) m^1_{\sigma(1)} m^2_{\sigma(2)} \dots m^n_{\sigma(n)}.$$

For any permutation $\sigma$, there is a unique inverse permutation $\sigma^{-1}$ that undoes $\sigma$. If $\sigma$ sends $i \to j$, then $\sigma^{-1}$ sends $j \to i$. In the two-line notation for a permutation, this corresponds to just flipping the permutation over. For example, if $\sigma = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$, then we can find $\sigma^{-1}$ by flipping the permutation and then putting the columns in order:

$$\sigma^{-1} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Since any permutation can be built up by transpositions, one can also find the inverse of a permutation $\sigma$ by undoing each of the transpositions used to build up $\sigma$; this shows that one can use the same number of transpositions to build $\sigma$ and $\sigma^{-1}$. In particular, $\operatorname{sgn}\sigma = \operatorname{sgn}\sigma^{-1}$.

Then:

$$\begin{aligned} \det M &= \sum_{\sigma} \operatorname{sgn}(\sigma) m^1_{\sigma(1)} m^2_{\sigma(2)} \dots m^n_{\sigma(n)} \\ &= \sum_{\sigma} \operatorname{sgn}(\sigma) m_1^{\sigma^{-1}(1)} m_2^{\sigma^{-1}(2)} \dots m_n^{\sigma^{-1}(n)} \\ &= \sum_{\sigma} \operatorname{sgn}(\sigma^{-1}) m_1^{\sigma^{-1}(1)} m_2^{\sigma^{-1}(2)} \dots m_n^{\sigma^{-1}(n)} \\ &= \sum_{\sigma} \operatorname{sgn}(\sigma) m_1^{\sigma(1)} m_2^{\sigma(2)} \dots m_n^{\sigma(n)} \\ &= \det M^T. \end{aligned}$$

The last equality is due to the existence of a unique inverse permutation: summing over permutations is the same as summing over all inverses of permutations. $\square$

**Example** Because of this theorem, we can see that expansion by minors also works over columns. Let $M = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 8 & 9 \end{pmatrix}$. Then $\det M = \det M^T = 1 \det \begin{pmatrix} 5 & 8 \\ 6 & 9 \end{pmatrix} = -3$.

## Determinant of the Inverse

Let $M$ and $N$ be $n \times n$ matrices.

We previously showed that

$$\det(MN) = \det M \det N, \text{ and } \det I = 1.$$

Then $1 = \det I = \det(MM^{-1}) = \det M \det M^{-1}$. As such we have:

**Theorem.**

$$\det M^{-1} = \frac{1}{\det M}$$

## Adjoint of a Matrix

A *cofactor* of $M$ is obtained choosing any entry $m^i_j$ of $M$ and then deleting the $i$th row and $j$th column of $M$, taking the determinant of the resulting matrix, and multiplying by$(-1)^{i+j}$. This is written cofactor$(m^i_j)$.

**Definition** For $M = (m^i_j)$ a square matrix, The *adjoint matrix* $\operatorname{adj} M$ is given by:

$$\operatorname{adj} M = (\text{cofactor}(m^i_j))^T$$

**Example**

$$\operatorname{adj}\begin{pmatrix} 3 & -1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \det\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} & -\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \det\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ -\det\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} & \det\begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix} & -\det\begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix} \\ \det\begin{pmatrix} -1 & -1 \\ 2 & 0 \end{pmatrix} & -\det\begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} & \det\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} \end{pmatrix}^T$$

Let's multiply $M \operatorname{adj} M$. For any matrix $N$, the $i, j$ entry of $MN$ is given by taking the dot product of the $i$th row of $M$ and the $j$th column of $N$.

Notice that the dot product of the $i$th row of $M$ and the $i$th column of $\operatorname{adj} M$ is just the expansion by minors of $\det M$ in the $i$th row.

Further, notice that the dot product of the $i$th row of $M$ and the $j$th column of $\operatorname{adj} M$ with $j \neq i$ is the same as expanding $M$ by minors, but with the $j$th row replaced by the $i$th row. Since the determinant of any matrix with a row repeated is zero, then these dot products are zero as well.

Then:

$$M \operatorname{adj} M = (\det M)I$$

Thus, when $\det M \neq 0$, the adjoint gives an explicit formula for $M^{-1}$.

**Theorem.**

$$M^{-1} = \frac{1}{\det M} \operatorname{adj} M$$

**Example** Continuing with the previous example,

$$\operatorname{adj} \begin{pmatrix} 3 & -1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ -1 & 3 & -1 \\ 1 & -3 & 7 \end{pmatrix}.$$

Now, multiply:

$$\begin{aligned} \begin{pmatrix} 3 & -1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 \\ -1 & 3 & -1 \\ 1 & -3 & 7 \end{pmatrix} &= \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} \\ \Rightarrow \begin{pmatrix} 3 & -1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1} &= \frac{1}{6} \begin{pmatrix} 2 & 0 & 2 \\ -1 & 3 & -1 \\ 1 & -3 & 7 \end{pmatrix} \end{aligned}$$

This process for finding the inverse matrix is sometimes called *Cramer's Rule*.

## Application: Volume of a Parallelpiped

Given three vectors $u, v, w$ in $\mathbb{R}^3$, the parallelpiped determined by the three vectors is the box whose edges are parallel to $u, v$, and $w$.

From calculus, we know that the volume of this object is $|u \cdot (v \times w)|$. This is the same as expansion by minors of the matrix whose columns are $u, v, w$. Then:

$$\text{Volume} = |\det \begin{pmatrix} u & v & w \end{pmatrix} |$$

## References

Hefferon, Chapter Four, Section I.1 and I.3

Wikipedia:

- Determinant
- Elementary Matrix
- Cramer's Rule

## Review Questions

1. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Show:

$$\det M = \frac{1}{2}(\operatorname{tr} M)^2 - \frac{1}{2}\operatorname{tr}(M^2)$$

   Suppose $M$ is a $3 \times 3$ matrix. Find and Verify a similar formular for $\det M$ in terms of $\operatorname{tr}(M^3)$, $(\operatorname{tr} M)(\operatorname{tr}(M^2))$, and $(\operatorname{tr} M)^3$.

2. Suppose $M = LU$ is an $LU$ decomposition. Explain how you would efficiently compute $\det M$ in this case.

3. In computer science, the *complexity* of an algorithm is computed (roughly) by counting the number of times a given operation is performed. Suppose adding or subtracting any two numbers takes $a$ seconds, and multiplying two numbers takes $m$ seconds. Then, for example, computing $2 \cdot 6 - 5$ would take $a + m$ seconds.

   *i.* How many additions and multiplications does it take to compute the determinant of a general $2 \times 2$ matrix?

   *ii.* Write a formula for the number of additions and multiplications it takes to compute the determinant of a general $n \times n$ matrix using the definition of the determinant. Assume that finding (and multiplying by) the sign of a permutation is free.

   *iii.* How many additions and multiplications does it take to compute the determinant of a general $3 \times 3$ matrix using expansion by minors? Assuming $m = 2a$, is this faster than computing the determinant from the definition?