16. Eigenvalues, Eigenvectors II

Last time, we developed the idea of eigenvalues and eigenvectors in the case of linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$. In this Section, we will develop the idea more generally.

Definition For a linear transformation $L : V \to V$, then λ is an *eigenvalue* of L with eigenvector $v \neq 0_V$ if

$$Lv = \lambda v.$$

This equation says that the direction of v is invariant (unchanged) under L.

Let V be a finite-dimensional vector space (we'll explain what it means to be finite-dimensional in more detail later; for now, take this to mean $\mathbb{R}^n$), and $L : V \to V$.

Matrix of a Linear Transformation Any vector in $\mathbb{R}^n$ can be written as a linear combination of the *standard basis vectors* $\{e_i | i \in \{1, \ldots, n\}\}$. The vector e_i has a one in the ith position, and zeros everywhere else. Then to find the matrix of any linear transformation $L : \mathbb{R}^n \to \mathbb{R}^n$, it suffices to know what $L(e_i)$ is for every i.

For any matrix M, observe that Me_i is equal to the ith column of M. Then if the ith column of M equals $L(e_i)$ for every i, then $Mv = L(v)$ for every $v \in \mathbb{R}^n$. Then the matrix representing L in the standard basis is just the matrix whose ith column is $L(e_i)$.

Since we can represent L by a square matrix M, and find eigenvalues λ_i and associated eigenvectors v_i by solving the homogeneous system

$$(M - \lambda_i I)v_i = 0.$$

This system has non-zero solutions if and only if the matrix

$$M - \lambda_i I$$

is singular, and so we require that

$$\det(\lambda I - M) = 0.$$

The left hand side of this equation is a polynomial in the variable λ called the *characteristic polynomial* $P_M(\lambda)$ of M. For an $n \times n$ matrix, the characteristic polynomial has degree n. Then

$$P_M(\lambda) = \lambda^n + c_1\lambda^{n-1} + \ldots + c_n.$$

Notice that $P_M(0) = \det(-M) = (-1)^n \det M$.

The *fundamental theorem of algebra* states that any polynomial can be factored into a product of linear terms over $\mathbb{C}$. Then there exists a collection of n complex numbers λ_i (possibly with repetition) such that

$$P_M(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\dots(\lambda - \lambda_n), \qquad P_M(\lambda_i) = 0$$

The eigenvalues λ_i of M are exactly the roots of $P_M(\lambda)$. These eigenvalues could be real or complex or zero, and they need not all be different. The number of times that any given root λ_i appears in the collection of eigenvalues is called its *multiplicity*.

Example Let L be the linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by $L(x, y, z) = (2x + y - z, x + 2y - z, -x - y + 2z)$. The matrix M representing L has columns Le_i for each i, so:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{L}{\mapsto} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Then the characteristic polynomial of L is[1]

$$\begin{aligned} P_M(\lambda) &= \det \begin{pmatrix} \lambda - 2 & -1 & 1 \\ -1 & \lambda - 2 & 1 \\ 1 & 1 & \lambda - 2 \end{pmatrix} \\ &= (\lambda - 2)[(\lambda - 2)^2 - 1] + [-(\lambda - 2) - 1] + [-(\lambda - 2) - 1] \\ &= (\lambda - 1)^2(\lambda - 4) \end{aligned}$$

Then L has eigenvalues $\lambda_1 = 1$ (with multiplicity 2), and $\lambda_2 = 4$ (with multiplicity 1).

To find the eigenvectors associated to each eigenvalue, we solve the homogeneous system $(M - \lambda_i I)X = 0$ for each i.

$\lambda = 4$: We set up the augmented matrix for the linear system:

$$\begin{aligned} \left(\begin{array}{ccc|c} -2 & 1 & -1 & 0 \\ 1 & -2 & -1 & 0 \\ -1 & -1 & -2 & 0 \end{array}\right) &\sim \left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right) \\ &\sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right). \end{aligned}$$

[1] It is often easier (and equivalent) to solve $\det(M - \lambda I) = 0$.

So we see that $z = t$, $y = -t$, and $z = -t$ gives a formula for eigenvectors in terms of the free parameter t. Any such eigenvector is of the form $t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$; thus L leaves a line through the origin invariant.

$\lambda = 1$: Again we set up an augmented matrix and find the solution set:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Then the solution set has two free parameters, s and t, such that $z = t$, $y = s$, and $x = -s + t$. Then L leaves invariant the set:

$$\{s \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} | s, t \in \mathbb{R}\}.$$

This set is a plane through the origin. So the multiplicity two eigenvalue has two independent eigenvectors, $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ that determine an invariant plane.

Eigenspaces

In the previous example, we found two eigenvectors $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ for L with eigenvalue 1. Notice that $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is also an eigenvector of L with eigenvalue 1. In fact, any linear combination $r \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ of these two eigenvectors will be another eigenvector with the same eigenvalue.

More generally, let $\{v_1, v_2, \ldots\}$ be eigenvectors of some linear transformation L with the same eigenvalue λ. Then any linear combination of the

v_i can be written $c_1v_1 + c_2v_2 + \ldots$ for some constants $\{c_1, c_2, \ldots\}$. Then:

$$\begin{aligned} L(c_1v_1 + c_2v_2 + \ldots) &= c_1Lv_1 + c_2Lv_2 + \ldots \text{ by linearity of } L \\ &= c_1\lambda v_1 + c_2\lambda v_2 + \ldots \text{ since } Lv_i = \lambda v_i \\ &= \lambda(c_1v_1 + c_2v_2 + \ldots). \end{aligned}$$

So every linear combination of the v_i is an eigenvector of L with the same eigenvalue λ.

The space of all vectors with eigenvalue λ is called an *eigenspace*. It is, in fact, a vector space contained within the larger vector space V: It contains 0_V, since $L0_V = 0_V = \lambda 0_V$, and is closed under addition and scalar multiplication by the above calculation. All other vector space properties are inherited from the fact that V itself is a vector space.

An eigenspace is an example of a *subspace* of V, a notion that we will explore further next time.

References

- Hefferon, Chapter Three, Section III.1: Representing Linear Maps with Matrices

- Hefferon, Chapter Five, Section II.3: Eigenvalues and Eigenvectors

Wikipedia:

- Eigen*

- Characteristic Polynomial

- Linear Transformations (and matrices thereof)

Review Questions

1. Explain why the characteristic polynomial of an $n \times n$ matrix has degree n. Make your explanation easy to read by starting with some simple examples, and then use properties of the determinant to give a *general* explanation.

2. Compute the characteristic polynomial $P_M(\lambda)$ of the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Now, since we can evaluate polynomials on square matrices,

we can plug M into its characteristic polynomial and find the matrix $P_M(M)$. What do you find from this computation? Investigate whether something similar holds for $n \times n$ matrices.