16. Subspaces and Spanning Sets

It is time to study vector spaces more carefully and answer some fundamental questions.

1. *Subspaces*: When is a subset of a vector space itself a vector space? (This is the notion of a *subspace*.)
2. *Linear Independence*: Given a collection of vectors, is there a way to tell whether they are independent, or if one is a linear combination of the others?
3. *Dimension*: Is there a consistent definition of how "big" a vector space is?
4. *Basis*: How do we label vectors? Can we write any vector as a sum of some basic set of vectors? How do we change our point of view from vectors labeled one way to vectors labeled in another way?

Let's start at the top.

Subspaces

Definition We say that a subset U of a vector space V is a *subspace* of V if U is a vector space under the inherited addition and scalar multiplication operations of V.

Example Consider a plane P in $\mathbb{R}^3$ through the origin:

$$ax + by + cz = 0$$

This plane can be expressed as the homogeneous system $\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$, $MX = 0$. If X_1 and X_2 are both solutions to $MX = 0$, then, by linearity of matrix multiplication, so is $\mu X_1 + \nu X_2$:

$$M(\mu X_1 + \nu X_2) = \mu M X_1 + \nu M X_2 = 0$$

So P is closed under addition and scalar multiplication. Additionally, P contains the origin (which can be derived from the above by setting $\mu = \nu = 0$). All other vector space requirements hold for P because they hold for all vectors in $\mathbb{R}^3$.

Lemma. *Let U be a non-empty subset of a vector space V. Then U is a subspace if and only if $\mu u_1 + \nu u_2 \in U$ for arbitrary u_1, u_2 in U, and arbitrary constants μ, ν.*

Proof. The proof is left as an exercise to the reader. □

Building Subspaces

Consider the set

$$U = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \} \subset \mathbb{R}^3.$$

Since U is only two vectors, it is clear that U is *not* a vector space. For example, the 0-vector is not in U, nor is U closed under vector addition.

But we know that any two vector define a plane. In this case, the vectors in U define the xy-plane in $\mathbb{R}^3$. We can consider the xy-plane as the set of all vectors that arise as a linear combination of the two vectors in U. Call this set of all linear combinations the *span* of U:

$$\text{span}(U) = \{x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} | x, y \in \mathbb{R}\}.$$

Any vector in the xy-plane is of the form

$$\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \text{span}(U).$$

Definition Let V be a vector space and $S = \{s_1, s_2, \ldots\} \subset V$ a subset of V. Then that *span of S* is the set:

$$\text{span}(U) = \{r_1 s_1 + r_2 s_2 + \ldots | r_i \in \mathbb{R}\}.$$

(Should we only allow finite sums for linear combinations?)

Example Let $V = \mathbb{R}^3$ and $X \subset V$ be the x-axis. Let $P = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and set $S = X \cup P$.

The elements of span(S) are linear combinations of vectors in the x-axis and the vector P.

Since the sum of any number of vectors along the x-axis is still a vector in the x-acis, then the elements of S are all of the form:

$$\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}.$$

Then span(S) is the xy-plane, which is a vector space.

('spanning set'=set of vectors whose span is a subspace, or the actual subspace?)

Lemma. *For any subset $S \subset V$,* span(S) *is a subspace of V.*

Proof. We need to show that span(S) is a vector space.

It suffices to show that span(S) is closed under linear combinations. Let $u, v \in \text{span}(S)$ and λ, μ be constants. By the definition of span(S), there are constants c_i and d_i such that:

$$\begin{aligned} u &= c_1 s_1 + c_2 s_2 + \ldots \\ v &= d_1 s_1 + d_2 s_2 + \ldots \\ \Rightarrow \lambda u + \mu v &= \lambda(c_1 s_1 + c_2 s_2 + \ldots) + \mu(d_1 s_1 + d_2 s_2 + \ldots) \\ &= (\lambda c_1 + \mu d_1) s_1 + (\lambda c_2 + \mu d_2) s_2 + \ldots \end{aligned}$$

This last sum is a linear combination of elements of S, and is thus in span(S). Then span(S) is closed under linear combinations, and is thus a subspace of V. □

Note that this proof consisted of little more than just writing out the definitions.

Example For which values of a does

$$\text{span}\{ \begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} a \\ 1 \\ 0 \end{pmatrix} \} = \mathbb{R}^3?$$

Given an arbitrary vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $\mathbb{R}^3$, we need to find constants r_1, r_2, r_3 such that

$$r_1 \begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix} + r_2 \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + r_3 \begin{pmatrix} a \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

We can write this as a linear system in the unknowns r_1, r_2, r_3 as follows:

$$\begin{pmatrix} 1 & 1 & a \\ 0 & 2 & 1 \\ a & -3 & 0 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

If the matrix $M = \begin{pmatrix} 1 & 1 & a \\ 0 & 2 & 1 \\ a & -3 & 0 \end{pmatrix}$ is invertible, then we can find a solution $M^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$ for *any* vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.

Then we should choose a so that M is invertible.

$$\det M = -2a^2 + 3 + a = -(2a - 3)(a + 1).$$

Then the span is $\mathbb{R}^3$ if and only if $a \neq -1, \frac{3}{2}$.

References

- Hefferon, Chapter Two, Section I.2: Subspaces and Spanning Sets

Wikipedia:

- Linear Subspace
- Linear Span

Review Questions

1. Suppose that V is a vector space and that $U \subset V$ is a subset of V. Show that

$$\mu u_1 + \nu u_2 \in U \text{ for all } u_1, u_2 \in U, \mu, \nu \in \mathbb{R}$$

implies that U is a subspace of V. (In other words, check all the vector space requirements for U.)

2. Let $P_3[x]$ be the vector space of degree 3 polynomials in the variable x. Check whether

$$x - x^3 \in \text{span}\{x^2, 2x + x^2, x + x^3\}$$

3. Let U and W be subspaces of V. Are:

   $i.$ $U \cup W$

   $ii.$ $U \cap W$

   also subspaces? Explain why or why not. Draw examples in $\mathbb{R}^3$.