18. Linear Independence

Consider a plane P that includes the origin in $\mathbb{R}^3$ and a collection $\{u, v, w\}$ of non-zero vectors in P. If no two of u, v and w are parallel, then certainly $P = \text{span}\{u, v, w\}$. But any two vectors determines a plane, so we should be able to span the plane using only two vectors. Then we could choose two of the vectors in $\{u, v, w\}$ whose span is P, and express the other as a linear combination of those two. Suppose u and v span P. Then there exist constants d^1, d^2 (not all zero) such that $w = d^1u + d^2v$. Since w can be expressed in terms of u and v we say that it is not independent. In other words, the relationship

$$c^1u + c^2v + c^3w = 0 \qquad c^i \in \mathbb{R}, \text{ some } c^i \neq 0$$

expresses the fact that u, v, w are not all independent.

Definition We say that the vectors $v_1, v_2, \ldots, v_n$ are *linearly dependent* if there exist constants $c_1, c_2, \ldots, c_n$ not all zero such that

$$c^1v_1 + c^2v_2 + \ldots + c^nv_n = 0.$$

Otherwise, the vectors $v_1, v_2, \ldots, v_n$ are *linearly independent.*

Example Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Are they linearly independent?

We need to see whether the system

$$c^1v_1 + c^2v_2 + c^3v_3 = 0.$$

has any solutions for c_1, c_2, c_3. We can rewrite this as a homogeneous system:

$$\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} c^1 \\ c^2 \\ c^3 \end{pmatrix} = 0$$

This system has solutions if and only if the matrix $M = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$ is singular, so we should find the determinant of M.

$$\det M = \det \begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 3 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = 0.$$

Then solutions exist. At this point we know that the vectors are linearly dependent. If we need to, we can find coefficients that demonstrate linear independence by solving the system of equations:

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Then $c^3 = \mu$, $c^2 = -\mu$, and $c^3 = -2\mu$. Then set $\mu = 1$ and obtain:

$$c^1 v_1 + c^2 v_2 + c^3 v_3 = 0 \Rightarrow -2v_1 - v_2 + v_3 = 0.$$

Theorem (Linear Dependence)**.** *A set of non-zero vectors* $\{v_1, \ldots, v_n\}$ *is linearly dependent if and only if one of the vectors* v_k *is expressible as a linear combination of the preceeding vectors.*

Proof. The theorem is an if and only if statement, so there are two things to show.

i. First, we show that if $v_k = c^1 v_1 + \ldots c^{k-1} v_{k-1}$ then the set is linearly dependent.

This is easy. We just rewrite the assumption:

$$c^1 v_1 + \ldots c^{k-1} v_{k-1} - v_k + 0v_{k+1} + \ldots + 0v_n = 0.$$

This is a vanishing linear combination of the vectors $\{v_1, \ldots, v_n\}$ with not all coefficients equal to zero, so $\{v_1, \ldots, v_n\}$ is a linearly dependent set.

ii. Now, we show that linear dependence implies that there exists k for which v_k is a linear combination of the vectors $\{v_1, \ldots, v_{k-1}\}$.

The assumption says that

$$c^1 v_1 + c^2 v_2 + \ldots + c^n v_n = 0.$$

Take k to be the largest number for which c_k is not equal to zero. Then:

$$c^1 v_1 + c^2 v_2 + \ldots + c^{k-1} v_{k-1} + c^k v_k = 0.$$

This tells us that $k > 1$, since otherwise we would have $c^1 v_1 = 0 \Rightarrow v_1 = 0$, contradicting the assumption that none of the v_i are the zero vector.

Then we can rearrange the equation:

$$\begin{aligned} c^1v_1 + c^2v_2 + \ldots + c^{k-1}v_{k-1} &= -c^kv_k \\ -\frac{c^1}{c^k}v_1 - \frac{c^2}{c^k}v_2 - \ldots - \frac{c^{k-1}}{c^k}v_{k-1} &= v_k. \end{aligned}$$

Then we have expressed v_k as a linear combination of the previous vectors, so we are done.

□

Example Consider the vector space $P_2(t)$ of polynomials of degree less than or equal to 2. Set:

$$\begin{aligned} v_1 &= 1+t \\ v_2 &= 1+t^2 \\ v_3 &= t+t^2 \\ v_4 &= 2+t+t^2 \\ v_5 &= 1+t+t^2 \end{aligned}$$

The set $\{v_1, \ldots, v_5\}$ is linearly dependent, because $v_4 = v_1 + v_2$. Now suppose vectors $v_1, \ldots, v_n$ are linearly dependent and

$$c^1v_1 + c^2v_2 + \ldots + c^nv_n = 0$$

with $c^1 \neq 0$. Then:

$$\text{span}\{v_1, \ldots, v_n\} = \text{span}\{v_2, \ldots, v_n\}$$

because any $x \in \text{span}\{v_1, \ldots, v_n\}$ is given by

$$\begin{aligned} x &= a^1v_1 + \ldots a^nv_n \\ &= a^1(-\frac{c^2}{c_1}v_2 - \ldots - \frac{c^n}{c_1}v_n) + a^2v_2 + \ldots + a^nv_n \\ &= (a^2 - a^1\frac{c^2}{c_1})v_2 + \ldots + (a^n - a^1\frac{c^n}{c_1})v_n. \end{aligned}$$

Then x is in $\text{span}\{v_2, \ldots, v_n\}$.

When we write a vector space as the span of a list of vectors, we would like that list to be as short as possible.

This can be achieved by iterating the above procedure.

Example In the above example, we found that $v_4 = v_1 + v_2$. In this case, any expression for a vector as a linear combination involving v_4 can be turned into a combination without v_4 by making the substitution $v_4 = v_1 + v_2$.

Then:

$$\begin{aligned} S &= \operatorname{span}\{1+t, 1+t^2, t+t^2, 2+t+t^2, 1+t+t^2\} \\ &= \operatorname{span}\{1+t, 1+t^2, t+t^2, 1+t+t^2\}. \end{aligned}$$

Now we can notice that $1 + t + t^2 = \frac{1}{2}([1+t] + [1+t^2] + [t+t^2])$. Then the vector $1+t+t^2 = v_5$ is also extraneous, since it can be expressed as a linear combination of the remaining three vecotrs, v_1, v_2, v_3. Then

$$S = \operatorname{span}\{1+t, 1+t^2, t+t^2\}.$$

Now there are no (non-zero) solutions to the linear system

$$c^1(1+t) + c^2(1+t^2) + c^3(t+t^2) = 0.$$

Then the remaining vectors $\{1+t, 1+t^2, t+t^2\}$ are linearly independent, and span the vector space S. Then these vectors are a minimal spanning set, which is called a *basis* for S.

Definition Given a vector space S and a set B of vectors such that:

- $\operatorname{span} B = S$, and
- There exists no subset $A \subset B$ such that $\operatorname{span} A = S$,

then B is called a *basis* of S.

Example Let B^3 be the space of 3×1 bit-valued matrices (i.e., column vectors). Is the following set linearly independent?

$$\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}$$

If the set is linearly dependent, then we can find non-zero solutions to the system:

$$c^1\begin{pmatrix}1\\1\\0\end{pmatrix} + c^2\begin{pmatrix}1\\0\\1\end{pmatrix} + c^3\begin{pmatrix}0\\1\\1\end{pmatrix} = 0,$$

which becomes the linear system

$$\begin{pmatrix}1 & 1 & 1\\ 1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}c^1\\ c^2\\ c^3\end{pmatrix} = 0.$$

Solutions exist if and only if the determinant of the matrix is non-zero. But:

$$\det\begin{pmatrix}1 & 1 & 1\\ 1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix} = 1\det\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix} - 1\det\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = -1 - 1 = 1 + 1 = 0$$

Then non-trivial solutions exist, and the set is thus not linearly independent.

Remark Do you see any similarity between these vectors and the vectors in $P_2(t)$ in the previous example?

References

- Hefferon, Chapter Two, Section II: Linear Independence
- Hefferon, Chapter Two, Section III.1: Basis

Wikipedia:

- Linear Independence
- Basis

Review Questions

1. Let B^n be the space of $n \times 1$ bit-valued matrices (i.e., column vectors).
    i. How many different vectors are there in B^n.
    ii. Find a collection S of vectors that span B^3 and are linearly independent. In other words, find a basis of B^3.
    iii. Write each other vector in B^3 as a linear combination of the vectors in the set S that you chose.
    iv. Would it be possible to span B^3 with only two vectors?

2. Let e_i be the vector in $\mathbb{R}^n$ with a 1 in the ith position and 0's in every other position. Let v be an arbitrary vector in $\mathbb{R}^n$.

   i. Prove that the collection $\{e_1, \ldots, e_n\}$ is linearly independent.

   ii. Show that $v = \sum_{i=1}^{n} (v \bullet e_i) e_i$.

   iii. What does this say about the span$\{e_1, \ldots, e_n\}$?