# 4. Solution Sets for Systems of Linear Equations

For a system of equations with $r$ equations and $k$ unknowns, one can have a number of different outcomes. For the sake of visualization, consider the case of $r$ equations in three variables. Geometrically, then, each of our equations is the equation of a plane in three-dimensional space. To find solutions to the system of equations, we look for the common intersection of the planes (if an intersection exists). Here we have five different possibilities:

1. **No solutions.** Some of the equations are contradictory, so no solutions exist.

2. **Unique Solution.** The planes have a unique point of intersection.

3. **Line.** The planes intersect in a common line; any point on that line then gives a solution to the system of equations.

4. **Plane.** Perhaps you only had one equation to begin with, or else all of the equations coincide geometrically. In this case, you have a plane of solutions, with two free parameters.

5. **All of $\mathbb{R}^3$.** If you start with no information, then any point in $\mathbb{R}^3$ is a solution. There are three free parameters.

In general, for systems of equations with $k$ unknowns, there are $k + 2$ possibile outcomes, corresponding to the number of free parameters in the solutions set, plus the possibility of no solutions. These types of solution sets are hard to visualize, but luckily 'hyperplanes' behave like planes in $\mathbb{R}^3$ in many ways.

## Non-Leading Variables

Variables that are not a pivot in the reduced row echelon form of a linear system are *free*. Se set them equal to arbitrary parameters $\mu_1, \mu_2, \ldots$.

**Example** $\left(\begin{array}{cccc|c} 1 & & 1 & -1 & 1 \\ & 1 & -1 & 1 & -1 \\ & & & & \end{array}\right)$

Here, $x_1$ and $x_2$ are the pivot variables and $x_3$ and $x_4$ are non-leading variables, and thus free. The solutions are then of the form $x_3 = \mu_1$, $x_4 = \mu_2$, $x_2 = 1 + \mu_1 - \mu_2$, $x_1 = 1 - \mu_1 + \mu_2$.

The preferred way to write a solution set is with set notation. Let $S$ be the set of solutions to the system. Then:

$$S = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \mu_1 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \}$$

It's worth noting that if we knew how to multiply matrices of any size, we could write the previous system as $MX = v$, where

$$M = \begin{pmatrix} 1 & & 1 & -1 \\ & 1 & -1 & 1 \end{pmatrix}, \qquad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \qquad v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Given two vectors we can *add* them term-by-term:

$$\begin{pmatrix} a^1 \\ a^2 \\ a^3 \\ \vdots \\ a^r \end{pmatrix} + \begin{pmatrix} b^1 \\ b^2 \\ b^3 \\ \vdots \\ b^r \end{pmatrix} = \begin{pmatrix} a^1 + b^1 \\ a^2 + b^2 \\ a^3 + b^3 \\ \vdots \\ a^r + b^r \end{pmatrix}$$

We can also multiply a vector by a scalar, like so:

$$\lambda \begin{pmatrix} a^1 \\ a^2 \\ a^3 \\ \vdots \\ a^r \end{pmatrix} = \begin{pmatrix} \lambda a^1 \\ \lambda a^2 \\ \lambda a^3 \\ \vdots \\ \lambda a^r \end{pmatrix}$$

Then yet another way to write the solution set for the example is:

$$X = X_0 + \mu_1 Y_1 + \mu_2 Y_2$$

where

$$X_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, Y_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, Y_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

**Definition** Let $X$ and $Y$ by vectors and $\alpha$ and $\beta$ be scalars. A function $f$ is *linear* if

$$f(\alpha X + \beta Y) = \alpha f(X) + \beta f(Y)$$

Eventually, we'll prove that matrix multiplication is linear. Then we will know that:

$$M(\alpha X + \beta Y) = \alpha M X + \beta M Y$$

Then the two equations $MX = v$ and $X = X_0 + \mu_1 Y_1 + \mu_2 Y_2$ together say that:

$$MX_0 + \mu_1 MY_1 + \mu_2 MY_2 = V$$

for *any* $\mu_1, \mu_2 \in \mathbb{R}$.

Choosing $\mu_1 = \mu_2 = 0$, we obtain $MX_0 = V$. Given the particular solution to the system, we can then deduce that $\mu_1 MY_1 + \mu_2 MY_2 = 0$. $X_0$ is an example of a *particular solution* to the system.

Setting $\mu_1 = 0, \mu_2 = 1$, and recalling the particular solution $MX_0 = V$, we obtain $MY_1 = 0$.

Likewise, setting $\mu_1 = 1, \mu_2 = 0$, we obtain $MY_2 = 0$.

$Y_1$ and $Y_2$ are *homogeneous* solutions to the system.

**Example** Consider the linear system with the augmented matrix we've been working with.

$$\begin{array}{rrrrl} x & & +z & -w & = 1 \\ & y & -z & +w & = 1 \end{array}$$

Recall that the system has the following solution set:

$$S = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mu_1 \begin{pmatrix} 1 \\ -1 \\ \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \}$$

Then $MX_0 = V$ says that $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ solves the system, which is certainly true.

$MY_1 = 0$ says that $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$ solves the system.

$$MY_2 = 0 \text{ says that } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \text{ solves the system.}$$

**Definition** Let $M$ a matrix and $V$ a vector. Given the linear system $MX = V$, we call $X_0$ a *particular solution* if $MX_0 = V$. We call $Y$ a *homogeneous solution* if $MY = 0$.

The linear system $MX = 0$ is called the (associated) *homogeneous system.*

If $X_0$ is a particular solution, then the general solution to the system is:

$$S = \{X_0 + y : MY = 0\}$$

In other words, the general solution = particular + homogeneous.

## Review Questions

1. Write down examples of augmented matrices corresponding to each of the five types of solution sets for systems of equations with three unknowns.

2. Let

$$M = \begin{pmatrix} a_1^1 & a_2^1 & \dots & a_k^1 \\ a_1^2 & a_2^2 & \dots & a_k^2 \\ \vdots & \vdots & & \vdots \\ a_1^r & a_2^r & \dots & a_k^r \end{pmatrix}, \qquad X = \begin{pmatrix} x^1 \\ x^2 \\ \dots \\ x^k \end{pmatrix}$$

Propose a rule for $MX$ so that $MX = 0$ is equivalent to the linear system:

$$\begin{array}{llll} a_1^1x^1 & +a_2^1x^2 & \dots +a_k^1x^k & = 0 \\ a_1^2x^1 & +a_2^2x^2 & \dots +a_k^2x^k & = 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_1^rx^1 & +a_2^rx^2 & \dots +a_k^rx^k & = 0 \end{array}$$

Does your rule for multiplying a matrix times a vector obey the linearity property? Prove it!

3. The *standard basis vector* $e_i$ is a column vector with a one in the $i$th row, and zeroes everywhere else. Using the rule for multiplying a matrix times a vector in the last problem, find a simple rule for multiplying $Me_i$, where $M$ is the general matrix defined in the last problem.