7. Linear Transformations

Recall that the key properties of vector spaces are vector addition and scalar multiplication. Now suppose we have two vector spaces V and W and a map L between them:

$$L : V \to W$$

Now, both V and W have notions of vector addition and scalar multiplication. It would be ideal if the map L *preserved* these operations. In other words, if adding vectors and then applying L were the same as applying L to two vectors and then adding them. Likewise, it would be nice if, when multiplying by a scalar, it didn't matter whether we multiplied before or after applying L. In formulas, this means that for any $u, v \in V$ and $c \in \mathbb{R}$:

$$\begin{aligned} L(u+v) &= L(u) + L(v) \\ L(cv) &= cL(v) \end{aligned}$$

Combining these two requirements into one equation, we get the definition of a linear function.

Definition A function $L : V \to W$ is linear if for all $u, v \in V$ and $r, s \in \mathbb{R}$ we have

$$L(ru + sv) = rL(u) + sL(v)$$

Notice that on the left the addition and scalar multiplication occur in V, while on the right the operations occur in W.

Example Take $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by:

$$L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ y+z \\ 0 \end{pmatrix}$$

Call $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Now check linearity.

$$
\begin{aligned}
L(ru+sv) &= L(r\begin{pmatrix}x\\y\\z\end{pmatrix}+s\begin{pmatrix}a\\b\\c\end{pmatrix})\\
&= L(\begin{pmatrix}rx\\ry\\rz\end{pmatrix}+\begin{pmatrix}sa\\sb\\sc\end{pmatrix})\\
&= L\begin{pmatrix}rx+sa\\ry+sb\\rz+sx\end{pmatrix}\\
&= \begin{pmatrix}rx+sa+ry+sb\\ry+sb+rz+sx\\0\end{pmatrix}
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
rL(u)+sL(v) &= rL\begin{pmatrix}x\\y\\z\end{pmatrix}+sL\begin{pmatrix}a\\b\\c\end{pmatrix}\\
&= r\begin{pmatrix}x+y\\y+z\\0\end{pmatrix}+s\begin{pmatrix}a+b\\b+c\\0\end{pmatrix}\\
&= \begin{pmatrix}rx+ry\\ry+rz\\0\end{pmatrix}+\begin{pmatrix}sa+sb\\sb+sc\\0\end{pmatrix}\\
&= \begin{pmatrix}rx+sa+ry+sb\\ry+sb+rz+sx\\0\end{pmatrix}
\end{aligned}
$$

Then the two sides of the linearity requirement are equal, so L is a linear transformation.

Remark We can write L using a matrix like so:

$$
L\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x+y\\y+z\\0\end{pmatrix}=\begin{pmatrix}1&1&0\\0&1&1\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}
$$

We previously checked that matrix multiplication on vectors obeyed the rule $M(ru+sv)=rMu+sMv$, so matrix multiplication is linear. As such, our check on L was guaranteed to work. In fact, matrix multiplication on vectors is a linear transformation.

Example Let V be the vector space of polynomials of finite degree with standard addition and scalar multiplication.

$$V = \{a_0 + a_1 x + \ldots + a_n x^n | n \in \mathbb{N}, a_i \in \mathbb{R}\}$$

Let $L : V \to V$ be the derivative $\frac{d}{dx}$. For p_1 and p_2 polynomials, the rules of differentiation tell us that

$$\frac{d}{dx}(rp_1 + sp_2) = r\frac{dp_1}{dx} + s\frac{dp_2}{dx}$$

Thus, the derivative is a linear function from the set of polynomials to itself.

We can represent a polynomial as a semi-infinite vector, like so:

$$a_0 + a_1 x + \ldots + a_n x^n \longleftrightarrow \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

Then we have:

$$\frac{d}{dx}(a_0 + a_1 x + \ldots + a_n x^n) = a_1 + 2a_2 x + \ldots + n a_n x^{n-1} \longleftrightarrow \begin{pmatrix} a_1 \\ 2a_2 \\ \vdots \\ na_n \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

One could then write the derivative as an infinite matrix:

$$\frac{d}{dx} \longleftrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 2 & 0 & \ldots \\ 0 & 0 & 0 & 3 & \ldots \\ \vdots & & & \vdots & \end{pmatrix}$$

Foreshadowing Dimension. You probably have some intuitive notion of what dimension means, though we haven't actually defined the idea of dimension mathematically yet. Some of the examples of vector spaces we have worked with have been finite dimensional. (For example, $\mathbb{R}^n$ will turn out to have dimension n.) The polynomial example above is an example of an infinite dimensional vector space.

Roughly speaking, dimension is the number of independent directions available. To figure out dimension, I stand at the origin, and pick a direction. If there are any vectors in my vector space that aren't in that direction, then I choose another direction that isn't in the line determined by the direction I chose. If there are any vectors in my vector space not in the plane determined by the first two directions, then I choose one of them as my next direction. In other words, I choose a collection of *independent* vectors in the vector space. The size of a minimal set of independent vectors is the dimension of the vector space.

For finite dimensional vector spaces, linear transformations can always be represented by matrices. For that reason, we will start studying matrices intensively in the next few lectures.

References

Hefferon, Chapter Three, Section II. (Note that Hefferon uses the term *homomorphism* for a linear map. 'Homomorphism' is a very general term which in mathematics means 'Structure-preserving map.' A linear map preserves the linear structure of a vector space, and is thus a type of homomorphism.)

Wikipedia:

- Linear Transformation
- Dimension

Review Questions

1. Show that the pair of conditions:

$$L(u+v) = L(u) + L(v) \tag{1}$$
$$L(cv) = cL(v) \tag{2}$$

is equivalent to the single condition:

$$L(ru+sv) = rL(u) + sL(v) \tag{3}$$

Your answer should have two parts. Show that $(1,2) \Rightarrow (3)$, and then show that $(3) \Rightarrow (1,2)$.

2. Let P_n be the space of degree n polynomials in the variable t. Suppose L is a linear transformation from $P_2 \to P_3$ such that $L(1) = 4, L(t) = t^3$, and $L(t^2) = t - 1$.

   - Find $L(1 + t + 2t^2)$.
   - Find $L(a + bt + ct^2)$.
   - Find all values a, b, c such that $L(a + bt + ct^2) = 1 + 3t + 2t^3$.

3. Show that integration is a linear transformation on the vector space of polynomials. What would a matrix for integration look like?